6e

COLLEGE ALGEBRA

JAMES STEWART ▼ LOTHAR REDLIN ▼ SALEEM WATSON

EXPONENTS AND RADICALS

$x^m x^n = x^{m+n}$ $\qquad$ $\dfrac{x^m}{x^n} = x^{m-n}$

$(x^m)^n = x^{mn}$ $\qquad$ $x^{-n} = \dfrac{1}{x^n}$

$(xy)^n = x^n y^n$ $\qquad$ $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

$x^{1/n} = \sqrt[n]{x}$ $\qquad$ $x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$

$\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$ $\qquad$ $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$

$\sqrt[m]{\sqrt[n]{x}} = \sqrt[n]{\sqrt[m]{x}} = \sqrt[mn]{x}$

SPECIAL PRODUCTS

$(x + y)(x - y) = x^2 - y^2$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x - y)^2 = x^2 - 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

FACTORING FORMULAS

$x^2 - y^2 = (x + y)(x - y)$

$x^2 + 2xy + y^2 = (x + y)^2$

$x^2 - 2xy + y^2 = (x - y)^2$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

QUADRATIC FORMULA

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

INEQUALITIES AND ABSOLUTE VALUE

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

If $a > 0$, then

$|x| = a$ means $x = a$ or $x = -a$.

$|x| < a$ means $-a < x < a$.

$|x| > a$ means $x > a$ or $x < -a$.

DISTANCE AND MIDPOINT FORMULAS

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of P_1P_2: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

LINES

Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ $\qquad$ $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Point-slope equation of line through $P_1(x_1, y_1)$ with slope m $\qquad$ $y - y_1 = m(x - x_1)$

Slope-intercept equation of line with slope m and y-intercept b $\qquad$ $y = mx + b$

Two-intercept equation of line with x-intercept a and y-intercept b $\qquad$ $\dfrac{x}{a} + \dfrac{y}{b} = 1$

The lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are

Parallel if the slopes are the same $\qquad$ $m_1 = m_2$

Perpendicular if the slopes are negative reciprocals $\qquad$ $m_1 = -1/m_2$

LOGARITHMS

$y = \log_a x$ means $a^y = x$

$\log_a a^x = x$ $\qquad$ $a^{\log_a x} = x$

$\log_a 1 = 0$ $\qquad$ $\log_a a = 1$

Common and natural logarithms

$\log x = \log_{10} x$ $\qquad$ $\ln x = \log_e x$

Laws of logarithms

$\log_a xy = \log_a x + \log_a y$

$\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

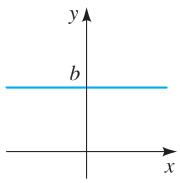
$\log_a x^b = b \log_a x$

Change of base formula

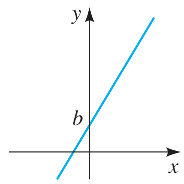
$$\log_b x = \frac{\log_a x}{\log_a b}$$

GRAPHS OF FUNCTIONS

Linear functions: $f(x) = mx + b$

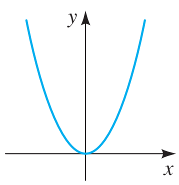


$f(x) = b$

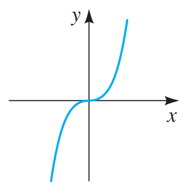


$f(x) = mx + b$

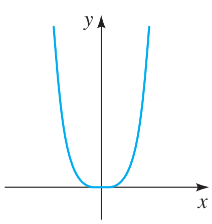
Power functions: $f(x) = x^n$



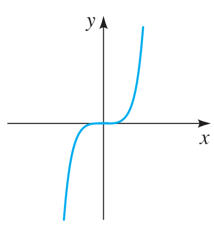
$f(x) = x^2$



$f(x) = x^3$

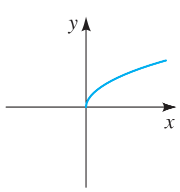


$f(x) = x^4$

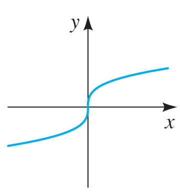


$f(x) = x^5$

Root functions: $f(x) = \sqrt[n]{x}$

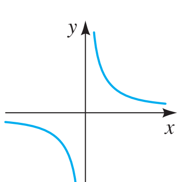


$f(x) = \sqrt{x}$

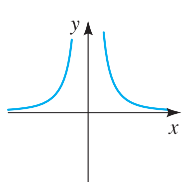


$f(x) = \sqrt[3]{x}$

Reciprocal functions: $f(x) = 1/x^n$

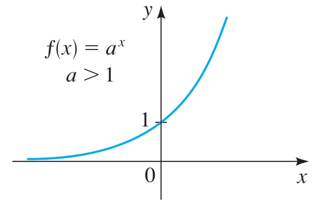


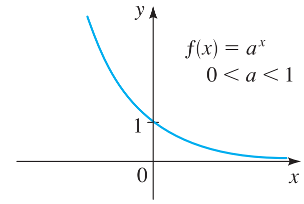
$f(x) = \dfrac{1}{x}$



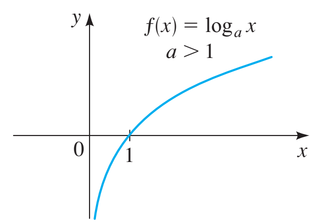
$f(x) = \dfrac{1}{x^2}$

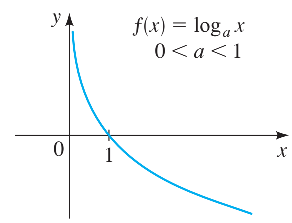
Exponential functions: $f(x) = a^x$



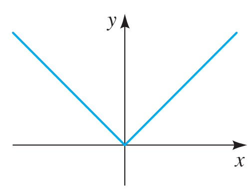


Logarithmic functions: $f(x) = \log_a x$



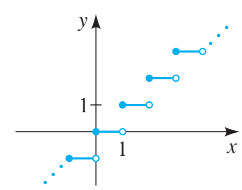


Absolute value function



$f(x) = |x|$

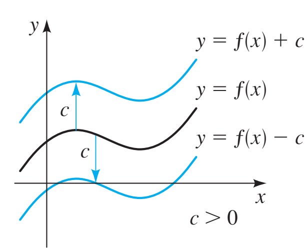
Greatest integer function



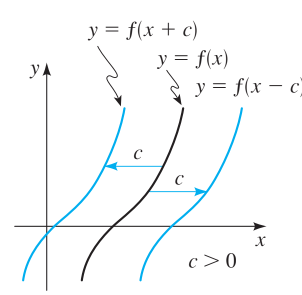
$f(x) = [\![x]\!]$

SHIFTING OF FUNCTIONS

Vertical shifting



Horizontal shifting



SIXTH EDITION

College Algebra

About the Authors

James Stewart received his MS from Stanford University and his PhD from the University of Toronto. He did research at the University of London and was influenced by the famous mathematician George Polya at Stanford University. Stewart is Professor Emeritus at McMaster University and is currently Professor of Mathematics at the University of Toronto. His research field is harmonic analysis and the connections between mathematics and music. James Stewart is the author of a bestselling calculus textbook series published by Brooks/Cole, Cengage Learning, including *Calculus, Calculus: Early Transcendentals*, and *Calculus: Concepts and Contexts*; a series of precalculus texts; and a series of high-school mathematics textbooks.

Lothar Redlin grew up on Vancouver Island, received a Bachelor of Science degree from the University of Victoria, and received a PhD from McMaster University in 1978. He subsequently did research and taught at the University of Washington, the University of Waterloo, and California State University, Long Beach. He is currently Professor of Mathematics at The Pennsylvania State University, Abington Campus. His research field is topology.

Saleem Watson received his Bachelor of Science degree from Andrews University in Michigan. He did graduate studies at Dalhousie University and McMaster University, where he received his PhD in 1978. He subsequently did research at the Mathematics Institute of the University of Warsaw in Poland. He also taught at The Pennsylvania State University. He is currently Professor of Mathematics at California State University, Long Beach. His research field is functional analysis.

Stewart, Redlin, and Watson have also published *Precalculus: Mathematics for Calculus, Trigonometry, Algebra and Trigonometry,* and (with Phyllis Panman) *College Algebra: Concepts and Contexts*.

About the Cover

The cover photograph shows the Turning Torso Tower in Malmö, Sweden. The spiraling tower of this apartment building consists of nine pentagonal segments, each containing five floors. Built from 2001 to 2005, it was designed by Santiago Calatrava, a Spanish architect. Calatrava has always been very interested in how mathematics can help him to realize the buildings he imagines. As a young student, he taught himself descriptive geometry from books in order to represent three-dimensional objects in two dimensions. Trained as both an engineer and an architect, he wrote a doctoral thesis in 1981 entitled "On the Foldability of Space Frames," which is filled with mathematics, especially geometric transformations. His strength as an engineer enables him to be daring in his architecture.

SIXTH EDITION

COLLEGE ALGEBRA

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THE PENNSYLVANIA STATE UNIVERSITY

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James Stewart, Lothar Redlin, Saleem Watson

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PREFACE

For many students a College Algebra course represents the first opportunity to discover the beauty and practical power of mathematics. Thus instructors are faced with the challenge of teaching the concepts and skills of algebra while at the same time imparting a sense of its utility in the real world. In this edition, as in the previous editions, our aim is to provide instructors and students with tools they can use to meet this challenge.

The emphasis is on understanding concepts. To help instructors foster conceptual understanding in their students, we use the rule of four: "Topics should be presented geometrically, numerically, algebraically, and verbally." A major part of conceptual understanding is an appreciation for the logical structure of algebra. For this reason every statement about algebra in this book is supported by a proof, an explanation, or an intuitive argument. Students also need to achieve a certain level of technical skill. Indeed, *conceptual understanding* and *technical skill* go hand in hand, each reinforcing the other. Above all, we present algebra as a problem-solving art with numerous applications to modeling and solving real-world problems.

In writing this Sixth Edition our purpose was to further enhance the utility of the book as an instructional tool for teachers and as a learning tool for students. We have made several major changes in this edition. These include a restructuring of the beginning chapters to allow for an earlier introduction to functions. Several chapters have been reorganized and rewritten (as described below) with the goal of further focusing the exposition on the main concepts. Each exercise set has been reexamined and enhanced where necessary. We have included more use of the powerful capabilities of the graphing calculator. In all these changes and numerous others, small and large, we have retained the main features that have contributed to the success of this book.

New to the Sixth Edition

- **Early Chapter on Functions** The chapter on functions now appears earlier in the book. The review material (in Chapters P and 1) has been streamlined and rewritten.
- **Diagnostic Test** A diagnostic test, designed to test preparedness for College Algebra, can be found at the beginning of the book (p. xxi).
- **Exercises** More than 20% of the exercises are new.
- **Book Companion Website** A new website, **www.stewartmath.com,** contains *Discovery Projects* for each chapter and *Focus on Problem Solving* sections that highlight different problem-solving principles outlined in the Prologue.
- **Chapter P: Prerequisites** This chapter now contains a section on basic equations, including linear equations and power equations.

- **Chapter 1: Equations and Graphs** This is a new chapter that includes an introduction to the coordinate plane and graphs of equations in two variables, as well as the material on solving equations. Including these topics in one chapter highlights the relationship between algebraic and graphical solutions of equations.
- **Chapter 2: Functions** This chapter now includes a discussion of "net change" in the value of a function. This concept emphasizes the dynamic nature of functions and prepares the way for the study of "average rate of change" of a function.
- **Chapter 3: Polynomial and Rational Functions** This chapter includes two new sections. The material on complex numbers has been moved to this chapter because the main use of complex numbers in this book is in the solution of polynomial equations. The section on variation now follows the section on rational functions, allowing for a graphical analysis of variation.
- **Chapter 4: Exponential and Logarithmic Functions** The material on the natural exponential function is now in a separate section. The section on modeling with exponential functions includes a subsection on converting a model from one base to another.
- **Chapter 5: Systems of Equations and Inequalities** This chapter now includes a separate section on nonlinear equations.
- **Chapter 6: Matrices and Determinants** This chapter now includes additional (optional) material on the use of the graphing calculator in working with matrices.
- **Chapter 9: Probability and Statistics** This chapter has been reorganized into four sections and includes new material on conditional probability and expected value.
- **Appendix A: Calculations and Significant Figures** This appendix contains information on working with approximate data and rounding final answers properly.
- **Appendix B: Graphing with a Graphing Calculator** This appendix contains general guidelines for interpreting graphs produced by a graphing calculator. It also discusses common pitfalls to avoid when using a graphing calculator.
- **Appendix C: Using the TI-83/84 Graphing Calculator** In this appendix we give step-by-step instructions for performing the basic calculator operations used in this book.

Teaching and Learning with the Help of This Book

We are keenly aware that good teaching comes in many forms and that there are many different approaches to teaching the concepts and skills of algebra. Moreover, students learn in different ways, absorbing ideas best in numerical, graphical, or verbal form. The organization of the topics in this book and the different types of exercises are designed to accommodate different teaching and learning styles.

Exercise Sets The most important way to foster conceptual understanding and hone technical skill is through the problems that the instructor assigns. To that end we have provided a wide selection of exercises.

- **Concept Exercises** These exercises ask students to use mathematical language to state fundamental facts about the topics of each section.
- **Skills Exercises** Each exercise set is carefully graded, progressing from basic skill-development exercises to more challenging problems requiring synthesis of previously learned material with new concepts.
- **Applications Exercises** We have included substantial applied problems that we believe will capture students' interest.
- **Discovery, Writing, and Group Learning** Each exercise set ends with a block of exercises labeled *Discovery* ■ *Discussion* ■ *Writing*. These exercises are designed to

encourage students to experiment, preferably in groups, with the concepts that were developed in the section and then to write about what they have learned rather than simply looking for "the answer."

- **Practice What You've Learned: Do Exercise . . .** At the end of each example in the text the student is directed to a similar exercise in the section that helps to reinforce the concepts and skills developed in that example (see, for instance, page 4).
- **Check Your Answer** Students are encouraged to check whether an answer they have obtained is reasonable. This is emphasized throughout the text in numerous *Check Your Answer* sidebars that accompany the examples (see, for instance, page 55).

GRAPHING CALCULATORS AND COMPUTERS We make use of graphing calculators and computers in examples and exercises throughout the book. Our calculator-oriented examples are always preceded by examples in which students must graph or calculate by hand, so that they can understand precisely what the calculator is doing when they later use it to simplify the routine, mechanical part of their work. The graphing calculator sections, subsections, examples, and exercises, all marked with the special symbol, are optional and may be omitted without loss of continuity. We use the following capabilities of the calculator.

- **Graphing, Regression, Matrix Algebra** The capabilities of the graphing calculator are used throughout the text to graph and analyze functions, families of functions, and sequences; to calculate and graph regression curves; to perform matrix algebra; to graph linear inequalities; to find partial sums of sequences; and for other powerful uses.
- **Using the TI-83/84 Graphing Calculator** Appendix C contains step-by-step instructions for performing all of the operations discussed in the text.
- **Simple Programs** We exploit the programming capabilities of the graphing calculator to simulate real-life situations, to sum series, or to compute the terms of a recursive sequence. (See, for instance, pages 520, 573, 577, and 667.)

FOCUS ON MODELING The "modeling" theme has been used throughout to unify and clarify the many applications of College Algebra. We have made a special effort to clarify the essential process of translating problems from English into the language of mathematics (see pages 248 and 423).

- **Constructing Models** There are numerous applied problems throughout the book in which students are given a model to analyze (see, for instance, page 263). But the material on modeling, in which students are required to *construct* mathematical models, has been organized into clearly defined sections and subsections (see, for example, pages 247, 384, and 421).
- **Focus on Modeling** Each chapter concludes with a *Focus on Modeling* section that explores the ways in which algebra can be used to model real-world situations. For example, the *Focus on Modeling* at the end of Chapter 1 introduces the basic idea of modeling a real-life situation by fitting lines to data. Other sections present ways in which polynomial, exponential, and logarithmic functions, and systems of inequalities can be used to model familiar phenomena from the sciences and from everyday life (see, for example, pages 340, 403, 458, and 620).

BOOK COMPANION WEBSITE A website that accompanies this book is located at **www.stewartmath.com**. The site includes many useful resources for teaching algebra, including the following:

- **Discovery Projects** Discovery Projects for each chapter are available on the website. Each project provides a challenging but accessible set of activities that enable students (perhaps working in groups) to explore in greater depth an interesting aspect of the topic they have just learned. (See, for instance, the Discovery Projects *Visualizing a Formula, Relations and Functions,* and *Will the Species Survive?*)

- **Focus on Problem Solving** Several *Focus on Problem Solving* sections are available on the website. Each such section highlights one of the problem-solving principles introduced in the Prologue and includes several challenging problems. (See, for instance, *Recognizing Patterns, Using Analogy, Introducing Something Extra, Taking Cases,* and *Working Backward.*)

Mathematical Vignettes Throughout the book we make use of the margins to provide historical notes, key insights, or applications of mathematics in the modern world. These serve to enliven the material and to show that mathematics is an important, vital activity, and that even at this elementary level it is fundamental to everyday life.

- **Mathematical Vignettes** These vignettes include biographies of interesting mathematicians and often include a key insight that the mathematician discovered and that is relevant to college algebra. (See, for instance, the vignettes on Viète, page 124; Salt Lake City, page 75; and radiocarbon dating, page 378.)
- **Mathematics in the Modern World** These vignettes emphasize the central role of mathematics in current advances in technology and the sciences (see pages 135, 442, 526, and 582, for example).

Review Sections and Chapter Tests Each chapter ends with an extensive review section that includes the following:

- **Learning Objective Summary** A summary of the learning objectives of the chapter is included at the end of the chapter. Each learning objective is linked to specific review exercises in which students can reinforce their understanding of that learning objective.
- **Review Exercises** The review exercises at the end of each chapter recapitulate the basic concepts and skills of the chapter and include exercises that combine the different ideas learned in the chapter.
- **Chapter Test** The review sections conclude with a Chapter Test designed to help students gauge their progress.
- **Cumulative Review Tests** The Cumulative Review Tests following Chapters 4, 7, and 9 combine skills and concepts from the preceding chapters and are designed to highlight the connections between the topics in these related chapters.
- **Answers** Brief answers to odd-numbered exercises in each section (including the review exercises), to all concept exercises, and to all questions in the chapter tests, are given in the back of the book.

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We are grateful to our colleagues who continually share with us their insights into teaching mathematics. We especially thank Andrew Bulman-Fleming for writing both the Study Guide and the Solutions Manual, Doug Shaw at the University of Northern Iowa for writing the Instructor's Guide, and Julian Allagan at Gainesville State College for checking the accuracy of the examples and exercises.

We thank Martha Emry, our production service and art editor; her energy, devotion, and experience were essential components in the creation of this book. We thank Barbara Willette, our copy editor, for her attention to every detail in the manuscript. We thank Jade Meyers and his staff at Matrix for their attractive and accurate graphs and Network Graphics for bringing many of our illustrations to life. We thank our designer Lisa Henry for the elegant and appropriate design for the interior of the book.

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Numerous other people were involved in the production of this book, including permissions editors, photo researchers, text designers, typesetters, compositors, proof readers, printers, and many more. We thank them all.

Above all, we thank our editor Gary Whalen. His vast editorial experience, his extensive knowledge of current issues in the teaching of mathematics, and especially his deep interest in mathematics textbooks have been invaluable resources in the writing of this book.

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ANCILLARIES

Instructor Resources

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ExamView® testing software allows instructors to quickly create, deliver, and customize tests for class in print and online formats and features automatic grading. It includes a test bank with hundreds of questions customized directly to the text. ExamView is available within the PowerLecture CD-ROM.

Solution Builder (www.cengage.com/solutionbuilder)
This online instructor database offers complete worked-out solutions to all exercises in the text, allowing the instructor to create customized, secure solutions printouts (in PDF format) matched exactly to the problems you assign in class.

PowerLecture with ExamView
(ISBN-10: 1-111-99029-8; ISBN-13: 978-1-111-99029-9)
This CD-ROM provides the instructor with dynamic media tools for teaching. Create, deliver, and customize tests (both print and online) in minutes with ExamView® Computerized Testing Featuring Algorithmic Equations. Easily build solution sets for homework or exams using Solution Builder's online solutions manual. Microsoft® PowerPoint® lecture slides and figures from the book are also included on this CD-ROM.

Enhanced WebAssign (ISBN-10: 0-538-73810-3; ISBN-13: 978-0-538-73810-1)
Exclusively from Cengage Learning, Enhanced WebAssign® offers an extensive online program for College Algebra to encourage the practice that is so critical for concept mastery. The meticulously crafted pedagogy and exercises in this text become even more effective in Enhanced WebAssign, supplemented by multimedia tutorial support and immediate feedback as students complete their assignments. Algorithmic problems allow the instructor to assign unique versions to each student. The Practice Another Version feature (activated at the instructor's discretion) allows students to attempt the questions with new sets of values until they feel confident enough to work the original problem. Students benefit from a YouBook with highlighting and search features; Personal Study Plans (based on diagnostic quizzing) that identify chapter topics they still need to master; and links to video solutions, interactive tutorials, and even live online help.

Student Resources

Printed

Student Solutions Manual (ISBN-10: 1-111-99024-7; ISBN-13: 978-1-111-99024-4)
Fully worked-out solutions to all of the odd-numbered exercises in the text are provided, giving students a way to check their answers and ensure that they took the correct steps to arrive at an answer.

Study Guide (ISBN-10: 1-111-99037-9; ISBN-13: 978-1-111-99037-4)
The Study Guide reinforces student understanding with detailed explanations, worked-out examples, and practice problems. Lists key ideas to master and builds problem-solving skills. There is a section in the Study Guide corresponding to each section in the text.

Media

Text-Specific DVD (ISBN-10: 1-111-99020-4; ISBN-13: 978-1-111-99020-6)
These text-specific instructional videos provide students with visual reinforcement of concepts and explanations given in easy-to-understand terms with detailed examples and sample problems. A flexible format offers versatility for quickly accessing topics or customizing lectures to self-paced, online, or hybrid courses. Closed captioning is provided for the hearing impaired.

Enhanced WebAssign (ISBN-10: 0-538-73810-3; ISBN-13: 978-0-538-73810-1)
Exclusively from Cengage Learning, Enhanced WebAssign® offers an extensive online program for College Algebra to encourage the practice that is so critical for concept mastery. Students will receive multimedia tutorial support as they complete their assignments. Students will also benefit from a YouBook with highlighting and search features; Personal Study Plans (based on diagnostic quizzing) that identify chapter topics they still need to master; and links to video solutions, interactive tutorials, and even live online help.

CengageBrain.com
Visit **www.cengagebrain.com** to access additional course materials and companion resources. At the CengageBrain.com home page, search for the ISBN of your title (from the back cover of your book) using the search box at the top of the page. This will take you to the product page where free companion resources can be found.

TO THE STUDENT

This textbook was written for you to use as a guide to mastering College Algebra. Here are some suggestions to help you get the most out of your course.

First of all, you should read the appropriate section of text *before* you attempt your homework problems. Reading a mathematics text is quite different from reading a novel, a newspaper, or even another textbook. You may find that you have to reread a passage several times before you understand it. Pay special attention to the examples, and work them out yourself with pencil and paper as you read. Then do the linked exercise(s) referred to in *Practice What You've Learned* at the end of each example. With this kind of preparation you will be able to do your homework much more quickly and with more understanding.

Don't make the mistake of trying to memorize every single rule or fact you may come across. Mathematics doesn't consist simply of memorization. Mathematics is a *problem-solving art*, not just a collection of facts. To master the subject you must solve problems—lots of problems. Do as many of the exercises as you can. Be sure to write your solutions in a logical, step-by-step fashion. Don't give up on a problem if you can't solve it right away. Try to understand the problem more clearly—reread it thoughtfully and relate it to what you have learned from your teacher and from the examples in the text. Struggle with it until you solve it. Once you have done this a few times you will begin to understand what mathematics is really all about.

Answers to the odd-numbered exercises, as well as all the answers to each chapter test, appear at the back of the book. If your answer differs from the one given, don't immediately assume that you are wrong. There may be a calculation that connects the two answers and makes both correct. For example, if you get $1/(\sqrt{2} - 1)$ but the answer given is $1 + \sqrt{2}$, your answer *is* correct, because you can multiply both numerator and denominator of your answer by $\sqrt{2} + 1$ to change it to the given answer. In rounding approximate answers, follow the guidelines in Appendix A, *Calculations and Significant Figures.*

The symbol ⊘ is used to warn against committing an error. We have placed this symbol in the margin to point out situations where we have found that many of our students make the same mistake.

ABBREVIATIONS

cm	centimeter	**mg**	milligram
dB	decibel	**MHz**	megahertz
F	farad	**mi**	mile
ft	foot	**min**	minute
g	gram	**mL**	milliliter
gal	gallon	**mm**	millimeter
h	hour	**N**	Newton
H	henry	**qt**	quart
Hz	Hertz	**oz**	ounce
in.	inch	**s**	second
J	Joule	**Ω**	ohm
kcal	kilocalorie	**V**	volt
kg	kilogram	**W**	watt
km	kilometer	**yd**	yard
kPa	kilopascal	**yr**	year
L	liter	**°C**	degree Celsius
lb	pound	**°F**	degree Fahrenheit
lm	lumen	**K**	Kelvin
M	mole of solute per liter of solution	$\Rightarrow$	implies
m	meter	$\Leftrightarrow$	is equivalent to

MATHEMATICAL VIGNETTES

MATHEMATICS IN THE MODERN WORLD

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ARE YOU READY FOR COLLEGE ALGEBRA?

To succeed in your College Algebra course you need to use some of the skills that you learned in your previous mathematics classes. In particular, you need to be familiar with the real number system, algebraic expressions, solving basic equations, and graphing. The following diagnostic tests are designed to assess your knowledge of these topics. After taking each test you can check your answers using the answer key on page xxiv. If you have difficulty with any topic, you can refresh your skills by studying the review materials from Chapters P and 1 that are referenced after each test.

A DIAGNOSTIC TEST: Real Numbers and Exponents

1. Perform the indicated operations. Write your final answer as an integer or as a fraction in lowest terms.

(a) $\frac{1}{3} + \frac{1}{2}$ **(b)** $2 - \frac{2}{3} + \frac{1}{4}$ **(c)** $4\left(2 - \frac{2}{3}\right)$ **(d)** $\dfrac{12}{\frac{4}{3} + \frac{1}{6}}$

2. Determine whether the given number is an integer, rational, or irrational.

(a) 10 **(b)** $\frac{16}{3}$ **(c)** 5^2 **(d)** $\sqrt{5}$

3. Is the inequality true or false?

(a) $-2 < 0$ **(b)** $5 \geq 5$ **(c)** $5 > 5$ **(d)** $3 \leq -10$ **(e)** $-2 > -6$

4. Express the inequality in interval notation.

(a) $-1 < x \leq 5$ **(b)** $x < 3$ **(c)** $x \geq 4$

5. Express the interval using inequalities.

(a) $(2, \infty)$ **(b)** $[-3, -1]$ **(c)** $[0, 9)$

6. Evaluate the expression without using a calculator.

(a) $(-3)^4$ **(b)** -3^4 **(c)** 3^{-4}

(d) $\dfrac{5^{12}}{5^{10}}$ **(e)** $\left(\dfrac{3}{4}\right)^{-2}$ **(f)** $16^{3/4}$

7. Simplify the expression. Write your final answer without negative exponents.

(a) $(4x^2y^3)(2xy^2)$ **(b)** $\left(\dfrac{5a^{1/2}}{a^2}\right)^2$ **(c)** $(x^{-2}y^{-3})(xy^2)^2$

Answers to Test A are on page xxiv. If you had difficulty with any of the questions on Test A, you should review the material covered in Sections P.2, P.3, and P.4.

B DIAGNOSTIC TEST: Algebraic Expressions

1. Expand and simplify.

(a) $4(x+3)+5(2x-1)$ **(b)** $(x+3)(x-5)$ **(c)** $(2x-1)(3x+2)$

(d) $(a-2b)(a+2b)$ **(e)** $(y-3)^2$ **(f)** $(2x+5)^2$

2. Factor the expression.

(a) $4x^2+2x$ **(b)** $3xy^2-6x^2y$ **(c)** $x^2+8x+15$

(d) x^2-x-2 **(e)** $2x^2+5x-12$ **(f)** x^2-16

3. Simplify the rational expression.

(a) $\dfrac{x^2+4x+3}{x^2-2x-3}$ **(b)** $\dfrac{2x^2-3x-2}{x^2-1}\cdot\dfrac{x+1}{2x+1}$

(c) $\dfrac{x^2-x}{x^2-9}-\dfrac{x+1}{x+3}$ **(d)** $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{2}{xy}}$

4. Rationalize the denominator and simplify.

(a) $\dfrac{\sqrt{3}}{\sqrt{7}}$ **(b)** $\dfrac{12}{3-\sqrt{5}}$

Answers to Test B are on page xxiv. If you had difficulty with any of the questions on Test B, you should review the material covered in Sections P.5, P.6, and P.7.

C DIAGNOSTIC TEST: Equations

1. Solve the linear equation.

(a) $3x-1=5$ **(b)** $2x+3=8$

(c) $2x=5x+6$ **(d)** $x+11=6-4x$

2. Solve the equation.

(a) $\frac{1}{3}x=6$ **(b)** $\frac{1}{2}x-\frac{3}{2}=\frac{7}{2}$

3. Find all real solutions of the equation.

(a) $x^2-7=0$ **(b)** $x^3+8=0$

(c) $2x^3-54=0$ **(d)** $x^4-16=0$

4. Solve the equation for the indicated variable.

(a) $4x+y=108$, for x **(b)** $8=\dfrac{mn}{k^2}$, for m

Answers to Test C are on page xxiv. If you had difficulty with any of the questions on Test C, you should review the material covered in Section P.8.

D DIAGNOSTIC TEST: The Coordinate Plane

1. Graph the following points in a coordinate plane.

(a) $(2, 4)$ **(b)** $(-1, 3)$ **(c)** $(3, -1)$

(d) $(0, 0)$ **(e)** $(5, 0)$ **(f)** $(0, -1)$

2. Find the distance between the given pair of points.

(a) $(1, 3), (5, 6)$ **(b)** $(-2, 0), (3, 12)$ **(c)** $(0, -4), (4, 0)$

3. Find the midpoint of the segment PQ.

(a) $P(3, 7), Q(5, 13)$ **(b)** $P(-2, 3), Q(8, -7)$

4. Graph the equation in a coordinate plane by plotting points.

(a) $y = x + 2$ **(b)** $y = 4 - x^2$

Answers to Test D are on page xxiv. If you had difficulty with any of the questions on Test D, you should review the material covered in Sections 1.1 and 1.2.

ANSWERS TO DIAGNOSTIC TESTS

A Answers

1. (a) $\frac{5}{6}$ **(b)** $\frac{19}{12}$ **(c)** $\frac{16}{3}$ **(d)** 8 **2. (a)** Integer and rational **(b)** Rational **(c)** Integer and rational **(d)** Irrational **3. (a)** True **(b)** True **(c)** False **(d)** False **(e)** True **4. (a)** $(-1, 5]$ **(b)** $(-\infty, 3)$ **(c)** $[4, \infty)$ **5. (a)** $x > 2$ **(b)** $-3 \le x \le -1$ **(c)** $0 \le x < 9$ **6. (a)** 81 **(b)** -81 **(c)** $\frac{1}{81}$ **(d)** 25 **(e)** $\frac{16}{9}$ **(f)** 8 **7. (a)** $8x^3y^5$ **(b)** $\dfrac{25}{a^3}$ **(c)** y

B Answers

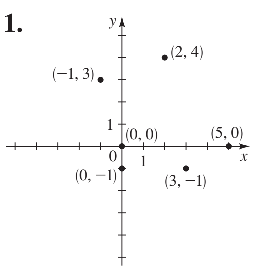
1. (a) $14x + 7$ **(b)** $x^2 - 2x - 15$ **(c)** $6x^2 + x - 2$ **(d)** $a^2 - 4b^2$ **(e)** $y^2 - 6y + 9$ **(f)** $4x^2 + 20x + 25$ **2. (a)** $2x(2x + 1)$ **(b)** $3xy(y - 2x)$ **(c)** $(x + 3)(x + 5)$ **(d)** $(x - 2)(x + 1)$ **(e)** $(2x - 3)(x + 4)$ **(f)** $(x - 4)(x + 4)$ **3. (a)** $\dfrac{x+3}{x-3}$ **(b)** $\dfrac{x-2}{x-1}$ **(c)** $\dfrac{1}{x-3}$ **(d)** $\dfrac{y-x}{2}$ **4. (a)** $\dfrac{\sqrt{21}}{7}$ **(b)** $9 + 3\sqrt{5}$

C Answers

1. (a) 2 **(b)** $\frac{5}{2}$ **(c)** -2 **(d)** -1 **2. (a)** 18 **(b)** 10 **3. (a)** $-\sqrt{7}, \sqrt{7}$ **(b)** -2 **(c)** 3 **(d)** $-2, 2$ **4. (a)** $x = 27 - \frac{1}{4}y$ **(b)** $m = \dfrac{8k^2}{n}$

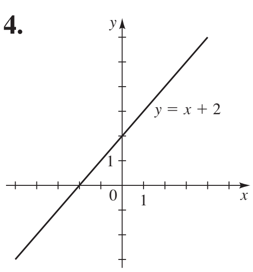
D Answers

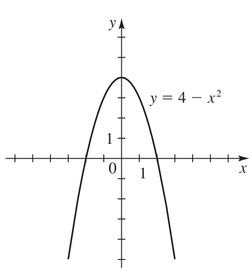
1.



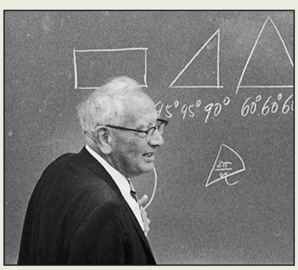
2. (a) 5 **(b)** 13 **(c)** $4\sqrt{2} \approx 5.66$ **3. (a)** $(4, 10)$ **(b)** $(3, -2)$

4.





PROLOGUE PRINCIPLES OF PROBLEM SOLVING



Chuck Painter/Stanford News Service

GEORGE POLYA (1887–1985) is famous among mathematicians for his ideas on problem solving. His lectures on problem solving at Stanford University attracted overflow crowds whom he held on the edges of their seats, leading them to discover solutions for themselves. He was able to do this because of his deep insight into the psychology of problem solving. His well-known book *How To Solve It* has been translated into 15 languages. He said that Euler (see page 300) was unique among great mathematicians because he explained how he found his results. Polya often said to his students and colleagues, "Yes, I see that your proof is correct, but how did you discover it?" In the preface to *How To Solve It,* Polya writes, "A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery."

The ability to solve problems is a highly prized skill in many aspects of our lives; it is certainly an important part of any mathematics course. There are no hard and fast rules that will ensure success in solving problems. However, in this Prologue we outline some general steps in the problem-solving process and we give principles that are useful in solving certain types of problems. These steps and principles are just common sense made explicit. They have been adapted from George Polya's insightful book *How To Solve It*.

1. Understand the Problem

The first step is to read the problem and make sure that you understand it. Ask yourself the following questions:

What is the unknown?
What are the given quantities?
What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram. Usually, it is necessary to

introduce suitable notation

In choosing symbols for the unknown quantities, we often use letters such as a, b, c, m, n, x, and y, but in some cases it helps to use initials as suggestive symbols, for instance, V for volume or t for time.

2. Think of a Plan

Find a connection between the given information and the unknown that enables you to calculate the unknown. It often helps to ask yourself explicitly: "How can I relate the given to the unknown?" If you don't see a connection immediately, the following ideas may be helpful in devising a plan.

► Try to Recognize Something Familiar

Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown.

▶ Try to Recognize Patterns

Certain problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, numerical, or algebraic. If you can see regularity or repetition in a problem, then you might be able to guess what the pattern is and then prove it.

▶ Use Analogy

Try to think of an analogous problem, that is, a similar or related problem but one that is easier than the original. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult one. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem is in three-dimensional geometry, you could look for something similar in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

▶ Introduce Something Extra

You might sometimes need to introduce something new—an auxiliary aid—to make the connection between the given and the unknown. For instance, in a problem for which a diagram is useful, the auxiliary aid could be a new line drawn in the diagram. In a more algebraic problem the aid could be a new unknown that relates to the original unknown.

▶ Take Cases

You might sometimes have to split a problem into several cases and give a different argument for each case. For instance, we often have to use this strategy in dealing with absolute value.

▶ Work Backward

Sometimes it is useful to imagine that your problem is solved and work backward, step by step, until you arrive at the given data. Then you might be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation $3x - 5 = 7$, we suppose that x is a number that satisfies $3x - 5 = 7$ and work backward. We add 5 to each side of the equation and then divide each side by 3 to get $x = 4$. Since each of these steps can be reversed, we have solved the problem.

▶ Establish Subgoals

In a complex problem it is often useful to set subgoals (in which the desired situation is only partially fulfilled). If you can attain or accomplish these subgoals, then you might be able to build on them to reach your final goal.

▶ Indirect Reasoning

Sometimes it is appropriate to attack a problem indirectly. In using **proof by contradiction** to prove that P implies Q, we assume that P is true and Q is false and try to see why this cannot happen. Somehow we have to use this information and arrive at a contradiction to what we absolutely know is true.

▶ Mathematical Induction

In proving statements that involve a positive integer n, it is frequently helpful to use the Principle of Mathematical Induction, which is discussed in Section 8.5.

3. Carry Out the Plan

In Step 2, a plan was devised. In carrying out that plan, you must check each stage of the plan and write the details that prove that each stage is correct.

4. Look Back

Having completed your solution, it is wise to look back over it, partly to see whether any errors have been made and partly to see whether you can discover an easier way to solve the problem. Looking back also familiarizes you with the method of solution, which may be useful for solving a future problem. Descartes said, "Every problem that I solved became a rule which served afterwards to solve other problems."

We illustrate some of these principles of problem solving with an example.

PROBLEM | Average Speed

A driver sets out on a journey. For the first half of the distance, she drives at the leisurely pace of 30 mi/h; during the second half she drives 60 mi/h. What is her average speed on this trip?

THINKING ABOUT THE PROBLEM

It is tempting to take the average of the speeds and say that the average speed for the entire trip is

$$\frac{30 + 60}{2} = 45 \text{ mi/h}$$

But is this simple-minded approach really correct?

Try a special case ▶

Let's look at an easily calculated special case. Suppose that the total distance traveled is 120 mi. Since the first 60 mi is traveled at 30 mi/h, it takes 2 h. The second 60 mi is traveled at 60 mi/h, so it takes one hour. Thus, the total time is $2 + 1 = 3$ hours and the average speed is

$$\frac{120}{3} = 40 \text{ mi/h}$$

So our guess of 45 mi/h was wrong.

SOLUTION

Understand the problem ▶

We need to look more carefully at the meaning of average speed. It is defined as

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}}$$

Introduce notation ▶

State what is given ▶

Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken for the first and second halves of the trip. Now we can write down the information we have been given. For the first half of the trip we have

$$30 = \frac{d}{t_1}$$

and for the second half we have

$$60 = \frac{d}{t_2}$$

Identify the unknown ▶

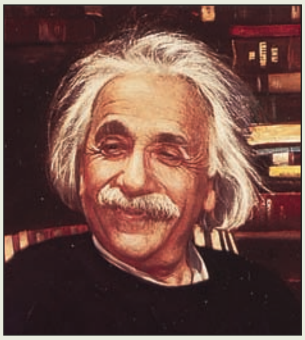
Now we identify the quantity that we are asked to find:

$$\text{average speed for entire trip} = \frac{\text{total distance}}{\text{total time}} = \frac{2d}{t_1 + t_2}$$

Connect the given with the unknown ▶

To calculate this quantity, we need to know t_1 and t_2, so we solve the above equations for these times:

$$t_1 = \frac{d}{30} \qquad t_2 = \frac{d}{60}$$



Don't feel bad if you can't solve these problems right away. Problems 1 and 4 were sent to Albert Einstein by his friend Wertheimer. Einstein (and his friend Bucky) enjoyed the problems and wrote back to Wertheimer. Here is part of his reply:

> Your letter gave us a lot of amusement. The first intelligence test fooled both of us (Bucky and me). Only on working it out did I notice that no time is available for the downhill run! Mr. Bucky was also taken in by the second example, but I was not. Such drolleries show us how stupid we are!

(See *Mathematical Intelligencer,* Spring 1990, page 41.)

Now we have the ingredients needed to calculate the desired quantity:

$$\begin{aligned}
\text{average speed} &= \frac{2d}{t_1 + t_2} = \frac{2d}{\dfrac{d}{30} + \dfrac{d}{60}} \\
&= \frac{60(2d)}{60\left(\dfrac{d}{30} + \dfrac{d}{60}\right)} && \text{Multiply numerator and denominator by 60} \\
&= \frac{120d}{2d + d} = \frac{120d}{3d} = 40
\end{aligned}$$

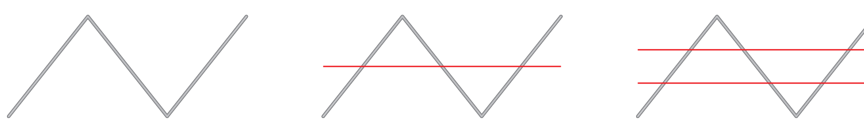
So the average speed for the entire trip is 40 mi/h. ■

PROBLEMS

1. **Distance, Time, and Speed** An old car has to travel a 2-mile route, uphill and down. Because it is so old, the car can climb the first mile—the ascent—no faster than an average speed of 15 mi/h. How fast does the car have to travel the second mile—on the descent it can go faster, of course—to achieve an average speed of 30 mi/h for the trip?

2. **Comparing Discounts** Which price is better for the buyer, a 40% discount or two successive discounts of 20%?

3. **Cutting up a Wire** A piece of wire is bent as shown in the figure. You can see that one cut through the wire produces four pieces and two parallel cuts produce seven pieces. How many pieces will be produced by 142 parallel cuts? Write a formula for the number of pieces produced by n parallel cuts.



4. **Amoeba Propagation** An amoeba propagates by simple division; each split takes 3 minutes to complete. When such an amoeba is put into a glass container with a nutrient fluid, the container is full of amoebas in one hour. How long would it take for the container to be filled if we start with not one amoeba, but two?

5. **Batting Averages** Player A has a higher batting average than player B for the first half of the baseball season. Player A also has a higher batting average than player B for the second half of the season. Is it necessarily true that player A has a higher batting average than player B for the entire season?

6. **Coffee and Cream** A spoonful of cream is taken from a pitcher of cream and put into a cup of coffee. The coffee is stirred. Then a spoonful of this mixture is put into the pitcher of cream. Is there now more cream in the coffee cup or more coffee in the pitcher of cream?

7. **Wrapping the World** A ribbon is tied tightly around the earth at the equator. How much more ribbon would you need if you raised the ribbon 1 ft above the equator everywhere? (You don't need to know the radius of the earth to solve this problem.)

8. **Ending Up Where You Started** A woman starts at a point P on the earth's surface and walks 1 mi south, then 1 mi east, then 1 mi north, and finds herself back at P, the starting point. Describe all points P for which this is possible. [*Hint:* There are infinitely many such points, all but one of which lie in Antarctica.]

Many more problems and examples that highlight different problem-solving principles are available at the book companion website: **www.stewartmath.com**. You can try them as you progress through the book.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CHAPTER P

PREREQUISITES

Making Good Decisions In our daily lives we are continually faced with situations in which we have to decide between different alternatives. We may need to decide on the best cell phone plan, which size pizza is the best deal, or whether to rent or buy a mathematics textbook. In many such situations algebra can reveal the best choice. In algebra we use letters to stand for numbers. This allows us to write equations that describe real-world situations. Of course, the letters in our equation must obey the same rules that numbers do. So in this chapter we review properties of numbers and algebraic expressions. You are probably already familiar with many of these properties, but it is helpful to get a fresh look at how these properties work together to solve real-world problems.

In the first section of this chapter we look at the central reason for studying algebra: its usefulness in describing (or modeling) real-world situations. In the *Focus on Modeling* at the end of the chapter we see how equations can help us make the best decisions in some everyday situations. This theme of using algebra to model real-world situations is further developed throughout the textbook.

P.1 Modeling the Real World with Algebra

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Use an algebra model ▶ Make an algebra model

In algebra we use letters to stand for numbers. This allows us to describe patterns that we see in the real world.

For example, if we let N stand for the number of hours you work and W stand for your hourly wage, then the formula

$$P = NW$$

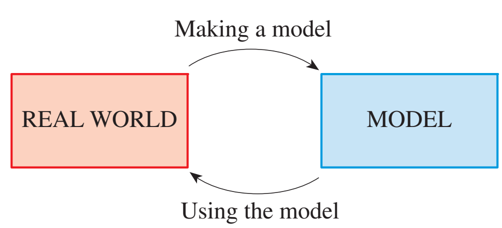
gives your pay P. The formula $P = NW$ is a description or *model* for pay. We can also call this formula an *algebra model*. We summarize the situation as follows:

Real World	Algebra Model
You work for an hourly *wage*. You would like to know your *pay* for any *number* of hours worked.	$P = NW$

The model $P = NW$ gives the pattern for finding the pay for *any* worker, with *any* hourly wage, working *any* number of hours. That's the power of algebra: By using letters to stand for numbers, we can write a single formula that describes many different situations.

We can now use the model $P = NW$ to answer questions such as "I make \$10 an hour, and I worked 35 hours; how much do I get paid?" or "I make \$8 an hour; how many hours do I need to work to get paid \$1000?"

In general, a **model** is a mathematical representation (such as a formula) of a real-world situation. **Modeling** is the process of making mathematical models. Once a model has been made, it can be used to answer questions about the thing being modeled.



The examples we study in this section are simple, but the methods are far reaching. This will become more apparent as we explore the applications of algebra in subsequent *Focus on Modeling* sections that follow each chapter.

▼ Using Algebra Models

We begin our study of modeling by using models that are given to us. In the next subsection we learn how to make our own models.

EXAMPLE 1 | Using a Model for Pay

Aaron makes \$9 an hour at his part-time job. Use the model $P = NW$ to answer the following questions:

(a) Aaron worked 35 hours last week. How much did he get paid?

(b) Aaron wants to earn enough money to buy a calculus text that costs \$126. How many hours does he need to work to earn this amount?

SOLUTION

(a) We know that $N = 35$ h and $W = \$9$. To find P, we substitute these values into the model.

$$\begin{aligned} P &= NW && \text{Model} \\ &= 35 \times 9 && \text{Substitute } N = 35,\ W = 9 \\ &= 315 && \text{Calculate} \end{aligned}$$

So Aaron was paid \$315.

(b) Aaron's hourly wage is $W = \$9$, and the amount of pay he needs to buy the book is $P = \$126$. To find N, we substitute these values into the model:

$$\begin{aligned} P &= NW && \text{Model} \\ 126 &= 9N && \text{Substitute } P = 126,\ W = 9 \\ \frac{126}{9} &= N && \text{Divide by 9} \\ N &= 14 && \text{Calculate} \end{aligned}$$

So Aaron must work 14 hours to buy this book.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **3** AND **9**

EXAMPLE 2 | Using an Elevation-Temperature Model

A mountain climber uses the model

$$T = 20 - 10h$$

to estimate the temperature T (in °C) at elevation h (in kilometers, km).

(a) Make a table that gives the temperature for each 1-km change in elevation, from elevation 0 km to elevation 5 km. How does temperature change as elevation increases?

(b) If the temperature is 5°C, what is the elevation?

SOLUTION

(a) Let's use the model to find the temperature at elevation $h = 3$ km:

$$\begin{aligned} T &= 20 - 10h && \text{Model} \\ &= 20 - 10(3) && \text{Substitute } h = 3 \\ &= -10 && \text{Calculate} \end{aligned}$$

So at an elevation of 3 km the temperature is -10°C. The other entries in the following table are calculated similarly.

Elevation (km)	Temperature (°C)
0	20°
1	10°
2	0°
3	−10°
4	−20°
5	−30°

We see that temperature decreases as elevation increases.

(b) We substitute $T = 5°C$ in the model and solve for h:

$$
\begin{aligned}
T &= 20 - 10h && \text{Model} \\
5 &= 20 - 10h && \text{Substitute } T = 5 \\
-15 &= -10h && \text{Subtract 20} \\
\frac{-15}{-10} &= h && \text{Divide by } -10 \\
1.5 &= h && \text{Calculator}
\end{aligned}
$$

The elevation is 1.5 km.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **13** ■

Making Algebra Models

In the next example we explore the process of making an algebra model for a real-life situation.

EXAMPLE 3 | Making a Model for Gas Mileage

12 mi/gal 40 mi/gal

The gas mileage of a car is the number of miles it can travel on one gallon of gas.

(a) Find a formula that models gas mileage in terms of the number of miles driven and the number of gallons of gasoline used.

(b) Henry's car used 10.5 gallons to drive 230 miles. Find its gas mileage.

THINKING ABOUT THE PROBLEM

Let's try a simple case. If a car uses 2 gallons to drive 100 miles, we easily see that

$$\text{gas mileage} = \frac{100}{2} = 50 \text{ mi/gal}$$

So gas mileage is the number of miles driven divided by the number of gallons used.

SOLUTION

(a) To find the formula we want, we need to assign symbols to the quantities involved:

In Words	In Algebra
Number of miles driven	N
Number of gallons used	G
Gas mileage (mi/gal)	M

We can express the model as follows:

$$\text{gas mileage} = \frac{\text{number of miles driven}}{\text{number of gallons used}}$$

$$M = \frac{N}{G} \qquad \text{Model}$$

(b) To get the gas mileage, we substitute $N = 230$ and $G = 10.5$ in the formula:

$$
\begin{aligned}
M &= \frac{N}{G} && \text{Model} \\
&= \frac{230}{10.5} && \text{Substitute } N = 230,\ G = 10.5 \\
&\approx 21.9 && \text{Calculator}
\end{aligned}
$$

The gas mileage for Henry's car is about 21.9 mi/gal.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **23** ■

P.1 EXERCISES

CONCEPTS

1. The model $L = 4S$ gives the total number of legs that S sheep have. Using this model, we find that 12 sheep have

$L =$ ________ legs.

2. Suppose gas costs \$3.50 a gallon. We make a model for the cost C of buying x gallons of gas by writing the formula

$C =$ ________.

SKILLS

3–12 ■ Use the model given to answer the questions about the object or process being modeled.

3. The sales tax T in a certain county is modeled by the formula $T = 0.06x$. Find the sales tax on an item whose price is \$120.

4. Mintonville School District residents pay a wage tax T that is modeled by the formula $T = 0.005x$. Find the wage tax paid by a resident who earns \$62,000 per year.

5. A bakery finds that the cost C (in dollars) of producing x apple pies is modeled by

$$C = 50 + 1.25x$$

Find the cost of producing 32 apple pies.

6. A company models its profit P (in dollars) on the sale of x golf balls by

$$P = 0.6x - 450$$

Find the profit on the sale of 1000 golf balls.

7. The distance d (in miles) driven by a car traveling at a speed of v miles per hour for t hours is given by

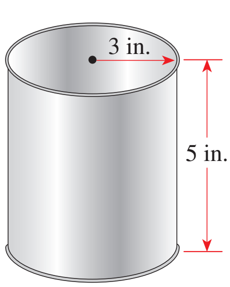
$$d = vt$$

If the car is driven at 70 mi/h for 3.5 h, how far has it traveled?

8. The volume V of a cylindrical can is modeled by the formula

$$V = \pi r^2 h$$

where r is the radius and h is the height of the can. Find the volume of a can with radius 3 in. and height 5 in.



9. The gas mileage M (in mi/gal) of a car is modeled by $M = N/G$, where N is the number of miles driven and G is the number of gallons of gas used.

(a) Find the gas mileage M for a car that drove 240 miles on 8 gallons of gas.

(b) A car with a gas mileage $M = 25$ mi/gal is driven 175 miles. How many gallons of gas are used?

10. A mountain climber models the temperature T (in °F) at elevation h (in ft) by

$$T = 70 - 0.003h$$

(a) Find the temperature T at an elevation of 1500 ft.

(b) If the temperature is 64°F, what is the elevation?

11. The portion of a floating iceberg that is below the water surface is much larger than the portion above the surface. The total volume V of an iceberg is modeled by

$$V = 9.5S$$

where S is the volume showing above the surface.

(a) Find the total volume of an iceberg if the volume showing above the surface is 4 km^3.

(b) Find the volume showing above the surface for an iceberg with total volume 19 km^3.

12. The power P measured in horsepower (hp) needed to drive a certain ship at a speed of s knots is modeled by

$$P = 0.06s^3$$

(a) Find the power needed to drive the ship at 12 knots.

(b) At what speed will a 7.5-hp engine drive the ship?

13. An ocean diver models the pressure P (in lb/in^2) at depth d (in ft) by

$$P = 14.7 + 0.45d$$

(a) Make a table that gives the pressure for each 10-ft change in depth, from a depth of 0 ft to 60 ft.

(b) If the pressure is 30 lb/in^2, what is the depth?

14. Arizonans use an average of 40 gallons of water per person each day.

(a) Find a model for the number of gallons W of water used by x Arizona residents each day.

(b) Make a table that gives the number of gallons of water used for each 1000-person increase in population, from 0 to 5000.

(c) Estimate the population of an Arizona town whose water usage is 140,000 gallons per day.

15–22 ■ Write an algebraic formula that models the given quantity.

15. The number N of days in w weeks

16. The number N of cents in q quarters

17. The average A of two numbers a and b

18. The average A of three numbers a, b, and c

19. The cost C of purchasing x gallons of gas at \$3.50 a gallon

20. The amount T of a 15% tip on a restaurant bill of x dollars

21. The distance d in miles that a car travels in t hours at 60 mi/h

22. The speed r of a boat that travels d miles in 3 hours

APPLICATIONS

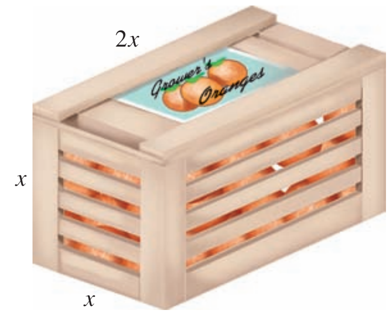
23. Cost of a Pizza A pizza parlor charges \$12 for a cheese pizza and \$1 for each topping.
- **(a)** How much does a 3-topping pizza cost?
- **(b)** Find a formula that models the cost C of a pizza with n toppings.
- **(c)** If a pizza costs \$16, how many toppings does it have?

$n = 1$ $\qquad$ $n = 4$

24. Renting a Car At a certain car rental agency a compact car rents for \$30 a day and 10¢ a mile.
- **(a)** How much does it cost to rent a car for 3 days if the car is driven 280 miles?
- **(b)** Find a formula that models the cost C of renting this car for n days if it is driven m miles.
- **(c)** If the cost for a 3-day rental was \$140, how many miles was the car driven?

25. Energy Cost for a Car The cost of the electricity needed to drive an all-electric car is about 4 cents per mile. The cost of the gasoline needed to drive the average gasoline-powered car is about 12 cents per mile.
- **(a)** Find a formula that models the energy cost C of driving x miles for (i) the all-electric car and (ii) the average gasoline-powered car.
- **(b)** Find the cost of driving 10,000 miles with each type of car.

26. Volume of Fruit Crate A fruit crate has square ends and is twice as long as it is wide (see the folowing figure).
- **(a)** Find the volume of the crate if its width is 20 inches.
- **(b)** Find a formula for the volume V of the crate in terms of its width x.



27. Cost of a Phone Call A phone card company charges a \$1 connection fee for each call and 10¢ per minute.
- **(a)** How much does a 10-minute call cost?
- **(b)** Find a formula that models the cost C of a phone call that lasts t minutes.
- **(c)** If a particular call cost \$2.20, how many minutes did the call last?
- **(d)** Find a formula that models the cost C (in cents) of a phone call that lasts t minutes if the connection fee is F cents and the rate is r cents per minute.

28. Grade Point Average In many universities students are given grade points for each credit unit according to the following scale:

A	4 points
B	3 points
C	2 points
D	1 point
F	0 point

For example, a grade of A in a 3-unit course earns $4 \times 3 = 12$ grade points and a grade of B in a 5-unit course earns $3 \times 5 = 15$ grade points. A student's grade point average (GPA) for these two courses is the total number of grade points earned divided by the number of units; in this case the GPA is $(12 + 15)/8 = 3.375$.
- **(a)** Find a formula for the GPA of a student who earns a grade of A in a units of course work, B in b units, C in c units, D in d units, and F in f units.
- **(b)** Find the GPA of a student who has earned a grade of A in two 3-unit courses, B in one 4-unit course, and C in three 3-unit courses.

P.2 The Real Numbers

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Classify real numbers ▶ Use properties of real numbers ▶ Work with fractions ▶ Graph numbers on a number line ▶ Use the order symbols $<$, $\leq$, $>$, $\geq$ ▶ Work with set and interval notation ▶ Work with absolute values ▶ Find distances on the real line

Let's review the types of numbers that make up the real number system. We start with the **natural numbers**:

$$1, 2, 3, 4, \ldots$$

The different types of real numbers were invented to meet specific needs. For example, natural numbers are needed for counting, negative numbers for describing debt or below-zero temperatures, rational numbers for concepts like "half a gallon of milk," and irrational numbers for measuring certain distances, like the diagonal of a square.

The **integers** consist of the natural numbers together with their negatives and 0:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$$

We construct the **rational numbers** by taking ratios of integers. Thus any rational number r can be expressed as

$$r = \frac{m}{n}$$

where m and n are integers and $n \neq 0$. Examples are

$$\tfrac{1}{2} \qquad -\tfrac{3}{7} \qquad 46 = \tfrac{46}{1} \qquad 0.17 = \tfrac{17}{100}$$

(Recall that division by 0 is always ruled out, so expressions like $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.) There are also real numbers, such as $\sqrt{2}$, that cannot be expressed as a ratio of integers and are therefore called **irrational numbers**. It can be shown, with varying degrees of difficulty, that these numbers are also irrational:

$$\sqrt{3} \qquad \sqrt{5} \qquad \sqrt[3]{2} \qquad \pi \qquad \frac{3}{\pi^2}$$

The set of all real numbers is usually denoted by the symbol $\mathbb{R}$. When we use the word *number* without qualification, we will mean "real number." Figure 1 is a diagram of the types of real numbers that we work with in this book.

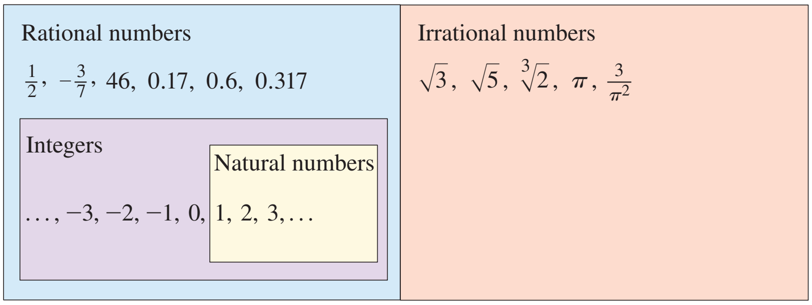


FIGURE 1 The real number system

A repeating decimal such as

$$x = 3.5474747\ldots$$

is a rational number. To convert it to a ratio of two integers, we write

$$\begin{aligned} 1000x &= 3547.47474747\ldots \\ 10x &= 35.47474747\ldots \\ \hline 990x &= 3512.0 \end{aligned}$$

Thus $x = \frac{3512}{990}$. (The idea is to multiply x by appropriate powers of 10 and then subtract to eliminate the repeating part.)

Every real number has a decimal representation. If the number is rational, then its corresponding decimal is repeating. For example,

$$\tfrac{1}{2} = 0.5000\ldots = 0.5\overline{0} \qquad \tfrac{2}{3} = 0.66666\ldots = 0.\overline{6}$$

$$\tfrac{157}{495} = 0.3171717\ldots = 0.3\overline{17} \qquad \tfrac{9}{7} = 1.285714285714\ldots = 1.\overline{285714}$$

(The bar indicates that the sequence of digits repeats forever.) If the number is irrational, the decimal representation is nonrepeating:

$$\sqrt{2} = 1.414213562373095\ldots \qquad \pi = 3.141592653589793\ldots$$

The word **algebra** comes from the 9th-century Arabic book *Hisâb al-Jabr w'al-Muqabala*, written by al-Khowarizmi. The title refers to transposing and combining terms, two processes that are used in solving equations. In Latin translations the title was shortened to *Aljabr*, from which we get the word *algebra*. The author's name itself made its way into the English language in the form of our word *algorithm*.

If we stop the decimal expansion of any number at a certain place, we get an approximation to the number. For instance, we can write

$$\pi \approx 3.14159265$$

where the symbol $\approx$ is read "is approximately equal to." The more decimal places we retain, the better our approximation.

▼ Properties of Real Numbers

We all know that $2 + 3 = 3 + 2$, and $5 + 7 = 7 + 5$, and $513 + 87 = 87 + 513$, and so on. In algebra we express all these (infinitely many) facts by writing

$$a + b = b + a$$

where a and b stand for any two numbers. In other words, "$a + b = b + a$" is a concise way of saying that "when we add two numbers, the order of addition doesn't matter." This fact is called the *Commutative Property* for addition. From our experience with numbers we know that the properties in the following box are also valid.

PROPERTIES OF REAL NUMBERS

Property	Example	Description
Commutative Properties		
$a + b = b + a$	$7 + 3 = 3 + 7$	When we add two numbers, order doesn't matter.
$ab = ba$	$3 \cdot 5 = 5 \cdot 3$	When we multiply two numbers, order doesn't matter.
Associative Properties		
$(a + b) + c = a + (b + c)$	$(2 + 4) + 7 = 2 + (4 + 7)$	When we add three numbers, it doesn't matter which two we add first.
$(ab)c = a(bc)$	$(3 \cdot 7) \cdot 5 = 3 \cdot (7 \cdot 5)$	When we multiply three numbers, it doesn't matter which two we multiply first.
Distributive Property		
$a(b + c) = ab + ac$	$2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5$	When we multiply a number by a sum of two numbers, we get the same result as we would get if we multiply the number by each of the terms and then add the results.
$(b + c)a = ab + ac$	$(3 + 5) \cdot 2 = 2 \cdot 3 + 2 \cdot 5$	

The Distributive Property applies whenever we multiply a number by a sum. Figure 2 explains why this property works for the case in which all the numbers are positive integers, but the property is true for any real numbers a, b, and c.

The Distributive Property is crucial because it describes the way addition and multiplication interact with each other.

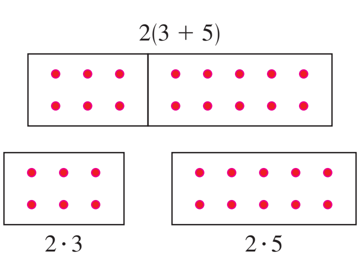


FIGURE 2 The Distributive Property

EXAMPLE 1 | Using the Distributive Property

(a) $2(x + 3) = 2 \cdot x + 2 \cdot 3$ — Distributive Property

$= 2x + 6$ — Simplify

(b) $(a + b)(x + y) = (a + b)x + (a + b)y$ — Distributive Property

$= (ax + bx) + (ay + by)$ — Distributive Property

$= ax + bx + ay + by$ — Associative Property of Addition

In the last step we removed the parentheses because, according to the Associative Property, the order of addition doesn't matter.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **17** AND **25**

Addition and Subtraction

Don't assume that $-a$ is a negative number. Whether $-a$ is negative or positive depends on the value of a. For example, if $a = 5$, then $-a = -5$, a negative number, but if $a = -5$, then $-a = -(-5) = 5$ (Property 2), a positive number.

The number 0 is special for addition; it is called the **additive identity** because $a + 0 = a$ for any real number a. Every real number a has a **negative**, $-a$, that satisfies $a + (-a) = 0$. **Subtraction** is the operation that undoes addition; to subtract a number from another, we simply add the negative of that number. By definition

$$a - b = a + (-b)$$

To combine real numbers involving negatives, we use the following properties.

PROPERTIES OF NEGATIVES

Property	Example
1. $(-1)a = -a$	$(-1)5 = -5$
2. $-(-a) = a$	$-(-5) = 5$
3. $(-a)b = a(-b) = -(ab)$	$(-5)7 = 5(-7) = -(5 \cdot 7)$
4. $(-a)(-b) = ab$	$(-4)(-3) = 4 \cdot 3$
5. $-(a + b) = -a - b$	$-(3 + 5) = -3 - 5$
6. $-(a - b) = b - a$	$-(5 - 8) = 8 - 5$

Property 6 states the intuitive fact that $a - b$ and $b - a$ are negatives of each other. Property 5 is often used with more than two terms:

$$-(a + b + c) = -a - b - c$$

EXAMPLE 2 | Using Properties of Negatives

Let x, y, and z be real numbers.

(a) $-(x + 2) = -x - 2$ — Property 5: $-(a + b) = -a - b$

(b) $-(x + y - z) = -x - y - (-z)$ — Property 5: $-(a + b) = -a - b$

$= -x - y + z$ — Property 2: $-(-a) = a$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **29**

Multiplication and Division

The number 1 is special for multiplication; it is called the **multiplicative identity** because $a \cdot 1 = a$ for any real number a. Every nonzero real number a has an **inverse**, $1/a$, that satisfies $a \cdot (1/a) = 1$. **Division** is the operation that undoes multiplication; to divide by a number, we multiply by the inverse of that number. If $b \neq 0$, then, by definition,

$$a \div b = a \cdot \frac{1}{b}$$

We write $a \cdot (1/b)$ as simply a/b. We refer to a/b as the **quotient** of a and b or as the **fraction** a over b; a is the **numerator** and b is the **denominator** (or **divisor**). To combine real numbers using the operation of division, we use the following properties.

PROPERTIES OF FRACTIONS

Property	Example	Description
1. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$	When **multiplying fractions**, multiply numerators and denominators.
2. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$	$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$	When **dividing fractions**, invert the divisor and multiply.
3. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	$\frac{2}{5} + \frac{7}{5} = \frac{2+7}{5} = \frac{9}{5}$	When **adding fractions** with the **same denominator**, add the numerators.
4. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$	$\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7 + 3 \cdot 5}{35} = \frac{29}{35}$	When **adding fractions** with **different denominators**, find a common denominator. Then add the numerators.
5. $\frac{ac}{bc} = \frac{a}{b}$	$\frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$	**Cancel** numbers that are **common factors** in numerator and denominator.
6. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$	$\frac{2}{3} = \frac{6}{9}$, so $2 \cdot 9 = 3 \cdot 6$	**Cross-multiply**.

When adding fractions with different denominators, we don't usually use Property 4. Instead we rewrite the fractions so that they have the smallest possible common denominator (often smaller than the product of the denominators), and then we use Property 3. This denominator is the **L**east **C**ommon **D**enominator (LCD) described in the next example.

EXAMPLE 3 | Using the LCD to Add Fractions

Evaluate: $\frac{5}{36} + \frac{7}{120}$

SOLUTION Factoring each denominator into prime factors gives

$$36 = 2^2 \cdot 3^2 \qquad \text{and} \qquad 120 = 2^3 \cdot 3 \cdot 5$$

We find the least common denominator (LCD) by forming the product of all the factors that occur in these factorizations, using the highest power of each factor.

Thus the LCD is $2^3 \cdot 3^2 \cdot 5 = 360$. So

$$\frac{5}{36} + \frac{7}{120} = \frac{5 \cdot 10}{36 \cdot 10} + \frac{7 \cdot 3}{120 \cdot 3} \qquad \text{Use common denominator}$$

$$= \frac{50}{360} + \frac{21}{360} = \frac{71}{360} \qquad \text{Property 3: Adding fractions with the same denominator}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **31**

The Real Line

The real numbers can be represented by points on a line, as shown in Figure 3. The positive direction (toward the right) is indicated by an arrow. We choose an arbitrary reference point O, called the **origin**, which corresponds to the real number 0. Given any convenient unit of measurement, each positive number x is represented by the point on the line a distance of x units to the right of the origin, and each negative number $-x$ is represented by the point x units to the left of the origin. The number associated with the point P is called the coordinate of P, and the line is then called a **coordinate line**, or a **real number line**, or simply a **real line**. Often we identify the point with its coordinate and think of a number as being a point on the real line.

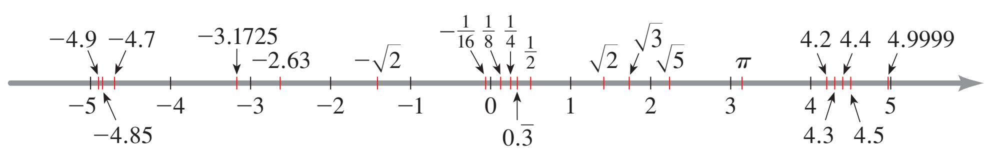


FIGURE 3 The real line

The real numbers are *ordered*. We say that ***a* is less than *b*** and write $a < b$ if $b - a$ is a positive number. Geometrically, this means that a lies to the left of b on the number line. Equivalently, we can say that ***b* is greater than *a*** and write $b > a$. The symbol $a \le b$ (or $b \ge a$) means that either $a < b$ or $a = b$ and is read "a is less than or equal to b." For instance, the following are true inequalities (see Figure 4):

$$7 < 7.4 < 7.5 \qquad -\pi < -3 \qquad \sqrt{2} < 2 \qquad 2 \le 2$$

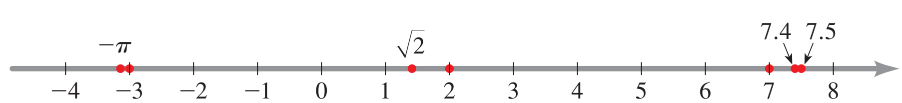


FIGURE 4

Sets and Intervals

A **set** is a collection of objects, and these objects are called the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $b \notin S$ means that b is not an element of S. For example, if Z represents the set of integers, then $-3 \in Z$ but $\pi \notin Z$.

Some sets can be described by listing their elements within braces. For instance, the set A that consists of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

We could also write A in **set-builder notation** as

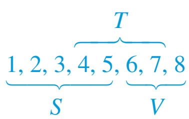
$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which is read "A is the set of all x such that x is an integer and $0 < x < 7$."

If S and T are sets, then their **union** $S \cup T$ is the set that consists of all elements that are in S *or* T (or in both). The **intersection** of S and T is the set $S \cap T$ consisting of all elements that are in both S *and* T. In other words, $S \cap T$ is the common part of S and T. The **empty set**, denoted by $\varnothing$, is the set that contains no element.

EXAMPLE 4 | Union and Intersection of Sets

If $S = \{1, 2, 3, 4, 5\}$, $T = \{4, 5, 6, 7\}$, and $V = \{6, 7, 8\}$, find the sets $S \cup T$, $S \cap T$, and $S \cap V$.



SOLUTION

$S \cup T = \{1, 2, 3, 4, 5, 6, 7\}$ — All elements in S or T

$S \cap T = \{4, 5\}$ — Elements common to both S and T

$S \cap V = \varnothing$ — S and V have no element in common

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **47**

FIGURE 5 The open interval (a, b)

FIGURE 6 The closed interval $[a, b]$

Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments. If $a < b$, then the **open interval** from a to b consists of all numbers between a and b and is denoted (a, b). The **closed interval** from a to b includes the endpoints and is denoted $[a, b]$. Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\} \qquad [a, b] = \{x \mid a \le x \le b\}$$

Note that parentheses ( ) in the interval notation and open circles on the graph in Figure 5 indicate that endpoints are *excluded* from the interval, whereas square brackets [ ] and solid circles in Figure 6 indicate that the endpoints are *included*. Intervals may also include one endpoint but not the other, or they may extend infinitely far in one direction or both. The following table lists the possible types of intervals.

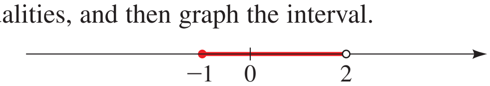
The symbol ∞ ("infinity") does not stand for a number. The notation (a, ∞), for instance, simply indicates that the interval has no endpoint on the right but extends infinitely far in the positive direction.

Notation	Set description	Graph
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \le x \le b\}$	
$[a, b)$	$\{x \mid a \le x < b\}$	
$(a, b]$	$\{x \mid a < x \le b\}$	
(a, ∞)	$\{x \mid a < x\}$	
$[a, \infty)$	$\{x \mid a \le x\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \le b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

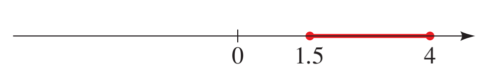
EXAMPLE 5 | Graphing Intervals

Express each interval in terms of inequalities, and then graph the interval.

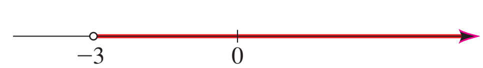
(a) $[-1, 2) = \{x \mid -1 \le x < 2\}$



(b) $[1.5, 4] = \{x \mid 1.5 \le x \le 4\}$

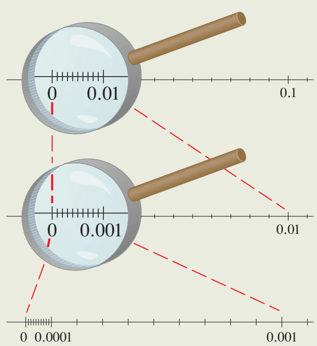


(c) $(-3, \infty) = \{x \mid -3 < x\}$



PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **53**

No Smallest or Largest Number in an Open Interval
Any interval contains infinitely many numbers—every point on the graph of an interval corresponds to a real number. In the closed interval $[0, 1]$, the smallest number is 0 and the largest is 1, but the open interval $(0, 1)$ contains no smallest or largest number. To see this, note that 0.01 is close to zero, but 0.001 is closer, 0.0001 is closer yet, and so on. We can always find a number in the interval $(0, 1)$ closer to zero than any given number. Since 0 itself is not in the interval, the interval contains no smallest number. Similarly, 0.99 is close to 1, but 0.999 is closer, 0.9999 closer yet, and so on. Since 1 itself is not in the interval, the interval has no largest number.



EXAMPLE 6 | Finding Unions and Intersections of Intervals

Graph each set.

(a) $(1, 3) \cap [2, 7]$ **(b)** $(1, 3) \cup [2, 7]$

SOLUTION

(a) The intersection of two intervals consists of the numbers that are in both intervals. Therefore

$$\begin{aligned}(1, 3) \cap [2, 7] &= \{x \mid 1 < x < 3 \text{ and } 2 \le x \le 7\}\\ &= \{x \mid 2 \le x < 3\} = [2, 3)\end{aligned}$$

This set is illustrated in Figure 7.

(b) The union of two intervals consists of the numbers that are in either one interval or the other (or both). Therefore

$$\begin{aligned}(1, 3) \cup [2, 7] &= \{x \mid 1 < x < 3 \text{ or } 2 \le x \le 7\}\\ &= \{x \mid 1 < x \le 7\} = (1, 7]\end{aligned}$$

This set is illustrated in Figure 8.

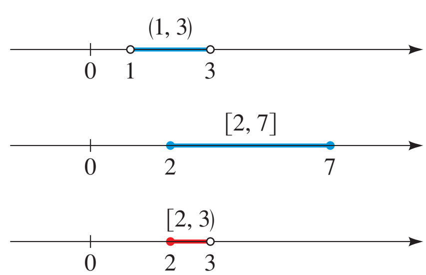


FIGURE 7 $(1, 3) \cap [2, 7] = [2, 3)$

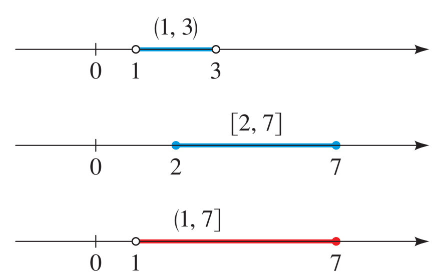


FIGURE 8 $(1, 3) \cup [2, 7] = (1, 7]$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **71**

Absolute Value and Distance

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line (see Figure 9). Distance is always positive or zero, so we have $|a| \ge 0$ for every number a. Remembering that $-a$ is positive when a is negative, we have the following definition.

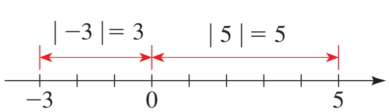


FIGURE 9

DEFINITION OF ABSOLUTE VALUE

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

EXAMPLE 7 | Evaluating Absolute Values of Numbers

(a) $|3| = 3$

(b) $|-3| = -(-3) = 3$

(c) $|0| = 0$

(d) $|3 - \pi| = -(3 - \pi) = \pi - 3 \qquad (\text{since } 3 < \pi \Rightarrow 3 - \pi < 0)$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **77**

When working with absolute values, we use the following properties.

PROPERTIES OF ABSOLUTE VALUE

Property	Example	Description
1. $\lvert a \rvert \geq 0$	$\lvert -3 \rvert = 3 \geq 0$	The absolute value of a number is always positive or zero.
2. $\lvert a \rvert = \lvert -a \rvert$	$\lvert 5 \rvert = \lvert -5 \rvert$	A number and its negative have the same absolute value.
3. $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	$\lvert -2 \cdot 5 \rvert = \lvert -2 \rvert \lvert 5 \rvert$	The absolute value of a product is the product of the absolute values.
4. $\left\lvert \dfrac{a}{b} \right\rvert = \dfrac{\lvert a \rvert}{\lvert b \rvert}$	$\left\lvert \dfrac{12}{-3} \right\rvert = \dfrac{\lvert 12 \rvert}{\lvert -3 \rvert}$	The absolute value of a quotient is the quotient of the absolute values.

What is the distance on the real line between the numbers -2 and 11? From Figure 10 we see that the distance is 13. We arrive at this by finding either $\lvert 11 - (-2) \rvert = 13$ or $\lvert (-2) - 11 \rvert = 13$. From this observation we make the following definition (see Figure 11).

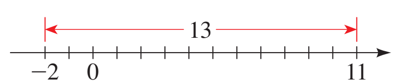


FIGURE 10

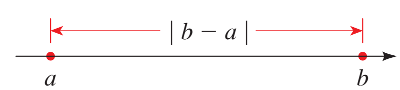


FIGURE 11 Length of a line segment is $\lvert b - a \rvert$

DISTANCE BETWEEN POINTS ON THE REAL LINE

If a and b are real numbers, then the **distance** between the points a and b on the real line is

$$d(a, b) = \lvert b - a \rvert$$

From Property 6 of negatives it follows that

$$\lvert b - a \rvert = \lvert a - b \rvert$$

This confirms that, as we would expect, the distance from a to b is the same as the distance from b to a.

EXAMPLE 8 | Distance Between Points on the Real Line

The distance between the numbers -8 and 2 is

$$d(a, b) = \lvert -8 - 2 \rvert = \lvert -10 \rvert = 10$$

We can check this calculation geometrically, as shown in Figure 12.

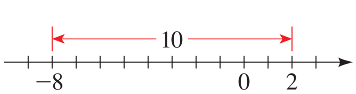


FIGURE 12

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **85**

P.2 EXERCISES

CONCEPTS

1. Give an example of each of the following:
 (a) A natural number
 (b) An integer that is not a natural number
 (c) A rational number that is not an integer
 (d) An irrational number

2. Complete each statement and name the property of real numbers you have used.
 (a) $ab =$ ________; ________ Property
 (b) $a + (b + c) =$ ________; ________ Property
 (c) $a(b + c) =$ ________; ________ Property

3. Express the set of real numbers between but not including 2 and 7 as follows:
 (a) In set-builder notation: ________________
 (b) In interval notation: ________________

4. Explain the difference between the following two sets of numbers:
$$A = [-2, 5] \qquad B = (-2, 5)$$

5. The symbol $|x|$ stands for the __________ of the number x. If x is not 0, then the sign of $|x|$ is always ________.

6. The absolute value of the difference between a and b is (geometrically) the ________ between a and b on the real number line.

SKILLS

7–8 ■ List the elements of the given set that are
 (a) natural numbers
 (b) integers
 (c) rational numbers
 (d) irrational numbers

7. $\{-3, 0, \frac{22}{7}, \sqrt{7}, 3.14, -\pi, 2.7\overline{6}, -1000, -\frac{2}{5}\}$
8. $\{3.3, 3.3333\ldots, -\sqrt{2}, 2.714, -500, 4\frac{2}{3}, \sqrt{25}, \frac{1234}{5678}, -\frac{9}{3}\}$

9–20 ■ State the property of real numbers being used.

9. $5 + 14 = 14 + 5$
10. $6 \cdot 13 = 13 \cdot 6$
11. $2(3 \cdot 7) = (2 \cdot 3)7$
12. $(5 + 8) + 12 = 5 + (8 + 12)$
13. $3(5 + 8) = 3 \cdot 5 + 3 \cdot 8$
14. $(2 + 7)9 = 2 \cdot 9 + 7 \cdot 9$
15. $(x + 2y) + 3z = x + (2y + 3z)$
16. $2(A + B) = 2A + 2B$
17. $(5x + 1)3 = 15x + 3$
18. $(x + a)(x + b) = (x + a)x + (x + a)b$
19. $2x(3 + y) = (3 + y)2x$
20. $7(a + b + c) = 7(a + b) + 7c$

21–24 ■ Rewrite the expression using the given property of real numbers.

21. Commutative Property of addition, $x + 3 = \square$
22. Associative Property of multiplication, $7(3x) = \square$
23. Distributive Property, $4(A + B) = \square$
24. Distributive Property, $5x + 5y = \square$

25–30 ■ Use properties of real numbers to write the expression without parentheses.

25. $3(x + y)$
26. $(a - b)8$
27. $4(2m)$
28. $\frac{4}{3}(-6y)$
29. $-\frac{5}{2}(2x - 4y)$
30. $(3a)(b + c - 2d)$

31–36 ■ Perform the indicated operations.

31. **(a)** $\frac{3}{10} + \frac{4}{15}$ **(b)** $\frac{1}{4} + \frac{1}{5}$
32. **(a)** $\frac{2}{3} - \frac{3}{5}$ **(b)** $1 + \frac{5}{8} - \frac{1}{6}$
33. **(a)** $\frac{2}{3}(6 - \frac{3}{2})$ **(b)** $0.25(\frac{8}{9} + \frac{1}{2})$
34. **(a)** $(3 + \frac{1}{4})(1 - \frac{4}{5})$ **(b)** $(\frac{1}{2} - \frac{1}{3})(\frac{1}{2} + \frac{1}{3})$
35. **(a)** $\dfrac{2}{\frac{2}{3}} - \dfrac{\frac{2}{3}}{2}$ **(b)** $\dfrac{\frac{1}{12}}{\frac{1}{8} - \frac{1}{9}}$
36. **(a)** $\dfrac{2 - \frac{3}{4}}{\frac{1}{2} - \frac{1}{3}}$ **(b)** $\dfrac{\frac{2}{5} + \frac{1}{2}}{\frac{1}{10} + \frac{3}{15}}$

37–38 ■ Place the correct symbol ($<$, $>$, or $=$) in the space.

37. **(a)** $3 \ \square \ \frac{7}{2}$ **(b)** $-3 \ \square \ -\frac{7}{2}$ **(c)** $3.5 \ \square \ \frac{7}{2}$
38. **(a)** $\frac{2}{3} \ \square \ 0.67$ **(b)** $\frac{2}{3} \ \square \ -0.67$ **(c)** $|0.67| \ \square \ |-0.67|$

39–44 ■ State whether each inequality is true or false.

39. **(a)** $5 < 7$ **(b)** $-5 < -7$
40. **(a)** $\sqrt{2} > 1.41$ **(b)** $1.41 \ge \sqrt{2}$
41. **(a)** $-2.1 \ge -2$ **(b)** $-2 \ge -2$
42. **(a)** $\dfrac{3}{5} > \dfrac{3}{4}$ **(b)** $-\dfrac{3}{5} > -\dfrac{3}{4}$
43. **(a)** $-8 > 1$ **(b)** $8 \le -8$
44. **(a)** $\pi > 3$ **(b)** $-\pi \ge -3$

45–46 ■ Write each statement in terms of inequalities.

45. **(a)** x is positive
 (b) t is less than 4
 (c) a is greater than or equal to π
 (d) x is less than $\frac{1}{3}$ and is greater than -5
 (e) The distance from p to 3 is at most 5

46. **(a)** y is negative
 (b) z is greater than 1
 (c) b is at most 8
 (d) w is positive and is less than or equal to 17
 (e) y is at least 2 units from π

47–50 ■ Find the indicated set if

$$A = \{1, 2, 3, 4, 5, 6, 7\} \qquad B = \{2, 4, 6, 8\}$$
$$C = \{7, 8, 9, 10\}$$

47. **(a)** $A \cup B$ **(b)** $A \cap B$

48. **(a)** $B \cup C$ **(b)** $B \cap C$

49. **(a)** $A \cup C$ **(b)** $A \cap C$

50. **(a)** $A \cup B \cup C$ **(b)** $A \cap B \cap C$

51–52 ■ Find the indicated set if

$$A = \{x \mid x \geq -2\} \qquad B = \{x \mid x < 4\}$$
$$C = \{x \mid -1 < x \leq 5\}$$

51. **(a)** $B \cup C$ **(b)** $B \cap C$

52. **(a)** $A \cap C$ **(b)** $A \cap B$

53–58 ■ Express the interval in terms of inequalities, and then graph the interval.

53. $(-3, 0)$ **54.** $(2, 8]$

55. $[2, 8)$ **56.** $[-6, -\frac{1}{2}]$

57. $[2, \infty)$ **58.** $(-\infty, 1)$

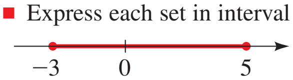
59–64 ■ Express the inequality in interval notation, and then graph the corresponding interval.

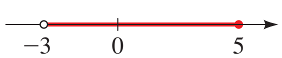
59. $x \leq 1$ **60.** $1 \leq x \leq 2$

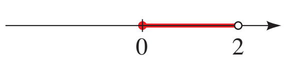
61. $-2 < x \leq 1$ **62.** $x \geq -5$

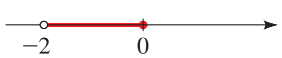
63. $x > -1$ **64.** $-5 < x < 2$

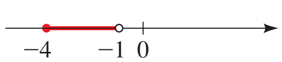
65–70 ■ Express each set in interval notation.

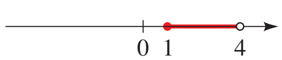
65. **(a)** 

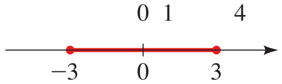
(b) 

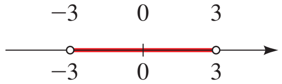
66. **(a)** 

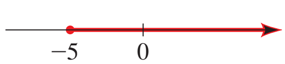
(b) 

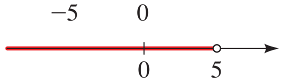
67. **(a)** 

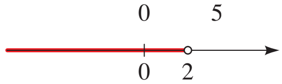
(b) 

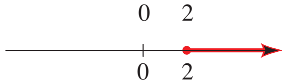
68. **(a)** 

(b) 

69. **(a)** 

(b) 

70. **(a)** 

(b) 

71–76 ■ Graph the set.

71. $(-2, 0) \cup (-1, 1)$ **72.** $(-2, 0) \cap (-1, 1)$

73. $[-4, 6] \cap [0, 8)$ **74.** $[-4, 6) \cup [0, 8)$

75. $(-\infty, -4) \cup (4, \infty)$ **76.** $(-\infty, 6] \cap (2, 10)$

77–82 ■ Evaluate each expression.

77. **(a)** $|100|$ **(b)** $|-73|$

78. **(a)** $|\sqrt{5} - 5|$ **(b)** $|10 - \pi|$

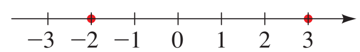
79. **(a)** $||-6| - |-4||$ **(b)** $\dfrac{-1}{|-1|}$

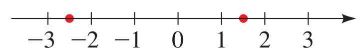
80. **(a)** $|2 - |-12||$ **(b)** $-1 - |1 - |-1||$

81. **(a)** $|(-2) \cdot 6|$ **(b)** $|(-\frac{1}{3})(-15)|$

82. **(a)** $\left|\dfrac{-6}{24}\right|$ **(b)** $\left|\dfrac{7 - 12}{12 - 7}\right|$

83–86 ■ Find the distance between the given numbers.

83. 

84. 

85. **(a)** 2 and 17 **(b)** -3 and 21 **(c)** $\frac{11}{8}$ and $-\frac{3}{10}$

86. **(a)** $\frac{7}{15}$ and $-\frac{1}{21}$ **(b)** -38 and -57 **(c)** -2.6 and -1.8

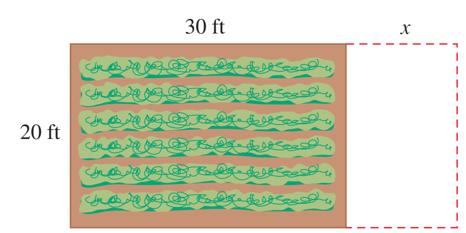
87–88 ■ Express each repeating decimal as a fraction. (See the margin note on page 7.)

87. **(a)** $0.\overline{7}$ **(b)** $0.2\overline{8}$ **(c)** $0.\overline{57}$

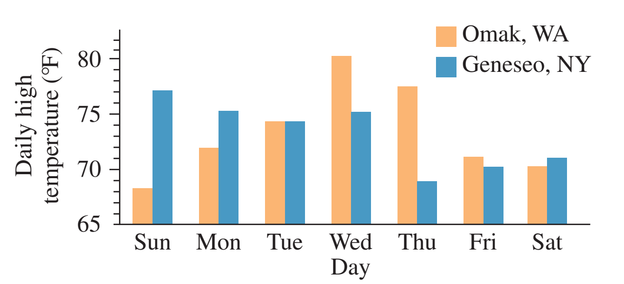
88. **(a)** $5.\overline{23}$ **(b)** $1.3\overline{7}$ **(c)** $2.1\overline{35}$

APPLICATIONS

89. Area of a Garden Mary's backyard vegetable garden measures 20 ft by 30 ft, so its area is $20 \times 30 = 600$ ft^2. She decides to make it longer, as shown in the figure, so that the area increases to $A = 20(30 + x)$. Which property of real numbers tells us that the new area can also be written $A = 600 + 20x$?



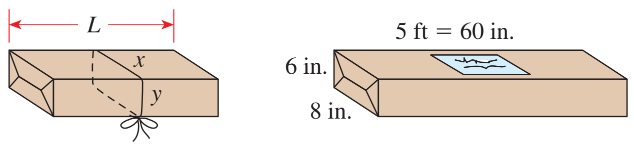
90. Temperature Variation The bar graph shows the daily high temperatures for Omak, Washington, and Geneseo, New York, during a certain week in June. Let T_O represent the temperature in Omak and T_G the temperature in Geneseo. Calculate $T_O - T_G$ and $|T_O - T_G|$ for each day shown. Which of these two values gives more information?



91. Mailing a Package The post office will only accept packages for which the length plus the "girth" (distance around) is no more than 108 inches. Thus, for the package in the figure, we must have

$$L + 2(x + y) \le 108$$

(a) Will the post office accept a package that is 6 in. wide, 8 in. deep, and 5 ft long? What about a package that measures 2 ft by 2 ft by 4 ft?
(b) What is the greatest acceptable length for a package that has a square base measuring 9 in. by 9 in?



DISCOVERY ■ DISCUSSION ■ WRITING

92. Signs of Numbers Let a, b, and c be real numbers such that $a > 0$, $b < 0$, and $c < 0$. Find the sign of each expression.

(a) $-a$ **(b)** $-b$ **(c)** bc
(d) $a - b$ **(e)** $c - a$ **(f)** $a + bc$
(g) $ab + ac$ **(h)** $-abc$ **(i)** ab^2

93. Sums and Products of Rational and Irrational Numbers Explain why the sum, the difference, and the product of two rational numbers are rational numbers. Is the product of two irrational numbers necessarily irrational? What about the sum?

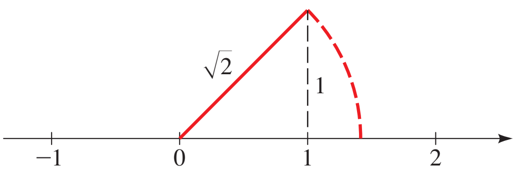
94. Combining Rational Numbers with Irrational Numbers Is $\frac{1}{2} + \sqrt{2}$ rational or irrational? Is $\frac{1}{2} \cdot \sqrt{2}$ rational or irrational? In general, what can you say about the sum of a rational and an irrational number? What about the product?

95. Limiting Behavior of Reciprocals Complete the following tables. What happens to the size of the fraction $1/x$ as x gets large? As x gets small?

x	$1/x$
1	
2	
10	
100	
1000	

x	$1/x$
1.0	
0.5	
0.1	
0.01	
0.001	

96. Irrational Numbers and Geometry Using the following figure, explain how to locate the point $\sqrt{2}$ on a number line. Can you locate $\sqrt{5}$ by a similar method? What about $\sqrt{6}$? List some other irrational numbers that can be located this way.



97. Commutative and Noncommutative Operations We have seen that addition and multiplication are both commutative operations.

(a) Is subtraction commutative?
(b) Is division of nonzero real numbers commutative?
(c) Are the actions of putting on your socks and putting on your shoes commutative?
(d) Are the actions of putting on your hat and putting on your coat commutative?
(e) Are the actions of washing laundry and drying it commutative?
(f) Give an example of a pair of actions that are commutative.
(g) Give an example of a pair of actions that are not commutative.

P.3 Integer Exponents and Scientific Notation

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Use exponential notation ▶ Simplify expressions using the Laws of Exponents ▶ Write numbers using scientific notation

In this section we review the rules for working with exponent notation. We also see how exponents can be used to represent very large and very small numbers.

Exponential Notation

A product of identical numbers is usually written in exponential notation. For example, $5 \cdot 5 \cdot 5$ is written as 5^3. In general, we have the following definition.

EXPONENTIAL NOTATION

If a is any real number and n is a positive integer, then the ***n*th power** of a is

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

The number a is called the **base**, and n is called the **exponent**.

EXAMPLE 1 | Exponential Notation

(a) $\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{32}$

(b) $(-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) = 81$

(c) $-3^4 = -(3 \cdot 3 \cdot 3 \cdot 3) = -81$

Note the distinction between $(-3)^4$ and -3^4. In $(-3)^4$ the exponent applies to -3, but in -3^4 the exponent applies only to 3.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **9**

We can state several useful rules for working with exponential notation. To discover the rule for multiplication, we multiply 5^4 by 5^2:

$$5^4 \cdot 5^2 = \underbrace{(5 \cdot 5 \cdot 5 \cdot 5)}_{4 \text{ factors}}\underbrace{(5 \cdot 5)}_{2 \text{ factors}} = \underbrace{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}_{6 \text{ factors}} = 5^6 = 5^{4+2}$$

It appears that *to multiply two powers of the same base, we add their exponents*. In general, for any real number a and any positive integers m and n, we have

$$a^m a^n = \underbrace{(a \cdot a \cdot \cdots \cdot a)}_{m \text{ factors}}\underbrace{(a \cdot a \cdot \cdots \cdot a)}_{n \text{ factors}} = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{m + n \text{ factors}} = a^{m+n}$$

Thus $a^m a^n = a^{m+n}$.

We would like this rule to be true even when m and n are 0 or negative integers. For instance, we must have

$$2^0 \cdot 2^3 = 2^{0+3} = 2^3$$

But this can happen only if $2^0 = 1$. Likewise, we want to have

$$5^4 \cdot 5^{-4} = 5^{4+(-4)} = 5^{4-4} = 5^0 = 1$$

and this will be true if $5^{-4} = 1/5^4$. These observations lead to the following definition.

ZERO AND NEGATIVE EXPONENTS

If $a \neq 0$ is any real number and n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}$$

EXAMPLE 2 | Zero and Negative Exponents

(a) $\left(\frac{4}{7}\right)^0 = 1$

(b) $x^{-1} = \dfrac{1}{x^1} = \dfrac{1}{x}$

(c) $(-2)^{-3} = \dfrac{1}{(-2)^3} = \dfrac{1}{-8} = -\dfrac{1}{8}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **11** ■

Rules for Working with Exponents

Familiarity with the following rules is essential for our work with exponents and bases. In the table the bases a and b are real numbers, and the exponents m and n are integers.

LAWS OF EXPONENTS

Law	Example	Description
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$	To multiply two powers of the same number, add the exponents.
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{3^5}{3^2} = 3^{5-2} = 3^3$	To divide two powers of the same number, subtract the exponents.
3. $(a^m)^n = a^{mn}$	$(3^2)^5 = 3^{2\cdot 5} = 3^{10}$	To raise a power to a new power, multiply the exponents.
4. $(ab)^n = a^n b^n$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2$	To raise a product to a power, raise each factor to the power.
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{3}{4}\right)^2 = \dfrac{3^2}{4^2}$	To raise a quotient to a power, raise both numerator and denominator to the power.
6. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$	$\left(\dfrac{3}{4}\right)^{-2} = \left(\dfrac{4}{3}\right)^2$	To raise a fraction to a negative power, invert the fraction and change the sign of the exponent.
7. $\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}$	$\dfrac{3^{-2}}{4^{-5}} = \dfrac{4^5}{3^2}$	To move a number raised to a power from numerator to denominator or from denominator to numerator, change the sign of the exponent.

PROOF OF LAW 3 If m and n are positive integers, we have

$$(a^m)^n = (\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})^n$$

$$= \underbrace{(\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})(\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}}) \cdots (\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})}_{n \text{ groups of factors}}$$

$$= \underbrace{a \cdot a \cdot \cdots \cdot a}_{mn \text{ factors}} = a^{mn}$$

The cases for which $m \le 0$ or $n \le 0$ can be proved using the definition of negative exponents. ■

PROOF OF LAW 4 If n is a positive integer, we have

$$(ab)^n = \underbrace{(ab)(ab)\cdots(ab)}_{n \text{ factors}} = \underbrace{(a \cdot a \cdot \cdots \cdot a)}_{n \text{ factors}} \cdot \underbrace{(b \cdot b \cdot \cdots \cdot b)}_{n \text{ factors}} = a^n b^n$$

Here we have used the Commutative and Associative Properties repeatedly. If $n \le 0$, Law 4 can be proved by using the definition of negative exponents. ■

You are asked to prove Laws 2, 5, 6, and 7 in Exercise 51 and 52.

EXAMPLE 3 | Using Laws of Exponents

(a) $x^4x^7 = x^{4+7} = x^{11}$ — Law 1: $a^m a^n = a^{m+n}$

(b) $y^4y^{-7} = y^{4-7} = y^{-3} = \dfrac{1}{y^3}$ — Law 1: $a^m a^n = a^{m+n}$

(c) $\dfrac{c^9}{c^5} = c^{9-5} = c^4$ — Law 2: $\dfrac{a^m}{a^n} = a^{m-n}$

(d) $(b^4)^5 = b^{4\cdot5} = b^{20}$ — Law 3: $(a^m)^n = a^{mn}$

(e) $(3x)^3 = 3^3x^3 = 27x^3$ — Law 4: $(ab)^n = a^nb^n$

(f) $\left(\dfrac{x}{2}\right)^5 = \dfrac{x^5}{2^5} = \dfrac{x^5}{32}$ — Law 5: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **19** AND **21** ■

EXAMPLE 4 | Simplifying Expressions with Exponents

Simplify:

(a) $(2a^3b^2)(3ab^4)^3$ **(b)** $\left(\dfrac{x}{y}\right)^3\left(\dfrac{y^2x}{z}\right)^4$

SOLUTION

(a)

$$\begin{aligned}(2a^3b^2)(3ab^4)^3 &= (2a^3b^2)[3^3a^3(b^4)^3] && \text{Law 4: } (ab)^n = a^nb^n\\ &= (2a^3b^2)(27a^3b^{12}) && \text{Law 3: } (a^m)^n = a^{mn}\\ &= (2)(27)a^3a^3b^2b^{12} && \text{Group factors with the same base}\\ &= 54a^6b^{14} && \text{Law 1: } a^ma^n = a^{m+n}\end{aligned}$$

(b)

$$\begin{aligned}\left(\frac{x}{y}\right)^3\left(\frac{y^2x}{z}\right)^4 &= \frac{x^3}{y^3}\,\frac{(y^2)^4x^4}{z^4} && \text{Laws 5 and 4}\\ &= \frac{x^3}{y^3}\,\frac{y^8x^4}{z^4} && \text{Law 3}\\ &= (x^3x^4)\left(\frac{y^8}{y^3}\right)\frac{1}{z^4} && \text{Group factors with the same base}\\ &= \frac{x^7y^5}{z^4} && \text{Laws 1 and 2}\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **27** AND **33** ■

When simplifying an expression, you will find that many different methods will lead to the same result; you should feel free to use any of the rules of exponents to arrive at your own method. In the next example we see how to simplify expressions with negative exponents.

EXAMPLE 5 | Simplifying Expressions with Negative Exponents

Eliminate negative exponents, and simplify each expression.

(a) $\dfrac{6st^{-4}}{2s^{-2}t^2}$ **(b)** $\left(\dfrac{y}{3z^3}\right)^{-2}$

MATHEMATICS IN THE MODERN WORLD

Although we are often unaware of its presence, mathematics permeates nearly every aspect of life in the modern world. With the advent of modern technology, mathematics plays an ever greater role in our lives. Today you were probably awakened by a digital alarm clock in a room whose temperature is controlled by a digital thermostat, made a phone call that used digital transmission, sent an e-mail message over the Internet, drove a car with digitally controlled fuel injection, and listened to music on a CD or MP3 player. Mathematics is crucially involved in each of these activities. In general, a property such as the intensity or frequency of sound, the oxygen level in a car's exhaust emission, the colors in an image, or the temperature in your bedroom is transformed into sequences of numbers by sophisticated mathematical algorithms. These numerical data, which usually consist of many millions of bits (the digits 0 and 1), are then transmitted and reinterpreted. Dealing with such huge amounts of data was not feasible until the invention of computers, machines whose logical processes were invented by mathematicians.

The contributions of mathematics in the modern world are not limited to technological advances. The logical processes of mathematics are now used to analyze complex problems in the social, political, and life sciences in new and surprising ways. Advances in mathematics continue to be made, some of the most exciting of these just within the past decade.

In other *Mathematics in the Modern World* vignettes, we will describe in more detail how mathematics affects us in our everyday activities.

SOLUTION

(a) We use Law 7, which allows us to move a number raised to a power from the numerator to the denominator (or vice versa) by changing the sign of the exponent:

t^{-4} moves to denominator and becomes t^4

$$\frac{6st^{-4}}{2s^{-2}t^2} = \frac{6ss^2}{2t^2t^4} \qquad \text{Law 7}$$

s^{-2} moves to numerator and becomes s^2

$$= \frac{3s^3}{t^6} \qquad \text{Law 1}$$

(b) We use Law 6, which allows us to change the sign of the exponent of a fraction by inverting the fraction:

$$\left(\frac{y}{3z^3}\right)^{-2} = \left(\frac{3z^3}{y}\right)^2 \qquad \text{Law 6}$$

$$= \frac{9z^6}{y^2} \qquad \text{Laws 5 and 4}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **35** ■

▼ Scientific Notation

Exponential notation is used by scientists as a compact way of writing very large numbers and very small numbers. For example, the nearest star beyond the sun, Proxima Centauri, is approximately 40,000,000,000,000 km away. The mass of a hydrogen atom is about 0.00000000000000000000000166 g. Such numbers are difficult to read and to write, so scientists usually express them in *scientific notation*.

SCIENTIFIC NOTATION

A positive number x is said to be written in **scientific notation** if it is expressed as follows:

$$x = a \times 10^n \qquad \text{where } 1 \le a < 10 \text{ and } n \text{ is an integer}$$

For instance, when we state that the distance to the star Proxima Centauri is 4×10^{13} km, the positive exponent 13 indicates that the decimal point should be moved 13 places to the *right*:

$$4 \times 10^{13} = 40{,}000{,}000{,}000{,}000$$

Move decimal point 13 places to the right

When we state that the mass of a hydrogen atom is 1.66×10^{-24} g, the exponent -24 indicates that the decimal point should be moved 24 places to the *left*:

$$1.66 \times 10^{-24} = 0.00000000000000000000000166$$

Move decimal point 24 places to the left

EXAMPLE 6 | Changing from Decimal Notation to Scientific Notation

Write each number in scientific notation.

(a) 56,920 **(b)** 0.000093

SOLUTION

(a) $\underbrace{56{,}920}_{\text{4 places}} = 5.692 \times 10^4$ **(b)** $0.\underbrace{0000}_{\text{5 places}}93 = 9.3 \times 10^{-5}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **39**

EXAMPLE 7 | Changing from Scientific Notation to Decimal Notation

Write each number in decimal notation.

(a) 6.97×10^9 **(b)** 4.6271×10^{-6}

SOLUTION

(a) $6.97 \times 10^9 = \underbrace{6{,}970{,}000{,}000}_{\text{9 places}}$ Move decimal 9 places to the right

(b) $4.6271 \times 10^{-6} = 0.\underbrace{000004}_{\text{6 places}}6271$ Move decimal 6 places to the left

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41**

To use scientific notation on a calculator, use a key labeled EE or EXP or EEX to enter the exponent. For example, to enter the number 3.629×10^{15} on a TI-83 or TI-84 calculator, we enter

3.629 2ND EE 15

and the display reads

3.629E15

Scientific notation is often used on a calculator to display a very large or very small number. For instance, if we use a calculator to square the number 1,111,111, the display panel may show (depending on the calculator model) the approximation

1.234568 12 or 1.234568 E12

Here the final digits indicate the power of 10, and we interpret the result as

$$1.234568 \times 10^{12}$$

EXAMPLE 8 | Calculating with Scientific Notation

If $a \approx 0.00046$, $b \approx 1.697 \times 10^{22}$, and $c \approx 2.91 \times 10^{-18}$, use a calculator to approximate the quotient ab/c.

SOLUTION We could enter the data using scientific notation, or we could use laws of exponents as follows:

$$\begin{aligned}\frac{ab}{c} &\approx \frac{(4.6 \times 10^{-4})(1.697 \times 10^{22})}{2.91 \times 10^{-18}} \\ &= \frac{(4.6)(1.697)}{2.91} \times 10^{-4+22+18} \\ &\approx 2.7 \times 10^{36}\end{aligned}$$

We state the answer rounded to two significant figures because the least accurate of the given numbers is stated to two significant figures. (See Appendix A, *Calculations and Significant Figures*.)

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45**

P.3 EXERCISES

CONCEPTS

1. Using exponential notation, we can write the product $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ as ________.

2. Is there a difference between $(-5)^4$ and -5^4?

3. In the expression 3^4 the number 3 is called the ________, and the number 4 is called the ________.

4. When we multiply two powers with the same base, we ________ the exponents. So $3^4 \cdot 3^5 =$ ________.

5. When we divide two powers with the same base, we ________ the exponents. So $\dfrac{3^5}{3^2} =$ ________.

6. When we raise a power to a new power, we ________ the exponents. So $(3^4)^2 =$ ________.

7. Express the following numbers without using exponents.

   (a) $2^{-1} =$ ________ **(b)** $2^{-3} =$ ________

   (c) $\left(\frac{1}{2}\right)^{-1} =$ ________ **(d)** $\dfrac{1}{2^{-3}} =$ ________

8. Scientists express very large or very small numbers using ________ notation. In scientific notation 8,300,000 is ________, and 0.0000327 is ________.

SKILLS

9–18 ■ Evaluate each expression.

9. **(a)** -3^2 **(b)** $(-3)^2$ **(c)** $\left(\frac{2}{3}\right)^3 \cdot (-3)^3$
10. **(a)** $(-5)^3$ **(b)** -5^3 **(c)** $(-5)^2 \cdot \left(\frac{2}{5}\right)^2$
11. **(a)** $\left(\frac{5}{3}\right)^0 \cdot 2^{-1}$ **(b)** $\dfrac{2^{-3}}{3^0}$ **(c)** $\left(\frac{1}{4}\right)^{-2}$
12. **(a)** $-2^{-3} \cdot (-2)^0$ **(b)** $-2^3 \cdot (-2)^0$ **(c)** $\left(\frac{-2}{3}\right)^{-3}$
13. **(a)** $5^3 \cdot 5$ **(b)** $3^2 \cdot 3^0$ **(c)** $(2^2)^3$
14. **(a)** $3^8 \cdot 3^5$ **(b)** $6^0 \cdot 6$ **(c)** $(5^4)^2$
15. **(a)** $(2^8)^0$ **(b)** $(-5)^0$ **(c)** -5^0
16. **(a)** $(3^{100})^0$ **(b)** -3^0 **(c)** $(-3)^0$
17. **(a)** $5^4 \cdot 5^{-2}$ **(b)** $\dfrac{10^7}{10^4}$ **(c)** $\dfrac{3^2}{3^4}$
18. **(a)** $3^{-3} \cdot 3^{-1}$ **(b)** $\dfrac{5^4}{5}$ **(c)** $\dfrac{7^2}{7^5}$

19–24 ■ Simplify each expression.

19. **(a)** $x^3 \cdot x^4$ **(b)** $(2y^2)^3$ **(c)** $y^{-2}y^7$
20. **(a)** $y^5 \cdot y^2$ **(b)** $(8x)^2$ **(c)** x^4x^{-3}
21. **(a)** $x^{-5} \cdot x^3$ **(b)** $w^{-2}w^{-4}w^5$ **(c)** $\dfrac{y^{10}y^0}{y^7}$
22. **(a)** $y^2 \cdot y^{-5}$ **(b)** $z^5z^{-3}z^{-4}$ **(c)** $\dfrac{x^6}{x^{10}}$
23. **(a)** $\dfrac{a^9a^{-2}}{a}$ **(b)** $(a^2a^4)^3$ **(c)** $(2x)^2(5x^6)$
24. **(a)** $\dfrac{z^2z^4}{z^3z^{-1}}$ **(b)** $(2a^3a^2)^4$ **(c)** $(-3z^2)^3(2z^3)$

25–38 ■ Simplify the expression and eliminate any negative exponents(s).

25. **(a)** $(3x^3y^2)(2y^3)$ **(b)** $(6a^5b)\left(\frac{1}{2}a^2b^4\right)$
26. **(a)** $(8m^{-2}n^4)\left(\frac{1}{2}n^{-2}\right)$ **(b)** $2pq^4(p^{-3}q^{-1})(7pq^{-2})$
27. **(a)** $(5z^{-2})^2(z^3)$ **(b)** $(3a^4b^{-2})^3(a^2b^{-1})$
28. **(a)** $(2y^{-2})^{-4}(2y^4)$ **(b)** $(3x^2y^{-1})^{-2}(x^{-1}y^{-3})^{-1}$
29. **(a)** $\dfrac{x^2y^{-1}}{x^{-5}}$ **(b)** $\dfrac{x^{-3}y^2}{x^{-2}y^{-1}}$
30. **(a)** $\dfrac{y^{-2}z^{-3}}{y^{-1}}$ **(b)** $\dfrac{y^4z^{-1}}{y^{-4}z^3}$
31. **(a)** $\left(\dfrac{a^3}{2b^2}\right)^3$ **(b)** $\left(\dfrac{2x^4}{y^{-1}}\right)^{-1}$
32. **(a)** $\left(\dfrac{4r^2t^{-1}}{r^{-4}t^2}\right)^2$ **(b)** $\left(\dfrac{x^3y^{-2}}{x^{-3}y^2}\right)^{-2}$
33. **(a)** $\left(\dfrac{a^2}{b}\right)^5\left(\dfrac{a^3b^2}{c^3}\right)^3$ **(b)** $\dfrac{(u^{-1}v^2)^2}{(u^3v^{-2})^3}$
34. **(a)** $\left(\dfrac{x^4z^2}{4y^5}\right)\left(\dfrac{2x^3y^2}{z^3}\right)^2$ **(b)** $\dfrac{(rs^2)^3}{(r^{-3}s^2)^2}$
35. **(a)** $\dfrac{8a^3b^{-4}}{2a^{-5}b^5}$ **(b)** $\left(\dfrac{y}{5x^{-2}}\right)^{-3}$
36. **(a)** $\dfrac{5xy^{-2}}{x^{-1}y^{-3}}$ **(b)** $\left(\dfrac{2a^{-1}b}{a^2b^{-3}}\right)^{-3}$
37. **(a)** $\left(\dfrac{3a}{b^3}\right)^{-1}$ **(b)** $\left(\dfrac{q^{-1}r^{-1}s^{-2}}{r^{-5}sq^{-8}}\right)^{-1}$
38. **(a)** $\left(\dfrac{s^2t^{-4}}{5s^{-1}t}\right)^{-2}$ **(b)** $\left(\dfrac{xy^{-2}z^{-3}}{x^2y^3z^{-4}}\right)^{-3}$

39–40 ■ Write each number in scientific notation.

39. **(a)** 69,300,000 **(b)** 7,200,000,000,000
    (c) 0.000028536 **(d)** 0.0001213
40. **(a)** 129,540,000 **(b)** 7,259,000,000
    (c) 0.0000000014 **(d)** 0.0007029

41–42 ■ Write each number in decimal notation.

41. **(a)** 3.19×10^5 **(b)** 2.721×10^8
    (c) 2.670×10^{-8} **(d)** 9.999×10^{-9}
42. **(a)** 7.1×10^{14} **(b)** 6×10^{12}
    (c) 8.55×10^{-3} **(d)** 6.257×10^{-10}

43–44 ■ Write the number indicated in each statement in scientific notation.

43. **(a)** A light-year, the distance that light travels in one year, is about 5,900,000,000,000 mi.

(b) The diameter of an electron is about 0.0000000000004 cm.
(c) A drop of water contains more than 33 billion billion molecules.

44. (a) The distance from the earth to the sun is about 93 million miles.
(b) The mass of an oxygen molecule is about 0.000000000000000000000053 g.
(c) The mass of the earth is about 5,970,000,000,000,000,000,000,000 kg.

45–50 ■ Use scientific notation, the Laws of Exponents, and a calculator to perform the indicated operations. State your answer rounded to the number of significant digits indicated by the given data.

45. $(7.2 \times 10^{-9})(1.806 \times 10^{-12})$

46. $(1.062 \times 10^{24})(8.61 \times 10^{19})$

47. $\dfrac{1.295643 \times 10^{9}}{(3.610 \times 10^{-17})(2.511 \times 10^{6})}$

48. $\dfrac{(73.1)(1.6341 \times 10^{28})}{0.0000000019}$

49. $\dfrac{(0.0000162)(0.01582)}{(594{,}621{,}000)(0.0058)}$ **50.** $\dfrac{(3.542 \times 10^{-6})^{9}}{(5.05 \times 10^{4})^{12}}$

51. Prove the given Laws of Exponents for the case in which m and n are positive integers and $m > n$.
(a) Law 2 **(b)** Law 5

52. Prove Laws 6 and 7 of Exponents.

APPLICATIONS

53. Distance to the Nearest Star Proxima Centauri, the star nearest to our solar system, is 4.3 light-years away. Use the information in Exercise 43(a) to express this distance in miles.

54. Speed of Light The speed of light is about 186,000 mi/s. Use the information in Exercise 44(a) to find how long it takes for a light ray from the sun to reach the earth.

55. Volume of the Oceans The average ocean depth is 3.7×10^{3} m, and the area of the oceans is 3.6×10^{14} m^2. What is the total volume of the ocean in liters? (One cubic meter contains 1000 liters.)

56. National Debt As of July 2010, the population of the United States was 3.070×10^{8}, and the national debt was 1.320×10^{13} dollars. How much was each person's share of the debt?

57. Number of Molecules A sealed room in a hospital, measuring 5 m wide, 10 m long, and 3 m high, is filled with pure oxygen. One cubic meter contains 1000 L, and 22.4 L of any gas contains 6.02×10^{23} molecules (Avogadro's number). How many molecules of oxygen are there in the room?

58. Body-Mass Index The body-mass index is a measure that medical researchers use to determine whether a person is overweight, underweight, or of normal weight. For a person who weighs W pounds and who is H inches tall, the body-mass index B is given by

$$B = 703\frac{W}{H^2}$$

A body-mass index is considered "normal" if it satisfies $18.5 \le B \le 24.9$, while a person with body-mass index $B \ge 30$ is considered obese.
(a) Calculate the body-mass index for each person listed in the table, then determine whether he or she is of normal weight, underweight, overweight, or obese.

Person	Weight	Height
Brian	295 lb	5 ft 10 in.
Linda	105 lb	5 ft 6 in.
Larry	220 lb	6 ft 4 in.
Helen	110 lb	5 ft 2 in.

(b) Determine your own body-mass index.

59. Interest on a CD A sum of \$5000 is invested in a 5-year certificate of deposit paying 3% interest per year, compounded monthly. After n years the amount of interest I that has accumulated is given by

$$I = 5000[(1.0025)^{12n} - 1]$$

Complete the following table, which gives the amount of interest accumulated after the given number of years.

Year	Total interest
1	\$152.08
2	308.79
3	
4	
5	

DISCOVERY ■ DISCUSSION ■ WRITING

60. How Big Is a Billion? If you had a million (10^6) dollars in a suitcase, and you spent a thousand (10^3) dollars each day, how many years would it take you to use all the money? If you spent at the same rate, how many years would it take you to empty a suitcase filled with a *billion* (10^9) dollars?

61. Easy Powers That Look Hard Calculate these expressions in your head. Use the Laws of Exponents to help you.

(a) $\dfrac{18^5}{9^5}$ **(b)** $20^6 \cdot (0.5)^6$

62. Distances Between Powers Which pair of numbers is closer together?

$$10^{10} \text{ and } 10^{50} \qquad \text{or} \qquad 10^{100} \text{ and } 10^{101}$$

63. Signs of Numbers Let a, b, and c be real numbers with $a > 0$, $b < 0$, and $c < 0$. Determine the sign of each expression.

(a) b^5 **(b)** b^{10} **(c)** ab^2c^3

(d) $(b - a)^3$ **(e)** $(b - a)^4$ **(f)** $\dfrac{a^3c^3}{b^6c^6}$

P.4 Rational Exponents and Radicals

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Simplify expressions involving radicals ▶ Simplify expressions involving rational exponents ▶ Express radicals using rational exponents ▶ Rationalize a denominator and express a quotient of radicals in standard form

In this section we learn to work with expressions that contain radicals or rational exponents.

▼ Radicals

We know what 2^n means whenever n is an integer. To give meaning to a power, such as $2^{4/5}$, whose exponent is a rational number, we need to discuss radicals.

The symbol $\sqrt{}$ means "the positive square root of." Thus

$$\sqrt{a} = b \qquad \text{means} \qquad b^2 = a \qquad \text{and} \qquad b \geq 0$$

It is true that the number 9 has two square roots, 3 and -3, but the notation $\sqrt{9}$ is reserved for the *positive* square root of 9 (sometimes called the *principal square root* of 9). If we want the negative root, we must write $-\sqrt{9}$, which is -3.

Since $a = b^2 \geq 0$, the symbol $\sqrt{a}$ makes sense only when $a \geq 0$. For instance,

$$\sqrt{9} = 3 \qquad \text{because} \qquad 3^2 = 9 \qquad \text{and} \qquad 3 \geq 0$$

Square roots are special cases of nth roots. The nth root of x is the number that, when raised to the nth power, gives x.

DEFINITION OF nth ROOT

If n is any positive integer, then the **principal nth root** of a is defined as follows:

$$\sqrt[n]{a} = b \qquad \text{means} \qquad b^n = a$$

If n is even, we must have $a \geq 0$ and $b \geq 0$.

Thus

$$\sqrt[4]{81} = 3 \qquad \text{because} \qquad 3^4 = 81 \qquad \text{and} \qquad 3 \geq 0$$

$$\sqrt[3]{-8} = -2 \qquad \text{because} \qquad (-2)^3 = -8$$

But $\sqrt{-8}$, $\sqrt[4]{-8}$, and $\sqrt[6]{-8}$ are not defined. (For instance, $\sqrt{-8}$ is not defined because the square of every real number is nonnegative.)

Notice that

$$\sqrt{4^2} = \sqrt{16} = 4 \qquad \text{but} \qquad \sqrt{(-4)^2} = \sqrt{16} = 4 = |-4|$$

So the equation $\sqrt{a^2} = a$ is not always true; it is true only when $a \geq 0$. However, we can always write $\sqrt{a^2} = |a|$. This last equation is true not only for square roots, but for any even root. This and other rules used in working with nth roots are listed in the following box. In each property we assume that all the given roots exist.

PROPERTIES OF *n*th ROOTS

Property	Example
1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$	$\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8}\sqrt[3]{27} = (-2)(3) = -6$
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\dfrac{16}{81}} = \dfrac{\sqrt[4]{16}}{\sqrt[4]{81}} = \dfrac{2}{3}$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = 3$
4. $\sqrt[n]{a^n} = a$ if n is odd	$\sqrt[3]{(-5)^3} = -5, \quad \sqrt[5]{2^5} = 2$
5. $\sqrt[n]{a^n} = \lvert a \rvert$ if n is even	$\sqrt[4]{(-3)^4} = \lvert -3 \rvert = 3$

EXAMPLE 1 | Simplifying Expressions Involving *n*th Roots

(a)
$$\begin{aligned} \sqrt[3]{x^4} &= \sqrt[3]{x^3 x} && \text{Factor out the largest cube} \\ &= \sqrt[3]{x^3}\sqrt[3]{x} && \text{Property 1: } \sqrt[3]{ab} = \sqrt[3]{a}\sqrt[3]{b} \\ &= x\sqrt[3]{x} && \text{Property 4: } \sqrt[3]{a^3} = a \end{aligned}$$

(b)
$$\begin{aligned} \sqrt[4]{81x^8y^4} &= \sqrt[4]{81}\sqrt[4]{x^8}\sqrt[4]{y^4} && \text{Property 1: } \sqrt[4]{abc} = \sqrt[4]{a}\sqrt[4]{b}\sqrt[4]{c} \\ &= 3\sqrt[4]{(x^2)^4}\,\lvert y \rvert && \text{Property 5: } \sqrt[4]{a^4} = \lvert a \rvert \\ &= 3x^2\lvert y \rvert && \text{Property 5: } \sqrt[4]{a^4} = \lvert a \rvert,\ \lvert x^2 \rvert = x^2 \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **27** AND **33**

It is frequently useful to combine like radicals in an expression such as $2\sqrt{3} + 5\sqrt{3}$. This can be done by using the Distributive Property. Thus

$$2\sqrt{3} + 5\sqrt{3} = (2 + 5)\sqrt{3} = 7\sqrt{3}$$

The next example further illustrates this process.

Avoid making the following error:

$$\sqrt{a+b} \cancel{=} \sqrt{a} + \sqrt{b}$$

For instance, if we let $a = 9$ and $b = 16$, then we see the error:

$$\begin{aligned} \sqrt{9+16} &\stackrel{?}{=} \sqrt{9} + \sqrt{16} \\ \sqrt{25} &\stackrel{?}{=} 3 + 4 \\ 5 &\stackrel{?}{=} 7 \quad \text{Wrong!} \end{aligned}$$

EXAMPLE 2 | Combining Radicals

(a)
$$\begin{aligned} \sqrt{32} + \sqrt{200} &= \sqrt{16 \cdot 2} + \sqrt{100 \cdot 2} && \text{Factor out the largest squares} \\ &= \sqrt{16}\sqrt{2} + \sqrt{100}\sqrt{2} && \text{Property 1} \\ &= 4\sqrt{2} + 10\sqrt{2} = 14\sqrt{2} && \text{Distributive Property} \end{aligned}$$

(b) If $b > 0$, then

$$\begin{aligned} \sqrt{25b} - \sqrt{b^3} &= \sqrt{25}\sqrt{b} - \sqrt{b^2}\sqrt{b} && \text{Property 1: } \sqrt{xy} = \sqrt{x}\sqrt{y} \\ &= 5\sqrt{b} - b\sqrt{b} && \text{Property 5; } b > 0 \\ &= (5 - b)\sqrt{b} && \text{Distributive Property} \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **39** AND **45**

Rational Exponents

To define what is meant by a *rational exponent* or, equivalently, a *fractional exponent* such as $a^{1/3}$, we need to use radicals. To give meaning to the symbol $a^{1/n}$ in a way that is consistent with the Laws of Exponents, we would have to have

$$(a^{1/n})^n = a^{(1/n)n} = a^1 = a$$

So by the definition of nth root,

$$a^{1/n} = \sqrt[n]{a}$$

In general, we define rational exponents as follows.

DEFINITION OF RATIONAL EXPONENTS

For any rational exponent m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = (\sqrt[n]{a})^m \qquad \text{or, equivalently,} \qquad a^{m/n} = \sqrt[n]{a^m}$$

If n is even, then we require that $a \geq 0$.

With this definition it can be proved that *the Laws of Exponents also hold for rational exponents* (see page 19).

EXAMPLE 3 | Using the Definition of Rational Exponents

(a) $4^{1/2} = \sqrt{4} = 2$

(b) $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$ Alternative solution: $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

(c) $125^{-1/3} = \dfrac{1}{125^{1/3}} = \dfrac{1}{\sqrt[3]{125}} = \dfrac{1}{5}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **51** AND **53**

EXAMPLE 4 | Using the Laws of Exponents with Rational Exponents

(a) $a^{1/3}a^{7/3} = a^{8/3}$ Law 1: $a^m a^n = a^{m+n}$

(b) $\dfrac{a^{2/5}a^{7/5}}{a^{3/5}} = a^{2/5+7/5-3/5} = a^{6/5}$ Law 1, Law 2: $\dfrac{a^m}{a^n} = a^{m-n}$

(c)
$$\begin{aligned}(2a^3b^4)^{3/2} &= 2^{3/2}(a^3)^{3/2}(b^4)^{3/2} && \text{Law 4: } (abc)^n = a^n b^n c^n\\ &= (\sqrt{2})^3 a^{3(3/2)} b^{4(3/2)} && \text{Law 3: } (a^m)^n = a^{mn}\\ &= 2\sqrt{2}a^{9/2}b^6\end{aligned}$$

(d)
$$\begin{aligned}\left(\frac{2x^{3/4}}{y^{1/3}}\right)^3\left(\frac{y^4}{x^{-1/2}}\right) &= \frac{2^3(x^{3/4})^3}{(y^{1/3})^3}\cdot(y^4x^{1/2}) && \text{Laws 5, 4, and 7}\\ &= \frac{8x^{9/4}}{y}\cdot y^4x^{1/2} && \text{Law 3}\\ &= 8x^{11/4}y^3 && \text{Laws 1 and 2}\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **61**, **65**, **67** AND **71**

EXAMPLE 5 | Simplifying by Writing Radicals as Rational Exponents

(a) $\dfrac{1}{\sqrt[3]{x^4}} = \dfrac{1}{x^{4/3}} = x^{-4/3}$

(b) $(2\sqrt{x})(3\sqrt[3]{x}) = (2x^{1/2})(3x^{1/3})$ — Definition of rational exponents

$= 6x^{1/2+1/3} = 6x^{5/6}$ — Law 1

(c) $\sqrt{x\sqrt{x}} = (xx^{1/2})^{1/2}$ — Definition of rational exponents

$= (x^{3/2})^{1/2}$ — Law 1

$= x^{3/4}$ — Law 3

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **73**, **77**, AND **87**

Rationalizing the Denominator; Standard Form

It is often useful to eliminate the radical in a denominator by multiplying both numerator and denominator by an appropriate expression. This procedure is called **rationalizing the denominator**. If the denominator is of the form $\sqrt{a}$, we multiply numerator and denominator by $\sqrt{a}$. In doing this we multiply the given quantity by 1, so we do not change its value. For instance,

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \cdot 1 = \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

Note that the denominator in the last fraction contains no radical. In general, if the denominator is of the form $\sqrt[n]{a^m}$ with $m < n$, then multiplying the numerator and denominator by $\sqrt[n]{a^{n-m}}$ will rationalize the denominator, because (for $a > 0$)

$$\sqrt[n]{a^m}\sqrt[n]{a^{n-m}} = \sqrt[n]{a^{m+n-m}} = \sqrt[n]{a^n} = a$$

A fractional expression whose denominator contains no radicals is said to be in **standard form**.

EXAMPLE 6 | Rationalizing Denominators

Put each fractional expression into standard form by rationalizing the denominator.

(a) $\dfrac{2}{\sqrt{3}}$ **(b)** $\dfrac{1}{\sqrt[3]{5}}$ **(c)** $\dfrac{1}{\sqrt[5]{x^2}}$

SOLUTION

This equals 1

(a) $\dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$ — Multiply by $\dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{2\sqrt{3}}{3}$ — $\sqrt{3} \cdot \sqrt{3} = 3$

(b) $\dfrac{1}{\sqrt[3]{5}} = \dfrac{1}{\sqrt[3]{5}} \cdot \dfrac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}$ — Multiply by $\dfrac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}$

$= \dfrac{\sqrt[3]{25}}{5}$ — $\sqrt[3]{5} \cdot \sqrt[3]{5^2} = \sqrt[3]{5^3} = 5$

(c) $\dfrac{1}{\sqrt[5]{x^2}} = \dfrac{1}{\sqrt[5]{x^2}} \cdot \dfrac{\sqrt[5]{x^3}}{\sqrt[5]{x^3}}$ — Multiply by $\dfrac{\sqrt[5]{x^3}}{\sqrt[5]{x^3}}$

$= \dfrac{\sqrt[5]{x^3}}{x}$ — $\sqrt[5]{x^2} \cdot \sqrt[5]{x^3} = \sqrt[5]{x^5} = x$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **89**, **91**, AND **93**

P.4 EXERCISES

CONCEPTS

1. Using exponential notation, we can write $\sqrt[3]{5}$ as ________.

2. Using radicals, we can write $5^{1/2}$ as ________.

3. Is there a difference between $\sqrt{5^2}$ and $(\sqrt{5})^2$? Explain.

4. Explain what $4^{3/2}$ means, then calculate $4^{3/2}$ in two different ways:

$(4^{1/2})^{\square} =$ ________ or $(4^3)^{\square} =$ ________

5. Explain how we rationalize a denominator, then complete the following steps to rationalize $\frac{1}{\sqrt{3}}$:

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\square}{\square} = \frac{\square}{\square}$$

6. Find the missing power in the following calculation: $5^{1/3} \cdot 5^{\square} = 5$.

SKILLS

7–14 ■ Write each radical expression using exponents, and each exponential expression using radicals.

	Radical expression	Exponential expression
7.	$\frac{1}{\sqrt{7}}$	
8.	$\sqrt[3]{5^2}$	
9.		$4^{2/3}$
10.		$6^{-3/2}$
11.	$\sqrt[5]{5^3}$	
12.		$3^{-1.5}$
13.		$a^{2/5}$
14.	$\frac{1}{\sqrt{x^5}}$	
15.	$\sqrt[3]{y^4}$	
16.		$y^{-5/3}$

17–24 ■ Evaluate each expression.

17. **(a)** $\sqrt{16}$ **(b)** $\sqrt[4]{16}$ **(c)** $\sqrt[4]{\frac{1}{16}}$

18. **(a)** $\sqrt{64}$ **(b)** $\sqrt[3]{-64}$ **(c)** $\sqrt[5]{-32}$

19. **(a)** $3\sqrt[3]{16}$ **(b)** $\frac{\sqrt{18}}{\sqrt{81}}$ **(c)** $\sqrt{\frac{27}{4}}$

20. **(a)** $2\sqrt[3]{81}$ **(b)** $\frac{\sqrt{12}}{\sqrt{25}}$ **(c)** $\sqrt{\frac{18}{49}}$

21. **(a)** $\sqrt{7}\sqrt{28}$ **(b)** $\frac{\sqrt{48}}{\sqrt{3}}$ **(c)** $\sqrt[4]{24}\sqrt[4]{54}$

22. **(a)** $\sqrt{12}\sqrt{24}$ **(b)** $\frac{\sqrt{54}}{\sqrt{6}}$ **(c)** $\sqrt[3]{15}\sqrt[3]{75}$

23. **(a)** $\frac{\sqrt{216}}{\sqrt{6}}$ **(b)** $\sqrt[3]{2}\sqrt[3]{32}$ **(c)** $\sqrt[4]{\frac{1}{4}}\sqrt[4]{\frac{1}{64}}$

24. **(a)** $\sqrt[5]{\frac{1}{8}}\sqrt[5]{\frac{1}{4}}$ **(b)** $\sqrt[6]{\frac{1}{2}}\sqrt[6]{128}$ **(c)** $\frac{\sqrt[3]{4}}{\sqrt[3]{108}}$

25–38 ■ Simplify the expression. Assume that the letters denote any real numbers.

25. $\sqrt[4]{x^4}$ **26.** $\sqrt[5]{x^{10}}$

27. $\sqrt[5]{32y^6}$ **28.** $\sqrt[3]{8a^5}$

29. $\sqrt[4]{16x^8}$ **30.** $\sqrt[3]{x^3y^6}$

31. $\sqrt[3]{x^3y}$ **32.** $\sqrt{x^4y^4}$

33. $\sqrt{36r^2t^4}$ **34.** $\sqrt[4]{48a^7b^4}$

35. $\sqrt[5]{a^6b^7}$ **36.** $\sqrt[3]{a^2b}\sqrt[3]{a^4b}$

37. $\sqrt[3]{\sqrt{64x^6}}$ **38.** $\sqrt[4]{x^4y^2z^2}$

39–50 ■ Simplify the expression. Assume that all letters denote positive numbers.

39. $\sqrt{32} + \sqrt{18}$ **40.** $\sqrt{75} + \sqrt{48}$

41. $\sqrt{125} - \sqrt{45}$ **42.** $\sqrt[3]{54} - \sqrt[3]{16}$

43. $\sqrt[3]{108} - \sqrt[3]{32}$ **44.** $\sqrt{8} + \sqrt{50}$

45. $\sqrt{9a^3} - \sqrt{a}$ **46.** $\sqrt{16x} + \sqrt{x^5}$

47. $\sqrt[3]{x^4} + \sqrt[3]{8x}$ **48.** $\sqrt[3]{2y^4} - \sqrt[3]{2y}$

49. $\sqrt[3]{16a^5} - 3\sqrt[3]{2a^2}$ **50.** $4\sqrt{18rt^3} + 5\sqrt{32r^3t^5}$

51–56 ■ Evaluate each expression.

51. **(a)** $16^{1/4}$ **(b)** $-125^{1/3}$ **(c)** $9^{-1/2}$

52. **(a)** $27^{1/3}$ **(b)** $(-8)^{1/3}$ **(c)** $-(\frac{1}{8})^{1/3}$

53. **(a)** $32^{2/5}$ **(b)** $(\frac{4}{9})^{-1/2}$ **(c)** $(\frac{16}{81})^{3/4}$

54. **(a)** $125^{2/3}$ **(b)** $(\frac{25}{64})^{3/2}$ **(c)** $27^{-4/3}$

55. **(a)** $5^{2/3} \cdot 5^{1/3}$ **(b)** $\frac{3^{3/5}}{3^{2/5}}$ **(c)** $(\sqrt[3]{4})^3$

56. **(a)** $3^{2/7} \cdot 3^{12/7}$ **(b)** $\frac{7^{2/3}}{7^{5/3}}$ **(c)** $(\sqrt[5]{6})^{-10}$

57–60 ■ Evaluate the expression using $x = 3$, $y = 4$, and $z = -1$.

57. $\sqrt{x^2 + y^2}$ **58.** $\sqrt[4]{x^3 + 14y + 2z}$

59. $(9x)^{2/3} + (2y)^{2/3} + z^{2/3}$ **60.** $(xy)^{2z}$

61–72 ■ Simplify the expression and eliminate any negative exponent(s). Assume that all letters denote positive numbers.

61. **(a)** $x^{3/4}x^{5/4}$ **(b)** $y^{2/3}y^{4/3}$

62. **(a)** $r^{1/6}r^{5/6}$ **(b)** $a^{3/5}a^{3/10}$

63. **(a)** $(4b)^{1/2}(8b^{1/4})$ **(b)** $(3a^{3/4})^2(5a^{1/2})$

64. **(a)** $(27y)^{1/3}(y^{5/3})$ **(b)** $(32w)^{3/5}(2w^{1/5})^2$

65. **(a)** $\dfrac{w^{4/3}w^{2/3}}{w^{1/3}}$ **(b)** $\dfrac{s^{5/2}(2s^{5/4})^2}{s^{1/2}}$

66. **(a)** $\dfrac{x^{3/4}x^{7/4}}{x^{5/4}}$ **(b)** $\dfrac{(2y^{4/3})^2y^{-2/3}}{y^{7/3}}$

67. **(a)** $(8a^6b^{3/2})^{2/3}$ **(b)** $(4a^6b^8)^{3/2}$

68. **(a)** $(64a^6b^3)^{2/3}$ **(b)** $(16w^8z^{3/2})^{3/4}$

69. **(a)** $(8y^3)^{-2/3}$ **(b)** $(u^4v^6)^{-1/3}$

70. **(a)** $(x^{-5}y^{1/3})^{-3/5}$ **(b)** $(2x^3y^{-1/4})^2(8y^{-3/2})^{-1/3}$

71. **(a)** $\left(\dfrac{x^{-2/3}}{y^{1/2}}\right)\left(\dfrac{x^{-2}}{y^{-3}}\right)^{1/6}$ **(b)** $\left(\dfrac{4y^3z^{2/3}}{x^{1/2}}\right)^2\left(\dfrac{x^{-3}y^6}{8z^4}\right)^{1/3}$

72. **(a)** $\left(\dfrac{x^8y^{-4}}{16y^{4/3}}\right)^{-1/4}$ **(b)** $\left(\dfrac{-8y^{3/4}}{y^3z^6}\right)^{-1/3}$

73–88 ■ Simplify the expression and express the answer using rational exponents. Assume that all letters denote positive numbers.

73. $\sqrt[9]{x^5}$ **74.** $\dfrac{1}{\sqrt[5]{x^3}}$

75. $(\sqrt[6]{y^5})(\sqrt[3]{y^2})$ **76.** $\sqrt[4]{b^3}\sqrt{b}$

77. $(5\sqrt[3]{x})(2\sqrt[4]{x})$ **78.** $(2\sqrt{a})(\sqrt[3]{a^2})$

79. $\sqrt{4st^3}\sqrt[6]{s^3t^2}$ **80.** $\sqrt[5]{x^3y^2}\sqrt[10]{x^4y^{16}}$

81. $\dfrac{\sqrt[4]{x^7}}{\sqrt[4]{x^3}}$ **82.** $\dfrac{\sqrt[3]{8x^2}}{\sqrt{x}}$

83. $\dfrac{\sqrt{xy}}{\sqrt[4]{16xy}}$ **84.** $\dfrac{\sqrt{a^3b}}{\sqrt[4]{a^3b^2}}$

85. $\sqrt{\dfrac{16u^3v}{uv^5}}$ **86.** $\sqrt[3]{\dfrac{54x^2y^4}{2x^5y}}$

87. $\sqrt[3]{y\sqrt{y}}$ **88.** $\sqrt{s\sqrt{s^3}}$

89–94 ■ Put each fractional expression into standard form by rationalizing the denominator.

89. **(a)** $\dfrac{1}{\sqrt{6}}$ **(b)** $\dfrac{3}{\sqrt{2}}$ **(c)** $\dfrac{9}{\sqrt{3}}$

90. **(a)** $\dfrac{12}{\sqrt{3}}$ **(b)** $\dfrac{5}{\sqrt{2}}$ **(c)** $\dfrac{2}{\sqrt{6}}$

91. **(a)** $\dfrac{1}{\sqrt[3]{4}}$ **(b)** $\dfrac{1}{\sqrt[4]{3}}$ **(c)** $\dfrac{8}{\sqrt[5]{2}}$

92. **(a)** $\dfrac{1}{\sqrt[5]{2^3}}$ **(b)** $\dfrac{2}{\sqrt[4]{3}}$ **(c)** $\dfrac{3}{\sqrt[4]{2^3}}$

93. **(a)** $\dfrac{1}{\sqrt[3]{x}}$ **(b)** $\dfrac{1}{\sqrt[6]{x^5}}$ **(c)** $\dfrac{1}{\sqrt[7]{x^3}}$

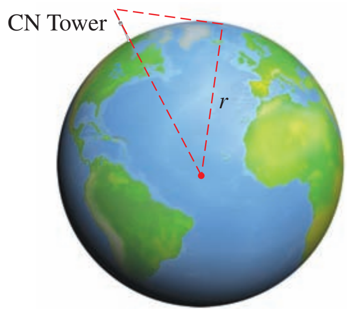
94. **(a)** $\dfrac{1}{\sqrt[3]{x^2}}$ **(b)** $\dfrac{1}{\sqrt[4]{x^3}}$ **(c)** $\dfrac{1}{\sqrt[3]{x^4}}$

APPLICATIONS

95. **How Far Can You See?** Because of the curvature of the earth, the maximum distance D that you can see from the top of a tall building of height h is estimated by the formula

$$D = \sqrt{2rh + h^2}$$

where $r = 3960$ mi is the radius of the earth and D and h are also measured in miles. How far can you see from the observation deck of the Toronto CN Tower, 1135 ft above the ground?



96. **Speed of a Skidding Car** Police use the formula $s = \sqrt{30fd}$ to estimate the speed s (in mi/h) at which a car is traveling if it skids d feet after the brakes are applied suddenly. The number f is the coefficient of friction of the road, which is a measure of the "slipperiness" of the road. The following table gives some typical estimates for f.

	Tar	Concrete	Gravel
Dry	1.0	0.8	0.2
Wet	0.5	0.4	0.1

(a) If a car skids 65 ft on wet concrete, how fast was it moving when the brakes were applied?
(b) If a car is traveling at 50 mi/h, how far will it skid on wet tar?

97. **Sailboat Races** The speed that a sailboat is capable of sailing is determined by three factors: its total length L, the surface area A of its sails, and its displacement V (the volume of water it displaces).

In general, a sailboat is capable of greater speed if it is longer, has a larger sail area, or displaces less water. To make sailing races fair, only boats in the same "class" can qualify to race together. For a certain race, a boat is considered to qualify if

$$0.30L + 0.38A^{1/2} - 3V^{1/3} \le 16$$

where L is measured in feet, A in square feet, and V in cubic feet. Use this inequality to answer the following questions.

(a) A sailboat has length 60 ft, sail area 3400 ft^2, and displacement 650 ft^3. Does this boat qualify for the race?

(b) A sailboat has length 65 ft and displaces 600 ft^3. What is the largest possible sail area that could be used and still allow the boat to qualify for this race?

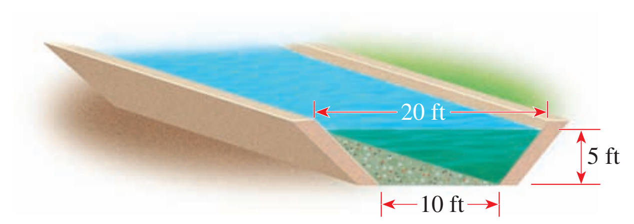
98. Flow Speed in a Channel The speed of water flowing in a channel, such as a canal or river bed, is governed by the Manning Equation,

$$V = 1.486\frac{A^{2/3}S^{1/2}}{p^{2/3}n}$$

Here V is the velocity of the flow in ft/s; A is the cross-sectional area of the channel in square feet; S is the downward slope of the channel; p is the wetted perimeter in feet (the distance from the top of one bank, down the side of the channel, across the bottom, and up to the top of the other bank); and n is the roughness coefficient (a measure of the roughness of the channel bottom). This equation is used to predict the capacity of flood channels to handle runoff from heavy rainfalls. For the canal shown in the figure, $A = 75\ \text{ft}^2$, $S = 0.050$, $p = 24.1$ ft, and $n = 0.040$.

(a) Find the speed at which water flows through the canal.

(b) How many cubic feet of water can the canal discharge per second? [*Hint:* Multiply V by A to get the volume of the flow per second.]



DISCOVERY ▪ DISCUSSION ▪ WRITING

99. Limiting Behavior of Powers Complete the following tables. What happens to the nth root of 2 as n gets large? What about the nth root of $\frac{1}{2}$?

n	$2^{1/n}$
1	
2	
5	
10	
100	

n	$\left(\frac{1}{2}\right)^{1/n}$
1	
2	
5	
10	
100	

Construct a similar table for $n^{1/n}$. What happens to the nth root of n as n gets large?

100. Comparing Roots Without using a calculator, determine which number is larger in each pair.

(a) $2^{1/2}$ or $2^{1/3}$ **(b)** $\left(\frac{1}{2}\right)^{1/2}$ or $\left(\frac{1}{2}\right)^{1/3}$

(c) $7^{1/4}$ or $4^{1/3}$ **(d)** $\sqrt[3]{5}$ or $\sqrt{3}$

P.5 ALGEBRAIC EXPRESSIONS

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Add and subtract polynomials ▶ Multiply algebraic expressions ▶ Use the Special Product Formulas

A **variable** is a letter that can represent any number from a given set of numbers. If we start with variables, such as x, y, and z and some real numbers, and we combine them using addition, subtraction, multiplication, division, powers, and roots, we obtain an **algebraic expression**. Here are some examples:

$$2x^2 - 3x + 4 \qquad \sqrt{x} + 10 \qquad \frac{y - 2z}{y^2 + 4}$$

A **monomial** is an expression of the form ax^k, where a is a real number and k is a nonnegative integer. A **binomial** is a sum of two monomials and a **trinomial** is a sum of three monomials. In general, a sum of monomials is called a *polynomial*. For example, the first expression listed above is a polynomial, but the other two are not.

POLYNOMIALS

A **polynomial** in the variable x is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers, and n is a nonnegative integer. If $a_n \neq 0$, then the polynomial has **degree** $\boldsymbol{n}$. The monomials $a_k x^k$ that make up the polynomial are called the **terms** of the polynomial.

Note that the degree of a polynomial is the highest power of the variable that appears in the polynomial.

Polynomial	Type	Terms	Degree
$2x^2 - 3x + 4$	trinomial	$2x^2, -3x, 4$	2
$x^8 + 5x$	binomial	$x^8, 5x$	8
$8 - x + x^2 - \frac{1}{2}x^3$	four terms	$-\frac{1}{2}x^3, x^2, -x, 8$	3
$5x + 1$	binomial	$5x, 1$	1
$9x^5$	monomial	$9x^5$	5
6	monomial	6	0

Adding and Subtracting Polynomials

We **add** and **subtract** polynomials using the properties of real numbers that were discussed in Section P.2. The idea is to combine **like terms** (that is, terms with the same variables raised to the same powers) using the Distributive Property. For instance,

Distributive Property

$ac + bc = (a + b)c$

$$5x^7 + 3x^7 = (5 + 3)x^7 = 8x^7$$

In subtracting polynomials, we have to remember that if a minus sign precedes an expression in parentheses, then the sign of every term within the parentheses is changed when we remove the parentheses:

$$-(b + c) = -b - c$$

[This is simply a case of the Distributive Property, $a(b + c) = ab + ac$, with $a = -1$.]

EXAMPLE 1 | Adding and Subtracting Polynomials

(a) Find the sum $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$.

(b) Find the difference $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$.

SOLUTION

(a) $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$

$= (x^3 + x^3) + (-6x^2 + 5x^2) + (2x - 7x) + 4$ Group like terms

$= 2x^3 - x^2 - 5x + 4$ Combine like terms

(b) $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$

$= x^3 - 6x^2 + 2x + 4 - x^3 - 5x^2 + 7x$ Distributive Property

$= (x^3 - x^3) + (-6x^2 - 5x^2) + (2x + 7x) + 4$ Group like terms

$= -11x^2 + 9x + 4$ Combine like terms

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **15**

Multiplying Algebraic Expressions

To find the **product** of polynomials or other algebraic expressions, we need to use the Distributive Property repeatedly. In particular, using it three times on the product of two binomials, we get

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$$

This says that we multiply the two factors by multiplying each term in one factor by each term in the other factor and adding these products. Schematically, we have

The acronym **FOIL** helps us remember that the product of two binomials is the sum of the products of the **F**irst terms, the **O**uter terms, the **I**nner terms, and the **L**ast terms.

$$(a + b)(c + d) = \underset{\text{F}}{\underset{\uparrow}{ac}} + \underset{\text{O}}{\underset{\uparrow}{ad}} + \underset{\text{I}}{\underset{\uparrow}{bc}} + \underset{\text{L}}{\underset{\uparrow}{bd}}$$

In general, we can multiply two algebraic expressions by using the Distributive Property and the Laws of Exponents.

EXAMPLE 2 | Multiplying Binomials Using FOIL

$$\begin{aligned}(2x + 1)(3x - 5) &= \underset{\text{F}}{\underset{\uparrow}{6x^2}} - \underset{\text{O}}{\underset{\uparrow}{10x}} + \underset{\text{I}}{\underset{\uparrow}{3x}} - \underset{\text{L}}{\underset{\uparrow}{5}} && \text{Distributive Property} \\ &= 6x^2 - 7x - 5 && \text{Combine like terms}\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **35**

When we multiply trinomials or other polynomials with more terms, we use the Distributive Property. It is also helpful to arrange our work in table form. The next example illustrates both methods.

EXAMPLE 3 | Multiplying Polynomials

Find the product: $(2x + 3)(x^2 - 5x + 4)$

SOLUTION 1: Using the Distributive Property

$$\begin{aligned}(2x + 3)(x^2 - 5x + 4) &= 2x(x^2 - 5x + 4) + 3(x^2 - 5x + 4) && \text{Distributive Property} \\ &= (2x \cdot x^2 - 2x \cdot 5x + 2x \cdot 4) + (3 \cdot x^2 - 3 \cdot 5x + 3 \cdot 4) && \text{Distributive Property} \\ &= (2x^3 - 10x^2 + 8x) + (3x^2 - 15x + 12) && \text{Laws of Exponents} \\ &= 2x^3 - 7x^2 - 7x + 12 && \text{Combine like terms}\end{aligned}$$

SOLUTION 2: Using Table Form

$$\begin{array}{rrrrl} & x^2 - & 5x + & 4 & \\ & & 2x + & 3 & \\ \hline & 3x^2 - & 15x + & 12 & \text{Multiply } x^2 - 5x + 4 \text{ by } 3 \\ 2x^3 - & 10x^2 + & 8x & & \text{Multiply } x^2 - 5x + 4 \text{ by } 2x \\ \hline 2x^3 - & 7x^2 - & 7x + & 12 & \text{Add like terms} \end{array}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **65**

Special Product Formulas

Certain types of products occur so frequently that you should memorize them. You can verify the following formulas by performing the multiplications.

See the *Discovery Project* referenced on page 36 for a geometric interpretation of some of these formulas.

> **SPECIAL PRODUCT FORMULAS**
>
> If A and B are any real numbers or algebraic expressions, then
>
> **1.** $(A + B)(A - B) = A^2 - B^2$ — Sum and difference of same terms
>
> **2.** $(A + B)^2 = A^2 + 2AB + B^2$ — Square of a sum
>
> **3.** $(A - B)^2 = A^2 - 2AB + B^2$ — Square of a difference
>
> **4.** $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ — Cube of a sum
>
> **5.** $(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$ — Cube of a difference

The key idea in using these formulas (or any other formula in algebra) is the **Principle of Substitution**: We may substitute any algebraic expression for any letter in a formula. For example, to find $(x^2 + y^3)^2$ we use Product Formula 2, substituting x^2 for A and y^3 for B, to get

$$(x^2 + y^3)^2 = (x^2)^2 + 2(x^2)(y^3) + (y^3)^2$$

$$(A + B)^2 = A^2 + 2AB + B^2$$

EXAMPLE 4 | Using the Special Product Formulas

Use a Special Product Formula to find each product.

(a) $(3x + 5)^2$ **(b)** $(x^2 - 2)^3$

SOLUTION

(a) Substituting $A = 3x$ and $B = 5$ in Product Formula 2, we get

$$(3x + 5)^2 = (3x)^2 + 2(3x)(5) + 5^2 = 9x^2 + 30x + 25$$

(b) Substituting $A = x^2$ and $B = 2$ in Product Formula 5, we get

$$\begin{aligned}(x^2 - 2)^3 &= (x^2)^3 - 3(x^2)^2(2) + 3(x^2)(2)^2 - 2^3 \\ &= x^6 - 6x^4 + 12x^2 - 8\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **43** AND **61**

EXAMPLE 5 | Using the Special Product Formulas

Find each product.

(a) $(2x - \sqrt{y})(2x + \sqrt{y})$ **(b)** $(x + y - 1)(x + y + 1)$

SOLUTION

(a) Substituting $A = 2x$ and $B = \sqrt{y}$ in Product Formula 1, we get

$$(2x - \sqrt{y})(2x + \sqrt{y}) = (2x)^2 - (\sqrt{y})^2 = 4x^2 - y$$

(b) If we group $x + y$ together and think of this as one algebraic expression, we can use Product Formula 1 with $A = x + y$ and $B = 1$:

$$\begin{aligned}(x + y - 1)(x + y + 1) &= [(x + y) - 1][(x + y) + 1] \\ &= (x + y)^2 - 1^2 && \text{Product Formula 1} \\ &= x^2 + 2xy + y^2 - 1 && \text{Product Formula 2}\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **59** AND **83**

P.5 EXERCISES

CONCEPTS

1. Which of the following expressions are polynomials?

(a) $2x^3 - \frac{1}{2}x + \sqrt{3}$ **(b)** $x^2 - \frac{1}{2} - 3\sqrt{x}$

(c) $\dfrac{1}{x^2 + 4x + 7}$ **(d)** $x^5 + 7x^2 - x + 100$

(e) $\sqrt[3]{8x^6 - 5x^3 + 7x - 3}$ **(f)** $\sqrt{3}x^4 + \sqrt{5}x^2 - 15x$

2. To add polynomials, we add ________ terms. So $(3x^2 + 2x + 4) + (8x^2 - x + 1) =$ ________________.

3. To subtract polynomials, we subtract ________ terms. So $(2x^3 + 9x^2 + x + 10) - (x^3 + x^2 + 6x + 8) =$ ________.

4. Explain how we multiply two polynomials, then perform the following multiplication: $(x + 2)(x + 3) =$ ______________.

5. The Special Product Formula for the "square of a sum" is $(A + B)^2 =$ ____________. So $(2x + 3)^2 =$ ____________.

6. The Special Product Formula for the product of the "sum and difference of terms" is $(A + B)(A - B) =$ ______________. So $(5 + x)(5 - x) =$ ______________.

SKILLS

7–12 ■ Complete the following table by stating whether the polynomial is a monomial, binomial, or trinomial, then list its terms and state its degree.

Polynomial	Type	Terms	Degree
7. $x^2 - 3x + 7$			
8. $2x^5 + 4x^2$			
9. -8			
10. $\frac{1}{2}x^7$			
11. $x - x^2 + x^3 - x^4$			
12. $\sqrt{2}x - \sqrt{3}$			

13–30 ■ Find the sum, difference, or product.

13. $(6x - 3) + (3x + 7)$ **14.** $(3 - 7x) - (11 + 4x)$

15. $(2x^2 - 5x) - (x^2 - 8x + 3)$

16. $(x^2 + 10x - 3) + (3x^2 - 12x + 5)$

17. $3(x - 1) + 4(x + 2)$

18. $8(2x + 5) - 7(x - 9)$

19. $(x^3 + 6x^2 - 4x + 7) - (3x^2 + 2x - 4)$

20. $4(x^2 - 3x + 5) - 3(x^2 - 2x + 1)$

21. $2x(x - 1)$ **22.** $3y(2y + 5)$

23. $x^2(x + 3)$ **24.** $-y(y^2 - 2)$

25. $2(2 - 5t) + t(t + 10)$ **26.** $5(3t - 4) - 2t(t - 3)$

27. $r(r^2 - 9) + 3r^2(2r - 1)$ **28.** $v^3(v - 9) - 2v^2(2 - 2v)$

29. $x^2(2x^2 - x + 1)$ **30.** $3x^3(x^4 - 4x^2 + 5)$

31–42 ■ Multiply the algebraic expressions using the FOIL method, and simplify.

31. $(x - 3)(x + 5)$ **32.** $(4 + x)(2 + x)$

33. $(s + 6)(2s + 3)$ **34.** $(2t + 3)(t - 1)$

35. $(3t - 2)(7t - 4)$ **36.** $(4s - 1)(2s + 5)$

37. $(3x + 5)(2x - 1)$ **38.** $(7y - 3)(2y - 1)$

39. $(x + 3y)(2x - y)$ **40.** $(4x - 5y)(3x - y)$

41. $(2r - 5s)(3r - 2s)$ **42.** $(6u + 5v)(u - 2v)$

43–64 ■ Multiply the algebraic expressions using a Special Product Formula, and simplify.

43. $(x + 5)^2$ **44.** $(x - 3)^2$

45. $(3y - 1)^2$ **46.** $(2y + 5)^2$

47. $(2u + v)^2$ **48.** $(x - 3y)^2$

49. $(2x + 3y)^2$ **50.** $(r - 2s)^2$

51. $(x^2 + 1)^2$ **52.** $(2 + y^3)^2$

53. $(x + 5)(x - 5)$ **54.** $(y - 3)(y + 3)$

55. $(3x - 4)(3x + 4)$ **56.** $(2y + 5)(2y - 5)$

57. $(x + 3y)(x - 3y)$ **58.** $(2u + v)(2u - v)$

59. $(\sqrt{x} + 2)(\sqrt{x} - 2)$ **60.** $(\sqrt{y} + \sqrt{2})(\sqrt{y} - \sqrt{2})$

61. $(y + 2)^3$ **62.** $(x - 3)^3$

63. $(1 - 2r)^3$ **64.** $(3 + 2y)^3$

65–84 ■ Perform the indicated operations, and simplify.

65. $(x + 2)(x^2 + 2x + 3)$ **66.** $(x + 1)(2x^2 - x + 1)$

67. $(2x - 5)(x^2 - x + 1)$ **68.** $(1 + 2x)(x^2 - 3x + 1)$

69. $\sqrt{x}(x - \sqrt{x})$ **70.** $x^{3/2}(\sqrt{x} - 1/\sqrt{x})$

71. $y^{1/3}(y^{2/3} + y^{5/3})$ **72.** $x^{1/4}(2x^{3/4} - x^{1/4})$

73. $(x^2 + y^2)^2$ **74.** $\left(c + \frac{1}{c}\right)^2$

75. $(x^2 - a^2)(x^2 + a^2)$ **76.** $(x^{1/2} + y^{1/2})(x^{1/2} - y^{1/2})$

77. $(\sqrt{a} - b)(\sqrt{a} + b)$

78. $(\sqrt{h^2 + 1} + 1)(\sqrt{h^2 + 1} - 1)$

79. $(1 + x^{2/3})(1 - x^{2/3})$ **80.** $(1 - b)^2(1 + b)^2$

81. $((x - 1) + x^2)((x - 1) - x^2)$

82. $(x + (2 + x^2))(x - (2 + x^2))$

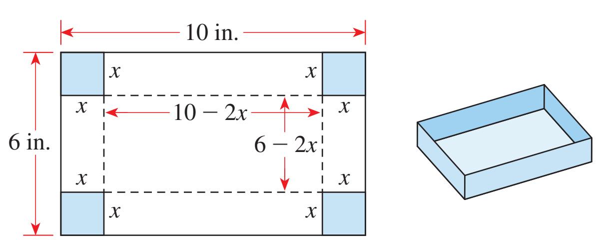
83. $(2x + y - 3)(2x + y + 3)$ **84.** $(x + y + z)(x - y - z)$

APPLICATIONS

85. Volume of a Box An open box is constructed from a 6 in. by 10 in. sheet of cardboard by cutting a square piece from each corner and then folding up the sides, as shown in the figure. The volume of the box is

$$V = x(6 - 2x)(10 - 2x)$$

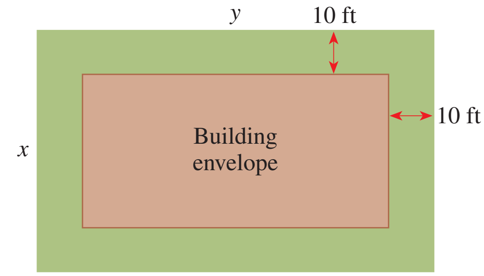
(a) Explain how the expression for V is obtained.
(b) Expand the expression for V. What is the degree of the resulting polynomial?
(c) Find the volume when $x = 1$ and when $x = 2$.



86. Building Envelope The building code in a certain town requires that a house be at least 10 ft from the boundaries of the lot. The buildable area (or *building envelope*) for the rectangular lot shown in the following figure is given by

$$A = (x - 20)(y - 20)$$

(a) Explain how the expression for A is obtained.
(b) Expand to express A as a polynomial in x and y.
(c) A contractor has a choice of purchasing one of two rectangular lots, each having the same area. One lot measures 100 ft by 400 ft; the other measures 200 ft by 200 ft. Which lot has the larger building envelope?



87. Interest on an Investment A 3-year certificate of deposit pays interest at a rate r compounded annually. If \$2000 is invested, then the amount at maturity is

$$A = 2000(1 + r)^3$$

(a) Expand the expression for A. What is the degree of the resulting polynomial?
(b) Find the amounts A for the values of r in the table.

Interest rate r	2%	3%	4.5%	6%	10%
Amount A					

88. Profit A wholesaler sells graphing calculators. For an order of x calculators his total cost in dollars is

$$C = 50 + 30x - 0.1x^2$$

and his total revenue is

$$R = 50x - 0.05x^2$$

(a) Find the profit P on an order of x calculators.
(b) Find the profit on an order of 10 calculators and on an order of 20 calculators.

DISCOVERY ■ DISCUSSION ■ WRITING

89. An Algebra Error Beginning algebra students sometimes make the following error when squaring a binomial:

$$(x + 5)^2 \cancel{=} x^2 + 25$$

(a) Substitute a value for x to verify that this is an error.
(b) What is the correct expansion for $(x + 5)^2$?

90. Degrees of Sums and Products of Polynomials Make up several pairs of polynomials, then calculate the sum and product of each pair. On the basis of your experiments and observations, answer the following questions.
(a) How is the degree of the product related to the degrees of the original polynomials?
(b) How is the degree of the sum related to the degrees of the original polynomials?
(c) Test your conclusions by finding the sum and product of the following polynomials:

$$2x^3 + x - 3 \qquad \text{and} \qquad -2x^3 - x + 7$$

DISCOVERY PROJECT **Visualizing a Formula**

In this project we discover geometric interpretations of some of the Special Product Formulas. You can find the project at the book companion website: **www.stewartmath.com**

P.6 FACTORING

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Factor out common factors ▶ Factor trinomials by trial and error ▶ Use the Special Factoring Formulas ▶ Factor algebraic expressions completely ▶ Factor by grouping terms

We use the Distributive Property to expand algebraic expressions. We sometimes need to reverse this process (again using the Distributive Property) by **factoring** an expression as a product of simpler ones. For example, we can write

FACTORING →

$$x^2 - 4 = (x - 2)(x + 2)$$

← EXPANDING

We say that $x - 2$ and $x + 2$ are **factors** of $x^2 - 4$.

▼ Common Factors

The easiest type of factoring occurs when the terms have a common factor.

EXAMPLE 1 | Factoring Out Common Factors

Factor each expression.

(a) $3x^2 - 6x$ **(b)** $8x^4y^2 + 6x^3y^3 - 2xy^4$

SOLUTION

(a) The greatest common factor of the terms $3x^2$ and $-6x$ is $3x$, so we have

$$3x^2 - 6x = 3x(x - 2)$$

(b) We note that

8, 6, and -2 have the greatest common factor 2
x^4, x^3, and x have the greatest common factor x
y^2, y^3, and y^4 have the greatest common factor y^2

So the greatest common factor of the three terms in the polynomial is $2xy^2$, and we have

$$\begin{aligned} 8x^4y^2 + 6x^3y^3 - 2xy^4 &= (2xy^2)(4x^3) + (2xy^2)(3x^2y) + (2xy^2)(-y^2) \\ &= 2xy^2(4x^3 + 3x^2y - y^2) \end{aligned}$$

CHECK YOUR ANSWER

(a) Multiplying gives

$$3x(x - 2) = 3x^2 - 6x \quad ✓$$

(b) Multiplying gives

$$\begin{aligned} &2xy^2(4x^3 + 3x^2y - y^2) \\ &= 8x^4y^2 + 6x^3y^3 - 2xy^4 \quad ✓ \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **7** AND **9** ■

EXAMPLE 2 | Factoring Out a Common Factor

Factor: $(2x + 4)(x - 3) - 5(x - 3)$

SOLUTION The two terms have the common factor $x - 3$.

$$\begin{aligned}(2x+4)(x-3) - 5(x-3) &= [(2x+4)-5](x-3) && \text{Distributive Property}\\ &= (2x-1)(x-3) && \text{Simplify}\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **11** ■

Factoring Trinomials

To factor a trinomial of the form $x^2 + bx + c$, we note that

$$(x+r)(x+s) = x^2 + (r+s)x + rs$$

so we need to choose numbers r and s so that $r + s = b$ and $rs = c$.

EXAMPLE 3 | Factoring $x^2 + bx + c$ by Trial and Error

Factor: $x^2 + 7x + 12$

SOLUTION We need to find two integers whose product is 12 and whose sum is 7. By trial and error we find that the two integers are 3 and 4. Thus the factorization is

$$x^2 + 7x + 12 = (x + \underbrace{3)(x + 4}_{\text{factors of }12})$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **13** ■

CHECK YOUR ANSWER

Multiplying gives

$(x+3)(x+4) = x^2 + 7x + 12$ ✓

To factor a trinomial of the form $ax^2 + bx + c$ with $a \neq 1$, we look for factors of the form $px + r$ and $qx + s$:

$$ax^2 + bx + c = (px + r)(qx + s) = pqx^2 + (ps + qr)x + rs$$

Therefore we try to find numbers p, q, r, and s such that $pq = a$, $rs = c$, $ps + qr = b$. If these numbers are all integers, then we will have a limited number of possibilities to try for p, q, r, and s.

factors of a

$$ax^2 + bx + c = (px + r)(qx + s)$$

factors of c

EXAMPLE 4 | Factoring $ax^2 + bx + c$ by Trial and Error

Factor: $6x^2 + 7x - 5$

SOLUTION We can factor 6 as $6 \cdot 1$ or $3 \cdot 2$, and -5 as $-5 \cdot 1$ or $5 \cdot (-1)$. By trying these possibilities, we arrive at the factorization

$$6x^2 + 7x - 5 = (\overbrace{3x + 5)(2x}^{\text{factors of }6} - 1)$$

$$(3x + \underbrace{5)(2x - 1}_{\text{factors of }-5})$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17** ■

CHECK YOUR ANSWER

Multiplying gives

$(3x+5)(2x-1) = 6x^2 + 7x - 5$ ✓

EXAMPLE 5 | Recognizing the Form of an Expression

Factor each expression.

(a) $x^2 - 2x - 3$ **(b)** $(5a+1)^2 - 2(5a+1) - 3$

SOLUTION

(a) $x^2 - 2x - 3 = (x - 3)(x + 1)$ Trial and error

(b) This expression is of the form

$$\square^2 - 2\square - 3$$

where $\square$ represents $5a + 1$. This is the same form as the expression in part (a), so it will factor as $(\square - 3)(\square + 1)$:

$$(5a + 1)^2 - 2(5a + 1) - 3 = [(5a + 1) - 3][(5a + 1) + 1]$$
$$= (5a - 2)(5a + 2)$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **19**

Special Factoring Formulas

Some special algebraic expressions can be factored using the following formulas. The first three are simply Special Product Formulas written backward.

FACTORING FORMULAS

Formula	**Name**
1. $A^2 - B^2 = (A - B)(A + B)$	Difference of squares
2. $A^2 + 2AB + B^2 = (A + B)^2$	Perfect square
3. $A^2 - 2AB + B^2 = (A - B)^2$	Perfect square
4. $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$	Difference of cubes
5. $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$	Sum of cubes

EXAMPLE 6 | Factoring Differences of Squares

Factor each expression.

(a) $4x^2 - 25$ **(b)** $(x + y)^2 - z^2$

SOLUTION

(a) Using the Difference of Squares Formula with $A = 2x$ and $B = 5$, we have

$$4x^2 - 25 = (2x)^2 - 5^2 = (2x - 5)(2x + 5)$$

$$A^2 \quad - \quad B^2 \quad = \quad (A \quad - \quad B)(A \quad + \quad B)$$

(b) We use the Difference of Squares Formula with $A = x + y$ and $B = z$:

$$(x + y)^2 - z^2 = (x + y - z)(x + y + z)$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **23** AND **27**

EXAMPLE 7 | Recognizing Perfect Squares

Factor each trinomial:

(a) $x^2 + 6x + 9$ **(b)** $4x^2 - 4xy + y^2$

SOLUTION

(a) Here $A = x$ and $B = 3$, so $2AB = 2 \cdot x \cdot 3 = 6x$. Since the middle term is $6x$, the trinomial is a perfect square. By the Perfect Square Formula we have

$$x^2 + 6x + 9 = (x + 3)^2$$

(b) Here $A = 2x$ and $B = y$, so $2AB = 2 \cdot 2x \cdot y = 4xy$. Since the middle term is $-4xy$, the trinomial is a perfect square. By the Perfect Square Formula we have

$$4x^2 - 4xy + y^2 = (2x - y)^2$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **29** AND **35**

EXAMPLE 8 | Factoring Differences and Sums of Cubes

Factor each polynomial:

(a) $27x^3 - 1$ **(b)** $x^6 + 8$

SOLUTION

(a) Using the Difference of Cubes Formula with $A = 3x$ and $B = 1$, we get

$$\begin{aligned} 27x^3 - 1 &= (3x)^3 - 1^3 = (3x - 1)[(3x)^2 + (3x)(1) + 1^2] \\ &= (3x - 1)(9x^2 + 3x + 1) \end{aligned}$$

(b) Using the Sum of Cubes Formula with $A = x^2$ and $B = 2$, we have

$$x^6 + 8 = (x^2)^3 + 2^3 = (x^2 + 2)(x^4 - 2x^2 + 4)$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **39** AND **43**

A trinomial is a perfect square if it is of the form

$$A^2 + 2AB + B^2 \qquad \text{or} \qquad A^2 - 2AB + B^2$$

So we **recognize a perfect square** if the middle term ($2AB$ or $-2AB$) is plus or minus twice the product of the square roots of the outer two terms.

Factoring an Expression Completely

When we factor an expression, the result can sometimes be factored further. In general, we first factor out common factors, then inspect the result to see whether it can be factored by any of the other methods of this section. We repeat this process until we have factored the expression completely.

EXAMPLE 9 | Factoring an Expression Completely

Factor each expression completely.

(a) $2x^4 - 8x^2$ **(b)** $x^5y^2 - xy^6$

SOLUTION

(a) We first factor out the power of x with the smallest exponent:

$$\begin{aligned} 2x^4 - 8x^2 &= 2x^2(x^2 - 4) && \text{Common factor is } 2x^2 \\ &= 2x^2(x - 2)(x + 2) && \text{Factor } x^2 - 4 \text{ as a difference of squares} \end{aligned}$$

(b) We first factor out the powers of x and y with the smallest exponents:

$$\begin{aligned} x^5y^2 - xy^6 &= xy^2(x^4 - y^4) && \text{Common factor is } xy^2 \\ &= xy^2(x^2 + y^2)(x^2 - y^2) && \text{Factor } x^4 - y^4 \text{ as a difference of squares} \\ &= xy^2(x^2 + y^2)(x + y)(x - y) && \text{Factor } x^2 - y^2 \text{ as a difference of squares} \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **85** AND **89**

In the next example we factor out variables with fractional exponents. This type of factoring occurs in calculus.

EXAMPLE 10 | Factoring Expressions with Fractional Exponents

Factor each expression.

(a) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ **(b)** $(2 + x)^{-2/3}x + (2 + x)^{1/3}$

SOLUTION

(a) Factor out the power of x with the *smallest exponent*, that is, $x^{-1/2}$:

$$\begin{aligned} 3x^{3/2} - 9x^{1/2} + 6x^{-1/2} &= 3x^{-1/2}(x^2 - 3x + 2) && \text{Factor out } 3x^{-1/2} \\ &= 3x^{-1/2}(x - 1)(x - 2) && \text{Factor the quadratic } x^2 - 3x + 2 \end{aligned}$$

(b) Factor out the power of $2 + x$ with the *smallest exponent*, that is, $(2 + x)^{-2/3}$:

$$\begin{aligned} (2 + x)^{-2/3}x + (2 + x)^{1/3} &= (2 + x)^{-2/3}[x + (2 + x)] && \text{Factor out } (2 + x)^{-2/3} \\ &= (2 + x)^{-2/3}(2 + 2x) && \text{Simplify} \\ &= 2(2 + x)^{-2/3}(1 + x) && \text{Factor out 2} \end{aligned}$$

CHECK YOUR ANSWER

To see that you have factored correctly, multiply using the Laws of Exponents.

(a) $3x^{-1/2}(x^2 - 3x + 2)$

$= 3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ ✓

(b) $(2 + x)^{-2/3}[x + (2 + x)]$

$= (2 + x)^{-2/3}x + (2 + x)^{1/3}$ ✓

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **53** AND **55**

Factoring by Grouping Terms

Polynomials with at least four terms can sometimes be factored by grouping terms. The following example illustrates the idea.

EXAMPLE 11 | Factoring by Grouping

Factor each polynomial.

(a) $x^3 + x^2 + 4x + 4$ **(b)** $x^3 - 2x^2 - 3x + 6$

SOLUTION

(a)
$$\begin{aligned} x^3 + x^2 + 4x + 4 &= (x^3 + x^2) + (4x + 4) && \text{Group terms} \\ &= x^2(x + 1) + 4(x + 1) && \text{Factor out common factors} \\ &= (x^2 + 4)(x + 1) && \text{Factor out } x + 1 \text{ from each term} \end{aligned}$$

(b)
$$\begin{aligned} x^3 - 2x^2 - 3x + 6 &= (x^3 - 2x^2) - (3x - 6) && \text{Group terms} \\ &= x^2(x - 2) - 3(x - 2) && \text{Factor out common factors} \\ &= (x^2 - 3)(x - 2) && \text{Factor out } x - 2 \text{ from each term} \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45**

P.6 EXERCISES

CONCEPTS

1. Consider the polynomial $2x^5 + 6x^4 + 4x^3$.

(a) How many terms does this polynomial have? ________

List the terms: ____________________

(b) What factor is common to each term? ________

Factor the polynomial: $2x^5 + 6x^4 + 4x^3 =$ __________.

2. To factor the trinomial $x^2 + 7x + 10$, we look for two integers whose product is ________ and whose sum is ________. These integers are ________ and ________, so the trinomial factors as ______________.

3. The Special Factoring Formula for the "difference of squares" is $A^2 - B^2 =$ ________. So $4x^2 - 25$ factors as ________.

4. The Special Factoring Formula for a "perfect square" is $A^2 + 2AB + B^2 =$ ________. So $x^2 + 10x + 25$ factors as ________.

SKILLS

5–12 ■ Factor out the common factor.

5. $5a - 20$

6. $-3b + 12$

7. $-2x^3 + 16x$

8. $2x^4 + 4x^3 - 14x^2$

9. $2x^2y - 6xy^2 + 3xy$

10. $-7x^4y^2 + 14xy^3 + 21xy^4$

11. $y(y - 6) + 9(y - 6)$

12. $(z + 2)^2 - 5(z + 2)$

13–20 ■ Factor the trinomial.

13. $x^2 + 2x - 3$

14. $x^2 - 6x + 5$

15. $x^2 + 2x - 15$

16. $2x^2 - 5x - 7$

17. $3x^2 - 16x + 5$

18. $5x^2 - 7x - 6$

19. $(3x + 2)^2 + 8(3x + 2) + 12$

20. $2(a + b)^2 + 5(a + b) - 3$

21–28 ■ Factor the difference of squares.

21. $x^2 - 25$

22. $9 - y^2$

23. $49 - 4z^2$

24. $9a^2 - 16$

25. $16y^2 - z^2$

26. $a^2 - 36b^2$

27. $(x + 3)^2 - y^2$

28. $x^2 - (y + 5)^2$

29–36 ■ Factor the perfect square.

29. $x^2 + 10x + 25$

30. $9 + 6y + y^2$

31. $z^2 - 12z + 36$

32. $w^2 - 16w + 64$

33. $4t^2 - 20t + 25$

34. $16a^2 + 24a + 9$

35. $9u^2 - 6uv + v^2$

36. $x^2 + 10xy + 25y^2$

37–44 ■ Factor the sum or difference of cubes.

37. $x^3 + 27$

38. $y^3 - 64$

39. $8a^3 - 1$

40. $8 + 27w^3$

41. $27x^3 + y^3$

42. $1 + 1000y^3$

43. $u^3 - v^6$

44. $8r^3 - 64t^6$

45–50 ■ Factor the expression by grouping terms.

45. $x^3 + 4x^2 + x + 4$

46. $3x^3 - x^2 + 6x - 2$

47. $2x^3 + x^2 - 6x - 3$

48. $-9x^3 - 3x^2 + 3x + 1$

49. $x^3 + x^2 + x + 1$

50. $x^5 + x^4 + x + 1$

51–58 ■ Factor the expression completely. Begin by factoring out the lowest power of each common factor.

51. $x^{5/2} - x^{1/2}$

52. $3x^{-1/2} + 4x^{1/2} + x^{3/2}$

53. $x^{-3/2} + 2x^{-1/2} + x^{1/2}$

54. $(x - 1)^{7/2} - (x - 1)^{3/2}$

55. $(x^2 + 1)^{1/2} + 2(x^2 + 1)^{-1/2}$

56. $x^{-1/2}(x + 1)^{1/2} + x^{1/2}(x + 1)^{-1/2}$

57. $2x^{1/3}(x - 2)^{2/3} - 5x^{4/3}(x - 2)^{-1/3}$

58. $3x^{-1/2}(x^2 + 1)^{5/4} - x^{3/2}(x^2 + 1)^{1/4}$

59–84 ■ Factor the expression.

59. $12x^3 + 18x$

60. $30x^3 + 15x^4$

61. $6y^4 - 15y^3$

62. $5ab - 8abc$

63. $x^2 - 2x - 8$

64. $x^2 - 14x + 48$

65. $y^2 - 8y + 15$

66. $z^2 + 6z - 16$

67. $2x^2 + 5x + 3$

68. $2x^2 + 7x - 4$

69. $9x^2 - 36x - 45$

70. $8x^2 + 10x + 3$

71. $6x^2 - 5x - 6$

72. $6 + 5t - 6t^2$

73. $x^2 - 36$

74. $4x^2 - 25$

75. $49 - 4y^2$

76. $4t^2 - 9s^2$

77. $t^2 - 6t + 9$

78. $x^2 + 10x + 25$

79. $4x^2 + 4xy + y^2$

80. $r^2 - 6rs + 9s^2$

81. $t^3 + 1$

82. $x^3 - 27$

83. $8x^3 - 125$

84. $125 + 27y^3$

85–96 ■ Factor the expression completely.

85. $x^3 + 2x^2 + x$

86. $3x^3 - 27x$

87. $x^4 + 2x^3 - 3x^2$

88. $3w^3 - 5w^4 - 2w^3$

89. $x^4y^3 - x^2y^5$

90. $18y^3x^2 - 2xy^4$

91. $x^6 - 8y^3$

92. $27a^3 + b^6$

93. $y^3 - 3y^2 - 4y + 12$

94. $y^3 - y^2 + y - 1$

95. $2x^3 + 4x^2 + x + 2$

96. $3x^3 + 5x^2 - 6x - 10$

97–106 ■ Factor the expression and simplify.

97. $(a + b)^2 - (a - b)^2$

98. $\left(1 + \frac{1}{x}\right)^2 - \left(1 - \frac{1}{x}\right)^2$

99. $x^2(x^2 - 1) - 9(x^2 - 1)$

100. $(a^2 - 1)b^2 - 4(a^2 - 1)$

101. $(x - 1)(x + 2)^2 - (x - 1)^2(x + 2)$

102. $(x + 1)^3x - 2(x + 1)^2x^2 + x^3(x + 1)$

103. $y^4(y + 2)^3 + y^5(y + 2)^4$

104. $n(x - y) + (n - 1)(y - x)$

105. $(a^2 + 1)^2 - 7(a^2 + 1) + 10$

106. $(a^2 + 2a)^2 - 2(a^2 + 2a) - 3$

107–112 ■ Factor the expression completely. (This type of expression arises in calculus in using the "product rule.")

107. $3x^2(4x - 12)^2 + x^3(2)(4x - 12)(4)$

108. $5(x^2 + 4)^4(2x)(x - 2)^4 + (x^2 + 4)^5(4)(x - 2)^3$

109. $3(2x - 1)^2(2)(x + 3)^{1/2} + (2x - 1)^3(\frac{1}{2})(x + 3)^{-1/2}$

110. $\frac{1}{3}(x + 6)^{-2/3}(2x - 3)^2 + (x + 6)^{1/3}(2)(2x - 3)(2)$

111. $(x^2 + 3)^{-1/3} - \frac{2}{3}x^2(x^2 + 3)^{-4/3}$

112. $\frac{1}{2}x^{-1/2}(3x + 4)^{1/2} + \frac{3}{2}x^{1/2}(3x + 4)^{-1/2}$

113. **(a)** Show that $ab = \frac{1}{2}[(a + b)^2 - (a^2 + b^2)]$.

(b) Show that $(a^2 + b^2)^2 - (a^2 - b^2)^2 = 4a^2b^2$.

(c) Show that

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$

(d) Factor completely: $4a^2c^2 - (a^2 - b^2 + c^2)^2$.

114. Verify Factoring Formulas 4 and 5 by expanding their right-hand sides.

APPLICATIONS

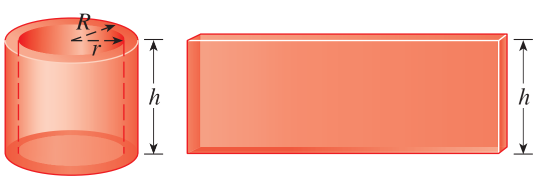
115. Volume of Concrete A culvert is constructed out of large cylindrical shells cast in concrete, as shown in the figure. Using the formula for the volume of a cylinder given on the inside back cover of this book, explain why the volume of the cylindrical shell is

$$V = \pi R^2 h - \pi r^2 h$$

Factor to show that

$$V = 2\pi \cdot \text{average radius} \cdot \text{height} \cdot \text{thickness}$$

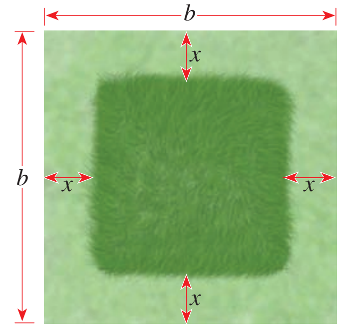
Use the "unrolled" diagram to explain why this makes sense geometrically.



116. Mowing a Field A square field in a certain state park is mowed around the edges every week. The rest of the field is kept unmowed to serve as a habitat for birds and small animals (see the figure). The field measures b feet by b feet, and the mowed strip is x feet wide.

(a) Explain why the area of the mowed portion is $b^2 - (b - 2x)^2$.

(b) Factor the expression in part (a) to show that the area of the mowed portion is also $4x(b - x)$.



DISCOVERY ■ DISCUSSION ■ WRITING

117. The Power of Algebraic Formulas Use the Difference of Squares Formula to factor $17^2 - 16^2$. Notice that it is easy to calculate the factored form in your head but not so easy to calculate the original form in this way. Evaluate each expression in your head:

(a) $528^2 - 527^2$ **(b)** $122^2 - 120^2$ **(c)** $1020^2 - 1010^2$

Now use the product formula $(A - B)(A + B) = A^2 - B^2$ to evaluate these products in your head:

(d) $49 \cdot 51$ **(e)** $998 \cdot 1002$

118. Differences of Even Powers

(a) Factor the expressions completely: $A^4 - B^4$ and $A^6 - B^6$.

(b) Verify that $18{,}335 = 12^4 - 7^4$ and that $2{,}868{,}335 = 12^6 - 7^6$.

(c) Use the results of parts (a) and (b) to factor the integers 18,335 and 2,868,335. Show that in both of these factorizations, all the factors are prime numbers.

119. Factoring $A^n - 1$ Verify the factoring formulas in the list by expanding and simplifying the right-hand side in each case.

$$A^2 - 1 = (A - 1)(A + 1)$$
$$A^3 - 1 = (A - 1)(A^2 + A + 1)$$
$$A^4 - 1 = (A - 1)(A^3 + A^2 + A + 1)$$

On the basis of the pattern displayed in this list, how do you think $A^5 - 1$ would factor? Verify your conjecture. Now generalize the pattern you have observed to obtain a factorization formula for $A^n - 1$, where n is a positive integer.

P.7 RATIONAL EXPRESSIONS

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find the domain of an algebraic expression ▶ Simplify rational expressions ▶ Add, subtract, multiply, and divide rational expressions ▶ Simplify compound fractions ▶ Rationalize a denominator or numerator ▶ Avoid common errors

A quotient of two algebraic expressions is called a **fractional expression**. Here are some examples:

$$\frac{2x}{x-1} \qquad \frac{y-2}{y^2+4} \qquad \frac{x^3-x}{x^2-5x+6} \qquad \frac{x}{\sqrt{x^2+1}}$$

A **rational expression** is a fractional expression in which both the numerator and denominator are polynomials. For example, the first three expressions in the above list are rational expressions, but the fourth is not, since its denominator contains a radical. In this section we learn how to perform algebraic operations on rational expressions.

▼ The Domain of an Algebraic Expression

Expression	Domain
$\dfrac{1}{x}$	$\{x \mid x \neq 0\}$
$\sqrt{x}$	$\{x \mid x \geq 0\}$
$\dfrac{1}{\sqrt{x}}$	$\{x \mid x > 0\}$

In general, an algebraic expression may not be defined for all values of the variable. The **domain** of an algebraic expression is the set of real numbers that the variable is permitted to have. The table in the margin gives some basic expressions and their domains.

EXAMPLE 1 | Finding the Domain of an Expression

Find the domains of the following expressions.

(a) $2x^2 + 3x - 1$ **(b)** $\dfrac{x}{x^2 - 5x + 6}$ **(c)** $\dfrac{\sqrt{x}}{x-5}$

SOLUTION

(a) This polynomial is defined for every x. Thus the domain is the set $\mathbb{R}$ of real numbers.

(b) We first factor the denominator:

$$\frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x-3)}$$

Denominator would be 0 if $x = 2$ or $x = 3$

Since the denominator is zero when $x = 2$ or 3, the expression is not defined for these numbers. The domain is $\{x \mid x \neq 2 \text{ and } x \neq 3\}$.

(c) For the numerator to be defined, we must have $x \geq 0$. Also, we cannot divide by zero, so $x \neq 5$.

Must have $x \geq 0$ to take square root

$$\frac{\sqrt{x}}{x-5}$$

Denominator would be 0 if $x = 5$

Thus the domain is $\{x \mid x \geq 0 \text{ and } x \neq 5\}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **13** ■

Simplifying Rational Expressions

To **simplify rational expressions**, we factor both numerator and denominator and use the following property of fractions:

$$\frac{AC}{BC} = \frac{A}{B}$$

This allows us to **cancel** common factors from the numerator and denominator.

EXAMPLE 2 | Simplifying Rational Expressions by Cancellation

Simplify: $\dfrac{x^2 - 1}{x^2 + x - 2}$

SOLUTION We first factor:

$$\begin{aligned} \frac{x^2 - 1}{x^2 + x - 2} &= \frac{(x - 1)(x + 1)}{(x - 1)(x + 2)} && \text{Factor} \\ &= \frac{x + 1}{x + 2} && \text{Cancel common factors} \end{aligned}$$

We can't cancel the x^2's in $\dfrac{x^2 - 1}{x^2 + x - 2}$ because x^2 is not a factor.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **25**

Multiplying and Dividing Rational Expressions

To **multiply rational expressions**, we use the following property of fractions:

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$$

This says that to multiply two fractions, we multiply their numerators and multiply their denominators.

EXAMPLE 3 | Multiplying Rational Expressions

Perform the indicated multiplication, and simplify: $\dfrac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \dfrac{3x + 12}{x - 1}$

SOLUTION We first factor:

$$\begin{aligned} \frac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \frac{3x + 12}{x - 1} &= \frac{(x - 1)(x + 3)}{(x + 4)^2} \cdot \frac{3(x + 4)}{x - 1} && \text{Factor} \\ &= \frac{3(x - 1)(x + 3)(x + 4)}{(x - 1)(x + 4)^2} && \text{Property of fractions} \\ &= \frac{3(x + 3)}{x + 4} && \text{Cancel common factors} \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **33**

To **divide rational expressions**, we use the following property of fractions:

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$$

This says that to divide a fraction by another fraction, we invert the divisor and multiply.

EXAMPLE 4 | Dividing Rational Expressions

Perform the indicated division, and simplify:

$$\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6}$$

SOLUTION

$$\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6} = \frac{x-4}{x^2-4} \cdot \frac{x^2+5x+6}{x^2-3x-4} \qquad \text{Invert divisor and multiply}$$

$$= \frac{(x-4)(x+2)(x+3)}{(x-2)(x+2)(x-4)(x+1)} \qquad \text{Factor}$$

$$= \frac{x+3}{(x-2)(x+1)} \qquad \text{Cancel common factors}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **39**

Adding and Subtracting Rational Expressions

Avoid making the following error:

$$\frac{A}{B+C} \;\cancel{=}\; \frac{A}{B} + \frac{A}{C}$$

For instance, if we let $A = 2$, $B = 1$, and $C = 1$, then we see the error:

$$\frac{2}{1+1} \stackrel{?}{=} \frac{2}{1} + \frac{2}{1}$$

$$\frac{2}{2} \stackrel{?}{=} 2 + 2$$

$$1 \stackrel{?}{=} 4 \qquad \text{Wrong!}$$

To **add or subtract rational expressions**, we first find a common denominator and then use the following property of fractions:

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}$$

Although any common denominator will work, it is best to use the **least common denominator** (LCD) as explained in Section P.2. The LCD is found by factoring each denominator and taking the product of the distinct factors, using the highest power that appears in any of the factors.

EXAMPLE 5 | Adding and Subtracting Rational Expressions

Perform the indicated operations, and simplify:

(a) $\dfrac{3}{x-1} + \dfrac{x}{x+2}$ **(b)** $\dfrac{1}{x^2-1} - \dfrac{2}{(x+1)^2}$

SOLUTION

(a) Here the LCD is simply the product $(x-1)(x+2)$.

$$\frac{3}{x-1} + \frac{x}{x+2} = \frac{3(x+2)}{(x-1)(x+2)} + \frac{x(x-1)}{(x-1)(x+2)} \qquad \text{Write fractions using LCD}$$

$$= \frac{3x+6+x^2-x}{(x-1)(x+2)} \qquad \text{Add fractions}$$

$$= \frac{x^2+2x+6}{(x-1)(x+2)} \qquad \text{Combine terms in numerator}$$

DIOPHANTUS lived in Alexandria about 250 A.D. His book *Arithmetica* is considered the first book on algebra. In it he gives methods for finding integer solutions of algebraic equations. *Arithmetica* was read and studied for more than a thousand years. Fermat (see page 107) made some of his most important discoveries while studying this book. Diophantus' major contribution is the use of symbols to stand for the unknowns in a problem. Although his symbolism is not as simple as what we use today, it was a major advance over writing everything in words. In Diophantus' notation the equation

$$x^5 - 7x^2 + 8x - 5 = 24$$

is written

$$\Delta K^{\gamma}\alpha\varsigma\eta ⫛ \Delta^{\gamma}\zeta\mathring{M}\varepsilon\iota^{\sigma}\kappa\delta$$

Our modern algebraic notation did not come into common use until the 17th century.

(b) The LCD of $x^2 - 1 = (x-1)(x+1)$ and $(x+1)^2$ is $(x-1)(x+1)^2$.

$$\begin{aligned}
\frac{1}{x^2-1} - \frac{2}{(x+1)^2} &= \frac{1}{(x-1)(x+1)} - \frac{2}{(x+1)^2} && \text{Factor} \\
&= \frac{(x+1) - 2(x-1)}{(x-1)(x+1)^2} && \text{Combine fractions using LCD} \\
&= \frac{x+1-2x+2}{(x-1)(x+1)^2} && \text{Distributive Property} \\
&= \frac{3-x}{(x-1)(x+1)^2} && \text{Combine terms in numerator}
\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **51** ■

Compound Fractions

A **compound fraction** is a fraction in which the numerator, the denominator, or both are themselves fractional expressions.

EXAMPLE 6 | Simplifying a Compound Fraction

Simplify: $\dfrac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}}$

SOLUTION 1 We combine the terms in the numerator into a single fraction. We do the same in the denominator. Then we invert and multiply.

$$\begin{aligned}
\frac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}} &= \frac{\dfrac{x+y}{y}}{\dfrac{x-y}{x}} \\
&= \frac{x+y}{y} \cdot \frac{x}{x-y} \\
&= \frac{x(x+y)}{y(x-y)}
\end{aligned}$$

SOLUTION 2 We find the LCD of all the fractions in the expression, then multiply numerator and denominator by it. In this example the LCD of all the fractions is xy. Thus

$$\begin{aligned}
\frac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}} &= \frac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}} \cdot \frac{xy}{xy} && \text{Multiply numerator and denominator by } xy \\
&= \frac{x^2 + xy}{xy - y^2} && \text{Simplify} \\
&= \frac{x(x+y)}{y(x-y)} && \text{Factor}
\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **71** ■

The next two examples show situations in calculus that require the ability to work with fractional expressions.

EXAMPLE 7 | Simplifying a Compound Fraction

Simplify: $\dfrac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h}$

SOLUTION We begin by combining the fractions in the numerator using a common denominator:

$$\frac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h} = \frac{\dfrac{a-(a+h)}{a(a+h)}}{h} \qquad \text{Combine fractions in the numerator}$$

$$= \frac{a-(a+h)}{a(a+h)} \cdot \frac{1}{h} \qquad \text{Invert divisor and multiply}$$

$$= \frac{a-a-h}{a(a+h)} \cdot \frac{1}{h} \qquad \text{Distributive Property}$$

$$= \frac{-h}{a(a+h)} \cdot \frac{1}{h} \qquad \text{Simplify}$$

$$= \frac{-1}{a(a+h)} \qquad \text{Cancel common factors}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **83**

EXAMPLE 8 | Simplifying a Compound Fraction

Simplify: $\dfrac{(1+x^2)^{1/2} - x^2(1+x^2)^{-1/2}}{1+x^2}$

SOLUTION 1 Factor $(1+x^2)^{-1/2}$ from the numerator:

Factor out the power of $1 + x^2$ with the *smallest* exponent, in this case $(1+x^2)^{-1/2}$.

$$\frac{(1+x^2)^{1/2} - x^2(1+x^2)^{-1/2}}{1+x^2} = \frac{(1+x^2)^{-1/2}[(1+x^2) - x^2]}{1+x^2}$$

$$= \frac{(1+x^2)^{-1/2}}{1+x^2} = \frac{1}{(1+x^2)^{3/2}}$$

SOLUTION 2 Since $(1+x^2)^{-1/2} = 1/(1+x^2)^{1/2}$ is a fraction, we can clear all fractions by multiplying numerator and denominator by $(1+x^2)^{1/2}$:

$$\frac{(1+x^2)^{1/2} - x^2(1+x^2)^{-1/2}}{1+x^2} = \frac{(1+x^2)^{1/2} - x^2(1+x^2)^{-1/2}}{1+x^2} \cdot \frac{(1+x^2)^{1/2}}{(1+x^2)^{1/2}}$$

$$= \frac{(1+x^2) - x^2}{(1+x^2)^{3/2}} = \frac{1}{(1+x^2)^{3/2}}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **91**

Rationalizing the Denominator or the Numerator

If a fraction has a denominator of the form $A + B\sqrt{C}$, we may rationalize the denominator by multiplying numerator and denominator by the **conjugate radical** $A - B\sqrt{C}$. This is effective because, by Product Formula 1 in Section P.5, the product of the denominator and its conjugate radical does not contain a radical:

$$(A + B\sqrt{C})(A - B\sqrt{C}) = A^2 - B^2C$$

EXAMPLE 9 | Rationalizing the Denominator

Rationalize the denominator: $\dfrac{1}{1 + \sqrt{2}}$

SOLUTION We multiply both the numerator and the denominator by the conjugate radical of $1 + \sqrt{2}$, which is $1 - \sqrt{2}$.

$$\frac{1}{1+\sqrt{2}} = \frac{1}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} \qquad \text{Multiply numerator and denominator by the conjugate radical}$$

$$= \frac{1-\sqrt{2}}{1^2 - (\sqrt{2})^2} \qquad \text{Product Formula 1: } (a+b)(a-b) = a^2 - b^2$$

$$= \frac{1-\sqrt{2}}{1-2} = \frac{1-\sqrt{2}}{-1} = \sqrt{2} - 1$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **95**

EXAMPLE 10 | Rationalizing the Numerator

Rationalize the numerator: $\dfrac{\sqrt{4+h} - 2}{h}$

SOLUTION We multiply numerator and denominator by the conjugate radical $\sqrt{4+h} + 2$:

$$\frac{\sqrt{4+h}-2}{h} = \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} \qquad \text{Multiply numerator and denominator by the conjugate radical}$$

$$= \frac{(\sqrt{4+h})^2 - 2^2}{h(\sqrt{4+h}+2)} \qquad \text{Product Formula 1: } (a+b)(a-b) = a^2 - b^2$$

$$= \frac{4+h-4}{h(\sqrt{4+h}+2)}$$

$$= \frac{h}{h(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2} \qquad \text{Cancel common factors}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **101**

Avoiding Common Errors

Don't make the mistake of applying properties of multiplication to the operation of addition. Many of the common errors in algebra involve doing just that. The following table states several properties of multiplication and illustrates the error in applying them to addition.

Correct multiplication property	**Common error with addition**
$(a \cdot b)^2 = a^2 \cdot b^2$	$(a + b)^2 \cancel{=} a^2 + b^2$
$\sqrt{a \cdot b} = \sqrt{a}\sqrt{b} \quad (a, b \geq 0)$	$\sqrt{a + b} \cancel{=} \sqrt{a} + \sqrt{b}$
$\sqrt{a^2 \cdot b^2} = a \cdot b \quad (a, b \geq 0)$	$\sqrt{a^2 + b^2} \cancel{=} a + b$
$\dfrac{1}{a} \cdot \dfrac{1}{b} = \dfrac{1}{a \cdot b}$	$\dfrac{1}{a} + \dfrac{1}{b} \cancel{=} \dfrac{1}{a + b}$
$\dfrac{ab}{a} = b$	$\dfrac{a + b}{a} \cancel{=} b$
$a^{-1} \cdot b^{-1} = (a \cdot b)^{-1}$	$a^{-1} + b^{-1} \cancel{=} (a + b)^{-1}$

To verify that the equations in the right-hand column are wrong, simply substitute numbers for a and b and calculate each side. For example, if we take $a = 2$ and $b = 2$ in the fourth error, we get different values for the left- and right-hand sides:

$$\underbrace{\frac{1}{a} + \frac{1}{b} = \frac{1}{2} + \frac{1}{2} = 1}_{\text{Left-hand side}} \qquad \underbrace{\frac{1}{a + b} = \frac{1}{2 + 2} = \frac{1}{4}}_{\text{Right-hand side}}$$

Since $1 \neq \frac{1}{4}$, the stated equation is wrong. You should similarly convince yourself of the error in each of the other equations. (See Exercise 119.)

P.7 EXERCISES

CONCEPTS

1. Which of the following are rational expressions?

(a) $\dfrac{3x}{x^2 - 1}$ **(b)** $\dfrac{\sqrt{x + 1}}{2x + 3}$ **(c)** $\dfrac{x(x^2 - 1)}{x + 3}$

2. To simplify a rational expression, we cancel *factors* that are common to the ________ and ________. So the expression $\dfrac{(x + 1)(x + 2)}{(x + 3)(x + 2)}$ simplifies to ________.

3. *True or false*?

(a) $\dfrac{x^2 + 3}{x^2 + 5}$ simplifies to $\dfrac{3}{5}$.

(b) $\dfrac{3x^2}{5x^2}$ simplifies to $\dfrac{3}{5}$.

4. (a) To multiply two rational expressions, we multiply their ________ together and multiply their ________ together. So $\dfrac{2}{x + 1} \cdot \dfrac{x}{x + 3}$ is the same as ________.

(b) To divide two rational expressions, we ________ the divisor, then multiply. So $\dfrac{3}{x + 5} \div \dfrac{x}{x + 2}$ is the same as ________.

5. Consider the expression $\dfrac{1}{x} - \dfrac{2}{x + 1} - \dfrac{x}{(x + 1)^2}$.

(a) How many terms does this expression have?
(b) Find the least common denominator of all the terms.
(c) Perform the addition and simplify.

6. *True or false*?

(a) $\dfrac{1}{2} + \dfrac{1}{x}$ is the same as $\dfrac{1}{2 + x}$.

(b) $\dfrac{1}{2} + \dfrac{1}{x}$ is the same as $\dfrac{x + 2}{2x}$.

SKILLS

7–16 ■ Find the domain of the expression.

7. $-x^2 + 3x + \frac{1}{2}$ **8.** $6x^4 - 5x^3 + 12x$

9. $3x^{-2}$ **10.** $x^{-4} + x^{-1/2}$

11. $\dfrac{2x + 1}{x - 4}$ **12.** $\dfrac{2t^2 - 5}{3t + 6}$

13. $\dfrac{x^2 + 1}{x^2 - x - 2}$ **14.** $\dfrac{\sqrt{2x}}{x + 1}$

15. $\sqrt{x + 3}$ **16.** $\dfrac{1}{\sqrt{x - 1}}$

17–30 ■ Simplify the rational expression.

17. $\dfrac{12x}{6x^2}$

18. $\dfrac{81x^3}{18x}$

19. $\dfrac{5y^2}{10y + y^2}$

20. $\dfrac{14t^2 - t}{7t}$

21. $\dfrac{3(x + 2)(x - 1)}{6(x - 1)^2}$

22. $\dfrac{4(x^2 - 1)}{12(x + 2)(x - 1)}$

23. $\dfrac{x - 2}{x^2 - 4}$

24. $\dfrac{x^2 - x - 2}{x^2 - 1}$

25. $\dfrac{x^2 + 6x + 8}{x^2 + 5x + 4}$

26. $\dfrac{x^2 - x - 12}{x^2 + 5x + 6}$

27. $\dfrac{y^2 + y}{y^2 - 1}$

28. $\dfrac{y^2 - 3y - 18}{2y^2 + 5y + 3}$

29. $\dfrac{2x^3 - x^2 - 6x}{2x^2 - 7x + 6}$

30. $\dfrac{1 - x^2}{x^3 - 1}$

31–38 ■ Perform the multiplication and simplify.

31. $\dfrac{4x}{x^2 - 4} \cdot \dfrac{x + 2}{16x}$

32. $\dfrac{x^2 - 25}{x^2 - 16} \cdot \dfrac{x + 4}{x + 5}$

33. $\dfrac{x^2 - 2x - 15}{x^2 - 9} \cdot \dfrac{x + 3}{x - 5}$

34. $\dfrac{x^2 + 2x - 3}{x^2 - 2x - 3} \cdot \dfrac{3 - x}{3 + x}$

35. $\dfrac{t - 3}{t^2 + 9} \cdot \dfrac{t + 3}{t^2 - 9}$

36. $\dfrac{x^2 - x - 6}{x^2 + 2x} \cdot \dfrac{x^3 + x^2}{x^2 - 2x - 3}$

37. $\dfrac{x^2 + 7x + 12}{x^2 + 3x + 2} \cdot \dfrac{x^2 + 5x + 6}{x^2 + 6x + 9}$

38. $\dfrac{x^2 + 2xy + y^2}{x^2 - y^2} \cdot \dfrac{2x^2 - xy - y^2}{x^2 - xy - 2y^2}$

39–46 ■ Perform the division and simplify.

39. $\dfrac{x + 3}{4x^2 - 9} \div \dfrac{x^2 + 7x + 12}{2x^2 + 7x - 15}$

40. $\dfrac{2x + 1}{2x^2 + x - 15} \div \dfrac{6x^2 - x - 2}{x + 3}$

41. $\dfrac{2x^2 + 3x + 1}{x^2 + 2x - 15} \div \dfrac{x^2 + 6x + 5}{2x^2 - 7x + 3}$

42. $\dfrac{4y^2 - 9}{2y^2 + 9y - 18} \div \dfrac{2y^2 + y - 3}{y^2 + 5y - 6}$

43. $\dfrac{\dfrac{x^3}{x + 1}}{\dfrac{x}{x^2 + 2x + 1}}$

44. $\dfrac{\dfrac{2x^2 - 3x - 2}{x^2 - 1}}{\dfrac{2x^2 + 5x + 2}{x^2 + x - 2}}$

45. $\dfrac{x/y}{z}$

46. $\dfrac{x}{y/z}$

47–66 ■ Perform the addition or subtraction and simplify.

47. $2 + \dfrac{x}{x + 3}$

48. $\dfrac{2x - 1}{x + 4} - 1$

49. $\dfrac{1}{x + 5} + \dfrac{2}{x - 3}$

50. $\dfrac{1}{x + 1} + \dfrac{1}{x - 1}$

51. $\dfrac{1}{x + 1} - \dfrac{1}{x + 2}$

52. $\dfrac{x}{x - 4} - \dfrac{3}{x + 6}$

53. $\dfrac{x}{(x + 1)^2} + \dfrac{2}{x + 1}$

54. $\dfrac{5}{2x - 3} - \dfrac{3}{(2x - 3)^2}$

55. $u + 1 + \dfrac{u}{u + 1}$

56. $\dfrac{2}{a^2} - \dfrac{3}{ab} + \dfrac{4}{b^2}$

57. $\dfrac{1}{x^2} + \dfrac{1}{x^2 + x}$

58. $\dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3}$

59. $\dfrac{2}{x + 3} - \dfrac{1}{x^2 + 7x + 12}$

60. $\dfrac{x}{x^2 - 4} + \dfrac{1}{x - 2}$

61. $\dfrac{1}{x + 3} + \dfrac{1}{x^2 - 9}$

62. $\dfrac{x}{x^2 + x - 2} - \dfrac{2}{x^2 - 5x + 4}$

63. $\dfrac{2}{x} + \dfrac{3}{x - 1} - \dfrac{4}{x^2 - x}$

64. $\dfrac{x}{x^2 - x - 6} - \dfrac{1}{x + 2} - \dfrac{2}{x - 3}$

65. $\dfrac{1}{x^2 + 3x + 2} - \dfrac{1}{x^2 - 2x - 3}$

66. $\dfrac{1}{x + 1} - \dfrac{2}{(x + 1)^2} + \dfrac{3}{x^2 - 1}$

67–82 ■ Simplify the compound fractional expression.

67. $\dfrac{2}{1 - \dfrac{1}{x}}$

68. $\dfrac{1}{3 + \dfrac{2}{x}}$

69. $\dfrac{1 - \dfrac{1}{x}}{1 + \dfrac{1}{x}}$

70. $\dfrac{1 - \dfrac{1}{x^2}}{x + \dfrac{1}{x^2}}$

71. $\dfrac{x - \dfrac{x}{y}}{y - \dfrac{y}{x}}$

72. $\dfrac{x + \dfrac{y}{x}}{y + \dfrac{x}{y}}$

73. $\dfrac{x + \dfrac{1}{x + 2}}{x - \dfrac{1}{x + 2}}$

74. $\dfrac{1 + \dfrac{1}{c - 1}}{1 - \dfrac{1}{c - 1}}$

75. $\dfrac{\dfrac{x + 2}{x - 1} - \dfrac{x - 3}{x - 2}}{x + 2}$

76. $\dfrac{\dfrac{x - 3}{x - 4} - \dfrac{x + 2}{x + 1}}{x + 3}$

77. $\dfrac{\dfrac{x}{y} - \dfrac{y}{x}}{\dfrac{1}{x^2} - \dfrac{1}{y^2}}$

78. $x - \dfrac{y}{\dfrac{x}{y} + \dfrac{y}{x}}$

79. $\dfrac{x^{-2} + y^{-2}}{x^{-1} + y^{-1}}$

80. $\dfrac{x^{-1} + y^{-1}}{(x + y)^{-1}}$

81. $1 - \dfrac{1}{1 - \dfrac{1}{x}}$

82. $1 + \dfrac{1}{1 + \dfrac{1}{1 + x}}$

83–88 ■ Simplify the fractional expression. (Expressions like these arise in calculus.)

83. $\dfrac{\dfrac{1}{1+x+h}-\dfrac{1}{1+x}}{h}$

84. $\dfrac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h}$

85. $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$

86. $\dfrac{(x+h)^3-7(x+h)-(x^3-7x)}{h}$

87. $\sqrt{1+\left(\dfrac{x}{\sqrt{1-x^2}}\right)^2}$

88. $\sqrt{1+\left(x^3-\dfrac{1}{4x^3}\right)^2}$

89–94 ■ Simplify the expression. (This type of expression arises in calculus when using the "quotient rule.")

89. $\dfrac{3(x+2)^2(x-3)^2-(x+2)^3(2)(x-3)}{(x-3)^4}$

90. $\dfrac{2x(x+6)^4-x^2(4)(x+6)^3}{(x+6)^8}$

91. $\dfrac{2(1+x)^{1/2}-x(1+x)^{-1/2}}{x+1}$

92. $\dfrac{(1-x^2)^{1/2}+x^2(1-x^2)^{-1/2}}{1-x^2}$

93. $\dfrac{3(1+x)^{1/3}-x(1+x)^{-2/3}}{(1+x)^{2/3}}$

94. $\dfrac{(7-3x)^{1/2}+\frac{3}{2}x(7-3x)^{-1/2}}{7-3x}$

95–100 ■ Rationalize the denominator.

95. $\dfrac{1}{2-\sqrt{3}}$

96. $\dfrac{2}{3-\sqrt{5}}$

97. $\dfrac{2}{\sqrt{2}+\sqrt{7}}$

98. $\dfrac{1}{\sqrt{x}+1}$

99. $\dfrac{y}{\sqrt{3}+\sqrt{y}}$

100. $\dfrac{2(x-y)}{\sqrt{x}-\sqrt{y}}$

101–106 ■ Rationalize the numerator.

101. $\dfrac{1-\sqrt{5}}{3}$

102. $\dfrac{\sqrt{3}+\sqrt{5}}{2}$

103. $\dfrac{\sqrt{r}+\sqrt{2}}{5}$

104. $\dfrac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$

105. $\sqrt{x^2+1}-x$

106. $\sqrt{x+1}-\sqrt{x}$

107–114 ■ State whether the given equation is true for all values of the variables. (Disregard any value that makes a denominator zero.)

107. $\dfrac{16+a}{16}=1+\dfrac{a}{16}$

108. $\dfrac{b}{b-c}=1-\dfrac{b}{c}$

109. $\dfrac{2}{4+x}=\dfrac{1}{2}+\dfrac{2}{x}$

110. $\dfrac{x+1}{y+1}=\dfrac{x}{y}$

111. $\dfrac{x}{x+y}=\dfrac{1}{1+y}$

112. $2\left(\dfrac{a}{b}\right)=\dfrac{2a}{2b}$

113. $\dfrac{-a}{b}=-\dfrac{a}{b}$

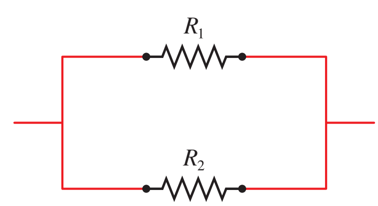
114. $\dfrac{1+x+x^2}{x}=\dfrac{1}{x}+1+x$

APPLICATIONS

115. Electrical Resistance If two electrical resistors with resistances R_1 and R_2 are connected in parallel (see the figure), then the total resistance R is given by

$$R=\dfrac{1}{\dfrac{1}{R_1}+\dfrac{1}{R_2}}$$

(a) Simplify the expression for R.
(b) If $R_1 = 10$ ohms and $R_2 = 20$ ohms, what is the total resistance R?



116. Average Cost A clothing manufacturer finds that the cost of producing x shirts is $500 + 6x + 0.01x^2$ dollars.
(a) Explain why the average cost per shirt is given by the rational expression

$$A=\dfrac{500+6x+0.01x^2}{x}$$

(b) Complete the table by calculating the average cost per shirt for the given values of x.

x	Average cost
10	
20	
50	
100	
200	
500	
1000	

DISCOVERY ■ DISCUSSION ■ WRITING

117. Limiting Behavior of a Rational Expression The rational expression

$$\frac{x^2 - 9}{x - 3}$$

is not defined for $x = 3$. Complete the tables and determine what value the expression approaches as x gets closer and closer to 3. Why is this reasonable? To see why, factor the numerator of the expression and simplify.

x	$\frac{x^2 - 9}{x - 3}$
2.80	
2.90	
2.95	
2.99	
2.999	

x	$\frac{x^2 - 9}{x - 3}$
3.20	
3.10	
3.05	
3.01	
3.001	

118. Is This Rationalization? In the expression $2/\sqrt{x}$ we would eliminate the radical if we were to square both numerator and denominator. Is this the same thing as rationalizing the denominator?

119. Algebraic Errors The left-hand column in the table lists some common algebraic errors. In each case, give an example using numbers that show that the formula is not valid. An example of this type, which shows that a statement is false, is called a *counterexample*.

Algebraic error	Counterexample
$\frac{1}{a} + \frac{1}{b} \cancel{=} \frac{1}{a + b}$	$\frac{1}{2} + \frac{1}{2} \neq \frac{1}{2 + 2}$
$(a + b)^2 \cancel{=} a^2 + b^2$	
$\sqrt{a^2 + b^2} \cancel{=} a + b$	
$\frac{a + b}{a} \cancel{=} b$	
$(a^3 + b^3)^{1/3} \cancel{=} a + b$	
$a^m/a^n \cancel{=} a^{m/n}$	
$a^{-1/n} \cancel{=} \frac{1}{a^n}$	

P.8 SOLVING BASIC EQUATIONS

LEARNING OBJECTIVES After completing this section, you will be able to:

Solve linear equations ▶ Solve power equations ▶ Solve for one variable in terms of others

Equations are the basic mathematical tool for solving real-world problems. In this section we learn how to solve equations.

An equation is a statement that two mathematical expressions are equal. For example,

$$3 + 5 = 8$$

is an equation. Most equations that we study in algebra contain variables, which are symbols (usually letters) that stand for numbers. In the equation

$$4x + 7 = 19$$

the letter x is the variable. We think of x as the "unknown" in the equation, and our goal is to find the value of x that makes the equation true. The values of the unknown that make the equation true are called the **solutions** or **roots** of the equation, and the process of finding the solutions is called **solving the equation**.

$x = 3$ is a solution of the equation $4x + 7 = 19$, because substituting $x = 3$ makes the equation true:

$\boxed{x = 3}$

$4(3) + 7 = 19$ ✓

Two equations with exactly the same solutions are called **equivalent equations**. To solve an equation, we try to find a simpler, equivalent equation in which the variable stands alone on one side of the equal sign. Here are the properties that we use to solve an equation. (In these properties, A, B, and C stand for any algebraic expressions, and the symbol $\Leftrightarrow$ means "is equivalent to.")

PROPERTIES OF EQUALITY

Property	Description
1. $A = B \quad \Leftrightarrow \quad A + C = B + C$	Adding the same quantity to both sides of an equation gives an equivalent equation.
2. $A = B \quad \Leftrightarrow \quad CA = CB \quad (C \neq 0)$	Multiplying both sides of an equation by the same nonzero quantity gives an equivalent equation.

These properties require that you *perform the same operation on both sides of an equation* when solving it. Thus if we say "*add* 4" when solving an equation, that is just a short way of saying "*add* 4 to each side of the equation."

Solving Linear Equations

The simplest type of equation is a *linear equation*, or first-degree equation, which is an equation in which each term is either a constant or a nonzero multiple of the variable.

LINEAR EQUATIONS

A **linear equation** in one variable is an equation that is equivalent to one of the form

$$ax + b = 0$$

where a and b are real numbers and x is the variable.

Here are some examples that illustrate the difference between linear and nonlinear equations.

Linear equations	Nonlinear equations	
$4x - 5 = 3$	$x^2 + 2x = 8$	Not linear; contains the square of the variable
$2x = \frac{1}{2}x - 7$	$\sqrt{x} - 6x = 0$	Not linear; contains the square root of the variable
$x - 6 = \dfrac{x}{3}$	$\dfrac{3}{x} - 2x = 1$	Not linear; contains the reciprocal of the variable

EXAMPLE 1 | Solving a Linear Equation

Solve the equation $7x - 4 = 3x + 8$.

SOLUTION We solve this by changing it to an equivalent equation with all terms that have the variable x on one side and all constant terms on the other:

$$
\begin{aligned}
7x - 4 &= 3x + 8 && \text{Given equation} \\
(7x - 4) + 4 &= (3x + 8) + 4 && \text{Add 4} \\
7x &= 3x + 12 && \text{Simplify} \\
7x - 3x &= (3x + 12) - 3x && \text{Subtract } 3x \\
4x &= 12 && \text{Simplify} \\
\tfrac{1}{4} \cdot 4x &= \tfrac{1}{4} \cdot 12 && \text{Multiply by } \tfrac{1}{4} \\
x &= 3 && \text{Simplify}
\end{aligned}
$$

Because it is important to CHECK YOUR ANSWER, we do this in many of our examples. In these checks, LHS stands for "left-hand side" and RHS stands for "right-hand side" of the original equation.

CHECK YOUR ANSWER

$x = 3$:

$x = 3$: LHS $= 7(3) - 4 = 17$

$x = 3$: RHS $= 3(3) + 8 = 17$

LHS = RHS ✓

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **21**

When a linear equation involves fractions, solving the equation is usually easier if we first multiply each side by the lowest common denominator (LCD) of the fractions, as we see in the following examples.

EXAMPLE 2 | Solving an Equation That Involves Fractions

Solve the equation $\frac{x}{6} + \frac{2}{3} = \frac{3}{4}x$.

SOLUTION The LCD of the denominators 6, 3, and 4 is 12, so we first multiply each side of the equation by 12 to clear the denominators:

$$12 \cdot \left(\frac{x}{6} + \frac{2}{3}\right) = 12 \cdot \frac{3}{4}x \qquad \text{Multiply by LCD}$$

$$2x + 8 = 9x \qquad \text{Distributive Property}$$

$$8 = 7x \qquad \text{Subtract } 2x$$

$$\frac{8}{7} = x \qquad \text{Divide by 7}$$

The solution is $x = \frac{8}{7}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **25**

In the next example we solve an equation that doesn't look like a linear equation, but it simplifies to one when we multiply by the LCD.

EXAMPLE 3 | An Equation Involving Fractional Expressions

Solve the equation $\frac{1}{x+1} + \frac{1}{x-2} = \frac{x+3}{x^2 - x - 2}$.

SOLUTION The LCD of the fractional expressions is $(x + 1)(x - 2) = x^2 - x - 2$. So as long as $x \neq -1$ and $x \neq 2$, we can multiply both sides of the equation by the LCD to get

$$(x+1)(x-2)\left(\frac{1}{x+1} + \frac{1}{x-2}\right) = (x+1)(x-2)\left(\frac{x+3}{x^2 - x - 2}\right) \qquad \text{Mulitiply by LCD}$$

$$(x - 2) + (x + 1) = x + 3 \qquad \text{Expand}$$

$$2x - 1 = x + 3 \qquad \text{Simplify}$$

$$x = 4 \qquad \text{Solve}$$

The solution is $x = 4$.

CHECK YOUR ANSWER

$x = 4$:

LHS $= \frac{1}{4+1} + \frac{1}{4-2} = \frac{1}{5} + \frac{1}{2} = \frac{7}{10}$

RHS $= \frac{4+3}{4^2 - 4 - 2} = \frac{7}{10}$

LHS = RHS ✓

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **47**

It is always important to check your answer, even if you never make a mistake in your calculations. This is because you sometimes end up with **extraneous solutions**, which are potential solutions that do not satisfy the original equation. The next example shows how this can happen.

EXAMPLE 4 | An Equation with No Solution

Solve the equation $2 + \dfrac{5}{x-4} = \dfrac{x+1}{x-4}$.

SOLUTION First, we multiply each side by the common denominator, which is $x - 4$:

$$(x-4)\left(2 + \frac{5}{x-4}\right) = (x-4)\left(\frac{x+1}{x-4}\right) \qquad \text{Multiply by } x-4$$

$$2(x-4) + 5 = x + 1 \qquad \text{Expand}$$

$$2x - 8 + 5 = x + 1 \qquad \text{Distributive Property}$$

$$2x - 3 = x + 1 \qquad \text{Simplify}$$

$$2x = x + 4 \qquad \text{Add 3}$$

$$x = 4 \qquad \text{Subtract } x$$

CHECK YOUR ANSWER

$x = 4$:

$$\text{LHS} = 2 + \frac{5}{4-4} = 2 + \frac{5}{0}$$

$$\text{RHS} = \frac{4+1}{4-4} = \frac{5}{0}$$

Impossible—can't divide by 0. LHS and RHS are undefined, so $x = 4$ is not a solution. ✗

But now if we try to substitute $x = 4$ back into the original equation, we would be dividing by 0, which is impossible. So this equation has *no solution*.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **49**

The first step in the preceding solution, multiplying by $x - 4$, had the effect of multiplying by 0. (Do you see why?) Multiplying each side of an equation by an expression that contains the variable may introduce extraneous solutions. That is why it is important to check every answer.

Solving Power Equations

Linear equations have variables only to the first power. Now let's consider some equations that involve squares, cubes, and other powers of the variable. Such equations will be studied more extensively in Sections 1.6 and 1.7. Here we just consider basic equations that can be simplified into the form $X^n = a$. Equations of this form are called **power equations** and can be solved by taking radicals of both sides of the equation.

"Algebra is a merry science," Uncle Jakob would say. "We go hunting for a little animal whose name we don't know, so we call it x. When we bag our game we pounce on it and give it its right name."

ALBERT EINSTEIN

SOLVING A POWER EQUATION

The power equation $X^n = a$ has the solution

$$X = \sqrt[n]{a} \qquad \text{if } n \text{ is odd}$$

$$X = \pm\sqrt[n]{a} \qquad \text{if } n \text{ is even and } a \geq 0$$

If n is even and $a < 0$, the equation has no real solution.

Here are some examples of solving power equations:

The equation $x^5 = 32$ has only one real solution: $x = \sqrt[5]{32} = 2$.

The equation $x^4 = 16$ has two real solutions: $x = \pm\sqrt[4]{16} = \pm 2$.

The equation $x^5 = -32$ has only one real solution: $x = \sqrt[5]{-32} = -2$.

The equation $x^4 = -16$ has no real solutions because $\sqrt[4]{-16}$ does not exist.

EXAMPLE 5 | Solving Power Equations

Solve each equation.

(a) $x^2 - 5 = 0$

(b) $(x - 4)^2 = 5$

SOLUTION

(a) $x^2 - 5 = 0$

$$x^2 = 5 \qquad \text{Add 5}$$

$$x = \pm\sqrt{5} \qquad \text{Take the square root}$$

The solutions are $x = \sqrt{5}$ and $x = -\sqrt{5}$.

(b) We can take the square root of each side of this equation as well:

$$(x - 4)^2 = 5$$

$$x - 4 = \pm\sqrt{5} \qquad \text{Take the square root}$$

$$x = 4 \pm \sqrt{5} \qquad \text{Add 4}$$

The solutions are $x = 4 + \sqrt{5}$ and $x = 4 - \sqrt{5}$.

Be sure to check that each answer satisfies the original equation.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **55** AND **61**

We will revisit equations like the ones in Example 5 in Section 1.6.

EXAMPLE 6 | Solving Power Equations

Find all real solutions for each equation.

(a) $x^3 = -8$

(b) $16x^4 = 81$

SOLUTION

(a) Since every real number has exactly one real cube root, we can solve this equation by taking the cube root of each side:

$$(x^3)^{1/3} = (-8)^{1/3}$$

$$x = -2$$

EUCLID (circa 300 B.C.) taught in Alexandria, Egypt. His *Elements* is the most widely influential scientific book in history. For 2000 years it was the standard introduction to geometry in the schools, and for many generations it was considered the best way to develop logical reasoning. Abraham Lincoln, for instance, studied the *Elements* as a way to sharpen his mind. The story is told that King Ptolemy once asked Euclid whether there was a faster way to learn geometry than through the *Elements*. Euclid replied that there is "no royal road to geometry"—meaning by this that mathematics does not respect wealth or social status. Euclid was revered in his own time and was referred to by the title "The Geometer" or "The Writer of the *Elements*." The greatness of the *Elements* stems from its precise, logical, and systematic treatment of geometry. For dealing with equality, Euclid lists the following rules, which he calls "common notions."

1. Things that are equal to the same thing are equal to each other.
2. If equals are added to equals, the sums are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things that coincide with one another are equal.
5. The whole is greater than the part.

(b) Here we must remember that if n is even, then every positive real number has *two* real nth roots, a positive one and a negative one:

If n is even, the equation $x^n = c$ $(c > 0)$ has two solutions, $x = c^{1/n}$ and $x = -c^{1/n}$.

$$\begin{aligned} x^4 &= \tfrac{81}{16} && \text{Divide by 16} \\ (x^4)^{1/4} &= \pm\left(\tfrac{81}{16}\right)^{1/4} && \text{Take the fourth root} \\ x &= \pm\tfrac{3}{2} \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **63** AND **65**

The next example shows how to solve an equation that involves a fractional power of the variable.

EXAMPLE 7 | Solving an Equation with a Fractional Power

Solve the equation $5x^{2/3} - 2 = 43$.

SOLUTION The idea is to first isolate the term with the fractional exponent, then raise both sides of the equation to the *reciprocal* of that exponent:

$$\begin{aligned} 5x^{2/3} - 2 &= 43 \\ 5x^{2/3} &= 45 && \text{Add 2} \\ x^{2/3} &= 9 && \text{Divide by 5} \\ x &= \pm 9^{3/2} && \text{Raise both sides to } \tfrac{3}{2} \text{ power} \\ x &= \pm 27 && \text{Simplify} \end{aligned}$$

If n is even, the equation $x^{n/m} = c$ has two solutions, $x = c^{m/n}$ and $x = -c^{m/n}$.

The solutions are $x = 27$ and $x = -27$.

CHECK YOUR ANSWERS

$x = 27$:

$$\begin{aligned} \text{LHS} &= 5(27)^{2/3} - 2 \\ &= 5(9) - 2 \\ &= 43 \\ \text{RHS} &= 43 \\ \text{LHS} &= \text{RHS} \quad ✓ \end{aligned}$$

$x = -27$:

$$\begin{aligned} \text{LHS} &= 5(-27)^{2/3} - 2 \\ &= 5(9) - 2 \\ &= 43 \\ \text{RHS} &= 43 \\ \text{LHS} &= \text{RHS} \quad ✓ \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **75**

Solving for One Variable in Terms of Others

Many formulas in the sciences involve several variables, and it is often necessary to express one of the variables in terms of the others. In the next example we solve for a variable in Newton's Law of Gravity.

EXAMPLE 8 | Solving for One Variable in Terms of Others

Solve for the variable M in the equation

This is Newton's Law of Gravity. It gives the gravitational force F between two masses m and M that are a distance r apart. The constant G is the universal gravitational constant.

$$F = G\frac{mM}{r^2}$$

SOLUTION Although this equation involves more than one variable, we solve it as usual by isolating M on one side and treating the other variables as we would numbers:

$$
\begin{aligned}
F &= \left(\frac{Gm}{r^2}\right)M && \text{Factor } M \text{ from RHS}\\
\left(\frac{r^2}{Gm}\right)F &= \left(\frac{r^2}{Gm}\right)\left(\frac{Gm}{r^2}\right)M && \text{Multiply by reciprocal of } \frac{Gm}{r^2}\\
\frac{r^2F}{Gm} &= M && \text{Simplify}
\end{aligned}
$$

The solution is $M = \dfrac{r^2F}{Gm}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **87**

EXAMPLE 9 | Solving for One Variable in Terms of Others

The surface area A of the closed rectangular box shown in Figure 1 can be calculated from the length l, the width w, and the height h according to the formula

$$A = 2lw + 2wh + 2lh$$

Solve for w in terms of the other variables in this equation.

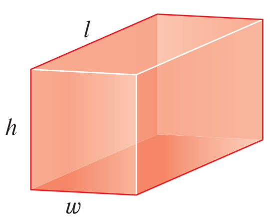


FIGURE 1 A closed rectangular box

SOLUTION Although this equation involves more than one variable, we solve it as usual by isolating w on one side, treating the other variables as we would numbers:

$$
\begin{aligned}
A &= (2lw + 2wh) + 2lh && \text{Collect terms involving } w\\
A - 2lh &= 2lw + 2wh && \text{Subtract } 2lh\\
A - 2lh &= (2l + 2h)w && \text{Factor } w \text{ from RHS}\\
\frac{A - 2lh}{2l + 2h} &= w && \text{Divide by } 2l + 2h
\end{aligned}
$$

The solution is $w = \dfrac{A - 2lh}{2l + 2h}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **89**

P.8 EXERCISES

CONCEPTS

1. Substituting $x = 3$ in the equation $4x - 2 = 10$ makes the equation true, so the number 3 is a ________ of the equation.

2. To solve an equation, we use the rules of algebra to put the variable alone on one side. Solve the equation $3x + 4 = 10$ by using the following steps:

$3x + 4 = 10$	Given equation
________	Subtract 4
________	Multiply by $\frac{1}{3}$

So the solution is $x =$ ________.

3. Which of the following equations are linear?

(a) $\dfrac{x}{2} + 2x = 10$ **(b)** $\dfrac{2}{x} - 2x = 1$

(c) $x + 7 = 5 - 3x$

4. Explain why each of the following equations is not linear.

(a) $x(x + 1) = 6$ **(b)** $\sqrt{x + 2} = x$

(c) $3x^2 - 2x - 1 = 0$

5. *True or false?*

(a) Adding the same number to each side of an equation always gives an equivalent equation.

(b) Multiplying each side of an equation by the same number always gives an equivalent equation.

(c) Squaring each side of an equation always gives an equivalent equation.

6. To solve the equation $x^3 = 125$, we take the ________ root of each side. So the solution is $x =$ ________.

SKILLS

7–14 ■ Determine whether the given value is a solution of the equation.

7. $4x + 7 = 9x - 3$
(a) $x = -2$ **(b)** $x = 2$

8. $2 - 5x = 8 + x$
(a) $x = -1$ **(b)** $x = 1$

9. $1 - [2 - (3 - x)] = 4x - (6 + x)$
(a) $x = 2$ **(b)** $x = 4$

10. $\dfrac{1}{x} - \dfrac{1}{x - 4} = 1$
(a) $x = 2$ **(b)** $x = 4$

11. $2x^{1/3} - 3 = 1$
(a) $x = -1$ **(b)** $x = 8$

12. $\dfrac{x^{3/2}}{x - 6} = x - 8$
(a) $x = 4$ **(b)** $x = 8$

13. $\dfrac{x - a}{x - b} = \dfrac{a}{b}$ $(b \neq 0)$
(a) $x = 0$ **(b)** $x = b$

14. $x^2 - bx + \dfrac{1}{4}b^2 = 0$
(a) $x = \dfrac{b}{2}$ **(b)** $x = \dfrac{1}{b}$

15–20 ■ Solve the given linear equation.

15. $3x + 7 = 0$
16. $12 - 5x = 0$
17. $7 - 2x = 15$
18. $4x - 95 = 1$
19. $\frac{1}{2}x + 7 = 3$
20. $2 + \frac{1}{3}x = -4$

21–52 ■ The given equation is either linear or equivalent to a linear equation. Solve the equation.

21. $x - 3 = 2x + 6$
22. $4x + 7 = 9x - 13$
23. $-7w = 15 - 2w$
24. $5t - 13 = 12 - 5t$
25. $\frac{1}{2}y - 2 = \frac{1}{3}y$
26. $\dfrac{z}{5} = \dfrac{3}{10}z + 7$
27. $2(1 - x) = 3(1 + 2x) + 5$
28. $5(x + 3) + 9 = -2(x - 2) - 1$
29. $4\left(y - \frac{1}{2}\right) - y = 6(5 - y)$
30. $r - 2[1 - 3(2r + 4)] = 61$
31. $x - \frac{1}{3}x - \frac{1}{2}x - 5 = 0$
32. $\dfrac{2}{3}y + \dfrac{1}{2}(y - 3) = \dfrac{y + 1}{4}$
33. $2x - \dfrac{x}{2} + \dfrac{x + 1}{4} = 6x$
34. $3x - \dfrac{5x}{2} = \dfrac{x + 1}{3} - \dfrac{1}{6}$
35. $(x - 1)(x + 2) = (x - 2)(x - 3)$
36. $x(x + 1) = (x + 3)^2$
37. $(x - 1)(4x + 5) = (2x - 3)^2$
38. $(t - 4)^2 = (t + 4)^2 + 32$
39. $\dfrac{1}{x} = \dfrac{4}{3x} + 1$
40. $\dfrac{2}{x} - 5 = \dfrac{6}{x} + 4$
41. $\dfrac{2x - 1}{x + 2} = \dfrac{4}{5}$
42. $\dfrac{2x - 7}{2x + 4} = \dfrac{2}{3}$
43. $\dfrac{2}{t + 6} = \dfrac{3}{t - 1}$
44. $\dfrac{6}{x - 3} = \dfrac{5}{x + 4}$
45. $\dfrac{3}{x + 1} - \dfrac{1}{2} = \dfrac{1}{3x + 3}$
46. $\dfrac{12x - 5}{6x + 3} = 2 - \dfrac{5}{x}$
47. $\dfrac{1}{z} - \dfrac{1}{2z} - \dfrac{1}{5z} = \dfrac{10}{z + 1}$
48. $\dfrac{1}{3 - t} + \dfrac{4}{3 + t} + \dfrac{15}{9 - t^2} = 0$
49. $\dfrac{x}{2x - 4} - 2 = \dfrac{1}{x - 2}$
50. $\dfrac{1}{x + 3} + \dfrac{5}{x^2 - 9} = \dfrac{2}{x - 3}$
51. $\dfrac{3}{x + 4} = \dfrac{1}{x} + \dfrac{6x + 12}{x^2 + 4x}$
52. $\dfrac{1}{x} - \dfrac{2}{2x + 1} = \dfrac{1}{2x^2 + x}$

53–76 ■ The given equation involves a power of the variable. Find all real solutions of the equation.

53. $x^2 = 49$
54. $x^2 = 18$
55. $x^2 - 24 = 0$
56. $x^2 - 7 = 0$
57. $8x^2 - 64 = 0$
58. $5x^2 - 125 = 0$
59. $x^2 + 16 = 0$
60. $6x^2 + 100 = 0$
61. $(x + 2)^2 = 4$
62. $3(x - 5)^2 = 15$
63. $x^3 = 27$
64. $x^5 + 32 = 0$
65. $x^4 - 16 = 0$
66. $64x^6 = 27$
67. $x^4 + 64 = 0$
68. $(x - 1)^3 + 8 = 0$
69. $(x + 2)^4 - 81 = 0$
70. $(x + 1)^4 + 16 = 0$
71. $3(x - 3)^3 = 375$
72. $4(x + 2)^5 = 1$
73. $\sqrt[3]{x} = 5$
74. $x^{4/3} - 16 = 0$
75. $2x^{5/3} + 64 = 0$
76. $6x^{2/3} - 216 = 0$

77–84 ■ Find the solution of the equation rounded to two decimals.

77. $3.02x + 1.48 = 10.92$
78. $8.36 - 0.95x = 9.97$
79. $2.15x - 4.63 = x + 1.19$
80. $3.95 - x = 2.32x + 2.00$
81. $3.16(x + 4.63) = 4.19(x - 7.24)$
82. $2.14(x - 4.06) = 2.27 - 0.11x$
83. $\dfrac{0.26x - 1.94}{3.03 - 2.44x} = 1.76$
84. $\dfrac{1.73x}{2.12 + x} = 1.51$

85–98 ■ Solve the equation for the indicated variable.

85. $r = \dfrac{12}{M}$; for M

86. $wd = rTH$; for T

87. $PV = nRT$; for R

88. $F = G\dfrac{mM}{r^2}$; for m

89. $P = 2l + 2w$; for w

90. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$; for R_1

91. $V = \frac{1}{3}\pi r^2 h$; for r

92. $F = G\dfrac{mM}{r^2}$; for r

93. $V = \frac{4}{3}\pi r^3$; for r

94. $a^2 + b^2 = c^2$; for b

95. $A = P\left(1 + \dfrac{i}{100}\right)^2$; for i

96. $a^2x + (a - 1) = (a + 1)x$; for x

97. $\dfrac{ax + b}{cx + d} = 2$; for x

98. $\dfrac{a + 1}{b} = \dfrac{a - 1}{b} + \dfrac{b + 1}{a}$; for a

APPLICATIONS

99. Shrinkage in Concrete Beams As concrete dries, it shrinks; the higher the water content, the greater the shrinkage. If a concrete beam has a water content of w kg/m^3, then it will shrink by a factor

$$S = \frac{0.032w - 2.5}{10{,}000}$$

where S is the fraction of the original beam length that disappears owing to shrinkage.

(a) A beam 12.025 m long is cast in concrete that contains 250 kg/m^3 water. What is the shrinkage factor S? How long will the beam be when it has dried?

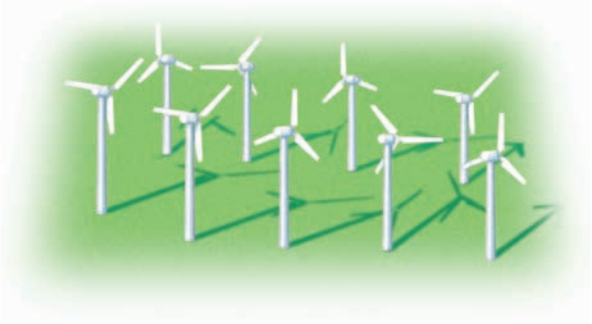
(b) A beam is 10.014 m long when wet. We want it to shrink to 10.009 m, so the shrinkage factor should be $S = 0.00050$. What water content will provide this amount of shrinkage?

100. Manufacturing Cost A toy maker finds that it costs $C = 450 + 3.75x$ dollars to manufacture x toy trucks. If the budget allows \$3600 in costs, how many trucks can be made?

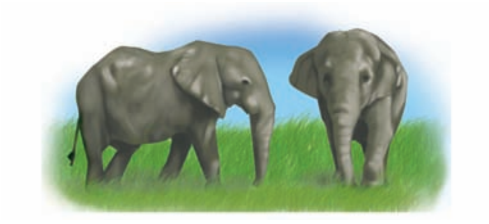
101. Power Produced by a Windmill When the wind blows with speed v km/h, a windmill with blade length 150 cm generates P watts (W) of power according to the formula $P = 15.6\,v^3$.

(a) How fast would the wind have to blow to generate 10,000 W of power?

(b) How fast would the wind have to blow to generate 50,000 W of power?



102. Food Consumption The average daily food consumption F of a herbivorous mammal with body weight x, where both F and x are measured in pounds, is given approximately by the equation $F = 0.3x^{3/4}$. Find the weight x of an elephant that consumes 300 lb of food per day.



DISCOVERY ■ DISCUSSION ■ WRITING

103. A Family of Equations The equation

$$3x + k - 5 = kx - k + 1$$

is really a **family of equations**, because for each value of k, we get a different equation with the unknown x. The letter k is called a **parameter** for this family. What value should we pick for k to make the given value of x a solution of the resulting equation?

(a) $x = 0$ **(b)** $x = 1$ **(c)** $x = 2$

104. Proof That 0 = 1? The following steps appear to give equivalent equations, which seem to prove that $1 = 0$. Find the error.

$x = 1$	Given
$x^2 = x$	Multiply by x
$x^2 - x = 0$	Subtract x
$x(x - 1) = 0$	Factor
$x = 0$	Divide by $x - 1$
$1 = 0$	Given $x = 1$

CHAPTER P | REVIEW

PROPERTIES AND FORMULAS

Properties of Real Numbers (p. 8)

Commutative: $a + b = b + a$
$ab = ba$

Associative: $(a + b) + c = a + (b + c)$
$(ab)c = a(bc)$

Distributive: $a(b + c) = ab + ac$

Absolute Value (p. 13)

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

$$|ab| = |a||b|$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

Distance between a and b is

$$d(a, b) = |b - a|$$

Exponents (p. 19)

$$a^m a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Radicals (p. 25)

$\sqrt[n]{a} = b$ means $b^n = a$

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

If n is odd, then $\sqrt[n]{a^n} = a$

If n is even, then $\sqrt[n]{a^n} = |a|$

$$a^{m/n} = \sqrt[n]{a^m}$$

Special Product Formulas (p. 34)

Sum and difference of same terms:

$$(A + B)(A - B) = A^2 - B^2$$

Square of a sum or difference:

$$(A + B)^2 = A^2 + 2AB + B^2$$

$$(A - B)^2 = A^2 - 2AB + B^2$$

Cube of a sum or difference:

$$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

$$(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$$

Special Factoring Formulas (p. 39)

Difference of squares:

$$A^2 - B^2 = (A + B)(A - B)$$

Perfect squares:

$$A^2 + 2AB + B^2 = (A + B)^2$$

$$A^2 - 2AB + B^2 = (A - B)^2$$

Sum or difference of cubes:

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

Rational Expressions (p. 44)

We can cancel common factors:

$$\frac{AC}{BC} = \frac{A}{B}$$

To multiply two fractions, we multiply their numerators together and their denominators together:

$$\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}$$

To divide fractions, we invert the divisor and multiply:

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C}$$

To add fractions, we find a common denominator:

$$\frac{A}{C} + \frac{B}{C} = \frac{A + B}{C}$$

Properties of Equality (p. 54)

$A = B \quad \Leftrightarrow \quad A + C = B + C$

$A = B \quad \Leftrightarrow \quad CA = CB \quad (C \neq 0)$

Linear Equations (p. 54)

A **linear equation** is an equation of the form $ax + b = 0$

Power Equations (p. 56)

A **power equation** is an equation of the form $X^n = a$. Its solutions are

$$X = \sqrt[n]{a} \quad \text{if } n \text{ is odd}$$

$$X = \pm\sqrt[n]{a} \quad \text{if } n \text{ is even}$$

If n is even and $a < 0$, the equation has no real solution.

LEARNING OBJECTIVES SUMMARY

Section	After completing this chapter, you should be able to . . .	Review Exercises
P.1	■ Use an algebra model	1–2
	■ Make an algebra model	1–2
P.2	■ Classify real numbers	3–4
	■ Use properties of real numbers	5–8
	■ Work with fractions	9–12
	■ Graph numbers on a number line	13–20
	■ Use the order symbols $<, \le, >, \ge$	17–20
	■ Work with set and interval notation	13–24
	■ Work with absolute values	25–28, 37–38
	■ Find distances on the real line	37–38
P.3	■ Use exponential notation	29–32, 43–50
	■ Simplify expressions using the Laws of Exponents	43–50
	■ Write numbers using scientific notation	51–54
P.4	■ Simplify expressions involving radicals	33–36, 39–42
	■ Simplify expressions involving rational exponents	46–49
	■ Express radicals using rational exponents	39–42, 47–48
	■ Rationalize a denominator and express a quotient of radicals in standard form	95–96
P.5	■ Add and subtract polynomials	73–79
	■ Multiply algebraic expressions	73–79
	■ Use the Special Product Formulas	74–79
P.6	■ Factor out common factors	55–56, 64–65
	■ Factor trinomials by trial and error	57–62
	■ Use the Special Factoring Formulas	63–67
	■ Factor algebraic expressions completely	55–72
	■ Factor by grouping terms	68–70
P.7	■ Find the domain of an algebraic expression	101–104
	■ Simplify rational expressions	80–93
	■ Add, subtract, multiply, and divide rational expressions	80–93
	■ Simplify compound fractions	91–92
	■ Rationalize a denominator or numerator	94–100
	■ Avoid common errors	105–110
P.8	■ Solve linear equations	111–122
	■ Solve power equations	123–132
	■ Solve for one variable in terms of others	133–136

EXERCISES

1–2 ■ Make and use an algebra model to solve the problem.

1. Elena regularly takes a multivitamin and mineral supplement. She purchases a bottle of 250 tablets and takes two tablets every day.
(a) Find a formula for the number of tablets T that are left in the bottle after she has been taking the tablets for x days.
(b) How many tablets are left after 30 days?
(c) How many days will it take for her to run out of tablets?

2. Alonzo's Delivery is having a sale on calzones. Each calzone costs \$2, and there is a \$3 delivery charge for phone-in orders.
(a) Find a formula for the total cost C of ordering x calzones for delivery.
(b) How much would it cost to have 4 calzones delivered?
(c) If you have \$15, how many calzones can you order?

3–4 ■ Determine whether each number is rational or irrational. If it is rational, determine whether it is a natural number, an integer, or neither.

3. **(a)** 16 **(b)** -16 **(c)** $\sqrt{16}$ **(d)** $\sqrt{2}$
(e) $\frac{8}{3}$ **(f)** $-\frac{8}{2}$

4. **(a)** -5 **(b)** $-\frac{25}{6}$ **(c)** $\sqrt{25}$ **(d)** 3π
(e) $\frac{24}{16}$ **(f)** 10^{20}

5–8 ■ State the property of real numbers being used.

5. $3 + 2x = 2x + 3$

6. $(a + b)(a - b) = (a - b)(a + b)$

7. $A(x + y) = Ax + Ay$

8. $(A + 1)(x + y) = (A + 1)x + (A + 1)y$

9–12 ■ Evaluate each expression. Express your answer as a fraction in lowest terms.

9. **(a)** $\dfrac{5}{6} + \dfrac{2}{3}$ **(b)** $\dfrac{5}{6} - \dfrac{2}{3}$

10. **(a)** $\dfrac{7}{10} - \dfrac{11}{15}$ **(b)** $\dfrac{7}{10} + \dfrac{11}{15}$

11. **(a)** $\dfrac{15}{8} \cdot \dfrac{12}{5}$ **(b)** $\dfrac{15}{8} \div \dfrac{12}{5}$

12. **(a)** $\dfrac{30}{7} \div \dfrac{12}{35}$ **(b)** $\dfrac{30}{7} \cdot \dfrac{12}{35}$

13–16 ■ Express the interval in terms of inequalities, and then graph the interval.

13. $[-2, 6)$ **14.** $(0, 10]$

15. $(-\infty, 4]$ **16.** $[-2, \infty)$

17–20 ■ Express the inequality in interval notation, and then graph the corresponding interval.

17. $x \ge 5$ **18.** $x < -3$

19. $-1 < x \le 5$ **20.** $0 \le x \le \frac{1}{2}$

21–24 ■ The sets A, B, C, and D are defined as follows:

$A = \{-1, 0, 1, 2, 3\}$ $B = \{\frac{1}{2}, 1, 4\}$
$C = \{x \mid 0 < x \le 2\}$ $D = (-1, 1]$

Find each of the following sets.

21. **(a)** $A \cup B$ **(b)** $A \cap B$

22. **(a)** $C \cup D$ **(b)** $C \cap D$

23. **(a)** $A \cap C$ **(b)** $B \cap D$

24. **(a)** $A \cap D$ **(b)** $B \cap C$

25–36 ■ Evaluate the expression.

25. $|7 - 10|$ **26.** $|-\frac{3}{2} - 5|$

27. $|3 - |-9||$ **28.** $1 - |1 - |-1||$

29. $2^{-3} - 3^{-2}$ **30.** $2^{1/2}8^{1/2}$

31. $216^{-1/3}$ **32.** $64^{2/3}$

33. $\dfrac{\sqrt{242}}{\sqrt{2}}$ **34.** $\sqrt[4]{4}\,\sqrt[4]{324}$

35. $\sqrt[3]{-125}$ **36.** $\sqrt{2}\sqrt{50}$

37–38 ■ Express the distance between the given numbers on the real line using an absolute value. Then evaluate this distance.

37. **(a)** 3 and 5 **(b)** 3 and -5

38. **(a)** 0 and -4 **(b)** 4 and -4

39–42 ■ Express the radical as a power with a rational exponent.

39. **(a)** $\sqrt[3]{7}$ **(b)** $\sqrt[5]{7^4}$ **40.** **(a)** $\sqrt[3]{5^7}$ **(b)** $(\sqrt[4]{5})^3$

41. **(a)** $\sqrt[6]{x^5}$ **(b)** $(\sqrt{x})^9$ **42.** **(a)** $\sqrt{y^3}$ **(b)** $(\sqrt[8]{y})^2$

43–50 ■ Simplify the expression.

43. $(2x^3y)^2(3x^{-1}y^2)$ **44.** $(a^2)^{-3}(a^3b)^2(b^3)^4$

45. $\dfrac{x^4(3x)^2}{x^3}$ **46.** $\left(\dfrac{r^2s^{4/3}}{r^{1/3}s}\right)^6$

47. $\sqrt[3]{(x^3y)^2y^4}$ **48.** $\sqrt{x^2y^4}$

49. $\dfrac{8r^{1/2}s^{-3}}{2r^{-2}s^4}$ **50.** $\left(\dfrac{ab^2c^{-3}}{2a^2b^{-4}}\right)^{-2}$

51. Write the number 78,250,000,000 in scientific notation.

52. Write the number 2.08×10^{-8} in decimal notation.

53. If $a \approx 0.00000293$, $b \approx 1.582 \times 10^{-14}$, and $c \approx 2.8064 \times 10^{12}$, use a calculator to approximate the number ab/c.

54. If your heart beats 80 times per minute and you live to be 90 years old, estimate the number of times your heart beats during your lifetime. State your answer in scientific notation.

55–72 ■ Factor the expression completely.

55. $2x^2y - 6xy^2$ **56.** $12x^2y^4 - 3xy^5 + 9x^3y^2$

57. $x^2 - 9x + 18$ **58.** $x^2 + 3x - 10$

59. $3x^2 - 2x - 1$ **60.** $6x^2 + x - 12$

61. $4t^2 - 13t - 12$ **62.** $x^4 - 2x^2 + 1$

63. $25 - 16t^2$ **64.** $2y^6 - 32y^2$

65. $a^4b^2 + ab^5$ **66.** $x^6 - 1$

67. $8x^3 + y^6$ **68.** $y^3 - 2y^2 - y + 2$

69. $4x^3 - 8x^2 + 3x - 6$ **70.** $3x^3 - 2x^2 + 18x - 12$

71. $(x + 1)^2 - 2(x + 1) + 1$ **72.** $(a + b)^2 + 2(a + b) - 15$

73–94 ■ Perform the indicated operations.

73. $(2x + 1)(3x - 2) - 5(4x - 1)$

74. $(2y - 7)(2y + 7)$

75. $(2a^2 - b)^2$

76. $(1 + x)(2 - x) - (3 - x)(3 + x)$

77. $(2x + 1)^3$ **78.** $x^3(x - 6)^2 + x^4(x - 6)$

79. $x^2(x - 2) + x(x - 2)^2$ **80.** $\dfrac{x^3 + 2x^2 + 3x}{x}$

81. $\dfrac{x^2 - 2x - 3}{2x^2 + 5x + 3}$

82. $\dfrac{t^3 - 1}{t^2 - 1}$

83. $\dfrac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \dfrac{3x + 12}{x - 1}$

84. $\dfrac{x^3/(x - 1)}{x^2/(x^3 - 1)}$

85. $\dfrac{x^2 - 2x - 15}{x^2 - 6x + 5} \div \dfrac{x^2 - x - 12}{x^2 - 1}$

86. $x - \dfrac{1}{x + 1}$

87. $\dfrac{1}{x - 1} - \dfrac{x}{x^2 + 1}$

88. $\dfrac{2}{x} + \dfrac{1}{x - 2} + \dfrac{3}{(x - 2)^2}$

89. $\dfrac{1}{x - 1} - \dfrac{2}{x^2 - 1}$

90. $\dfrac{1}{x + 2} + \dfrac{1}{x^2 - 4} - \dfrac{2}{x^2 - x - 2}$

91. $\dfrac{\dfrac{1}{x} - \dfrac{1}{2}}{x - 2}$

92. $\dfrac{\dfrac{1}{x} - \dfrac{1}{x + 1}}{\dfrac{1}{x} + \dfrac{1}{x + 1}}$

93. $\dfrac{3(x + h)^2 - 5(x + h) - (3x^2 - 5x)}{h}$

94. $\dfrac{\sqrt{x + h} - \sqrt{x}}{h}$ (rationalize the numerator)

95–100 ■ Rationalize the denominator, and simplify.

95. $\dfrac{1}{\sqrt{7}}$

96. $\dfrac{3}{\sqrt{6}}$

97. $\dfrac{12}{\sqrt{3} - 1}$

98. $\dfrac{14}{3 - \sqrt{2}}$

99. $\dfrac{x}{2 + \sqrt{x}}$

100. $\dfrac{\sqrt{x} + 1}{\sqrt{x} - 1}$

101–104 ■ Find the domain of the algebraic expression.

101. $\dfrac{x + 5}{x + 10}$

102. $\dfrac{2x}{x^2 - 9}$

103. $\dfrac{\sqrt{x}}{x^2 - 3x - 4}$

104. $\dfrac{\sqrt{x - 3}}{x^2 - 4x + 4}$

105–110 ■ State whether the given equation is true for all values of the variables. (Disregard any value that makes a denominator 0.)

105. $(x + y)^3 = x^3 + y^3$

106. $\dfrac{1 + \sqrt{a}}{1 - a} = \dfrac{1}{1 - \sqrt{a}}$

107. $\dfrac{12 + y}{y} = \dfrac{12}{y} + 1$

108. $\sqrt[3]{a + b} = \sqrt[3]{a} + \sqrt[3]{b}$

109. $\sqrt{a^2} = a$

110. $\dfrac{1}{x + 4} = \dfrac{1}{x} + \dfrac{1}{4}$

111–132 ■ Find all real solutions of the equation.

111. $3x + 12 = 24$

112. $5x - 7 = 42$

113. $7x - 6 = 4x + 9$

114. $8 - 2x = 14 + x$

115. $\frac{1}{3}x - \frac{1}{2} = 2$

116. $\frac{2}{3}x + \frac{3}{5} = \frac{1}{5} - 2x$

117. $2(x + 3) - 4(x - 5) = 8 - 5x$

118. $\dfrac{x - 5}{2} - \dfrac{2x + 5}{3} = \dfrac{5}{6}$

119. $\dfrac{x + 1}{x - 1} = \dfrac{2x - 1}{2x + 1}$

120. $\dfrac{x}{x + 2} - 3 = \dfrac{1}{x + 2}$

121. $\dfrac{x + 1}{x - 1} = \dfrac{3x}{3x - 6}$

122. $(x + 2)^2 = (x - 4)^2$

123. $x^2 = 144$

124. $4x^2 = 49$

125. $x^3 - 27 = 0$

126. $6x^4 + 15 = 0$

127. $(x + 1)^3 = -64$

128. $(x + 2)^2 - 2 = 0$

129. $\sqrt[3]{x} = -3$

130. $x^{2/3} - 4 = 0$

131. $4x^{3/4} - 500 = 0$

132. $(x - 2)^{1/5} = 2$

133–136 ■ Solve the equation for the indicated variable.

133. $A = \dfrac{x + y}{2}$; solve for x

134. $V = xy + yz + xz$; solve for y

135. $J = \dfrac{1}{t} + \dfrac{1}{2t} + \dfrac{1}{3t}$; solve for t

136. $F = k\dfrac{q_1 q_2}{r^2}$; solve for r

CHAPTER P TEST

1. A pizzeria charges \$9 for a medium plain cheese pizza plus \$1.50 for each extra topping.
   (a) Find a formula that models the cost C of a medium pizza with x toppings.
   (b) Use your model from part (a) to find the cost of a medium pizza with the following extra toppings: anchovies, ham, sausage, and pineapple.

2. Determine whether each number is rational or irrational. If it is rational, determine whether it is a natural number, an integer, or neither.
   (a) 5 **(b)** $\sqrt{5}$ **(c)** $-\frac{9}{3}$ **(d)** $-1{,}000{,}000$

3. Let $A = \{-2, 0, 1, 3, 5\}$ and $B = \{0, \frac{1}{2}, 1, 5, 7\}$. Find each of the following sets.
   (a) $A \cap B$ **(b)** $A \cup B$

4. **(a)** Graph the intervals $[-4, 2)$ and $[0, 3]$ on a real line.
   (b) Find the intersection and the union of the intervals in part (a), and graph each of them on a real line.
   (c) Use an absolute value to express the distance between -4 and 2 on the real line, and then evaluate this distance.

5. Evaluate each expression:
   (a) -2^6 **(b)** $(-2)^6$ **(c)** 2^{-6} **(d)** $\dfrac{7^{10}}{7^{12}}$
   (e) $\left(\dfrac{3}{2}\right)^{-2}$ **(f)** $\dfrac{\sqrt[5]{32}}{\sqrt{16}}$ **(g)** $\sqrt[4]{\dfrac{3^8}{2^{16}}}$ **(h)** $81^{-3/4}$

6. Write each number in scientific notation.
   (a) 186,000,000,000 **(b)** 0.0000003965

7. Simplify each expression. Write your final answer without negative exponents.
   (a) $\dfrac{a^2 b}{a^{-1} b^5}$ **(b)** $(3x^2 y^{1/2} x^{-2})^3$
   (c) $(3a^3b^3)(4ab^2)^2$ **(d)** $\sqrt{200} - \sqrt{32}$
   (e) $\sqrt{48x^4y^5}$ **(f)** $\sqrt[3]{\dfrac{125}{x^{-9}}}$
   (g) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$

8. Perform the indicated operations, and simplify.
   (a) $3(x + 6) + 4(2x - 5)$ **(b)** $(x + 3)(4x - 5)$ **(c)** $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$
   (d) $(2x + 3)^2$ **(e)** $(x + 2)^3$ **(f)** $x^2(x - 3)(x + 3)$

9. Factor each expression completely.
   (a) $4x^2 - 25$ **(b)** $2x^2 + 5x - 12$ **(c)** $x^3 - 3x^2 - 4x + 12$
   (d) $x^4 + 27x$ **(e)** $(2x - y)^2 - 10(2x - y) + 25$ **(f)** $x^3y - 4xy$

10. Simplify the rational expression.
   (a) $\dfrac{x^2 + 3x + 2}{x^2 - x - 2}$ **(b)** $\dfrac{2x^2 - x - 1}{x^2 - 9} \cdot \dfrac{x + 3}{2x + 1}$
   (c) $\dfrac{x^2}{x^2 - 4} - \dfrac{x + 1}{x + 2}$ **(d)** $\dfrac{\dfrac{y}{x} - \dfrac{x}{y}}{\dfrac{1}{y} - \dfrac{1}{x}}$

11. Rationalize the denominator, and simplify.
   (a) $\dfrac{6}{\sqrt[3]{4}}$ **(b)** $\dfrac{\sqrt{6}}{2 + \sqrt{3}}$

12. Find all real solutions of each equation.

(a) $4x - 3 = 2x + 7$ **(b)** $8x^3 = -125$

(c) $x^{2/3} - 64 = 0$ **(d)** $\dfrac{x}{2x - 5} = \dfrac{x + 3}{2x - 1}$

(e) $3(x + 1)^2 - 18 = 0$

13. Einstein's famous equation $E = mc^2$ gives the relationship between energy E and mass m. In this equation c represents the speed of light. Solve the equation to express c in terms of E and m.

Making the Best Decisions

When you buy a car, subscribe to a cell phone plan, or put an addition on your house, you need to make decisions. Such decisions are usually difficult because they require you to choose between several good alternatives. For example, there are many good car models, but which one has the optimal combination of features for the amount of money you want to spend? In this *Focus* we explore how to construct and use algebraic models of real-life situations to help make the best (or optimal) decisions.

EXAMPLE 1 | Buying a Car

Ben wants to buy a new car, and he has narrowed his choices to two models.

Model A sells for \$12,500, gets 25 mi/gal, and costs \$350 a year for insurance.

Model B sells for \$21,000, gets 48 mi/gal, and costs \$425 a year for insurance.

Ben drives about 36,000 miles a year, and gas costs about \$4.00 a gallon.

(a) Find a formula for the total cost of owning Model A for any number of years.

(b) Find a formula for the total cost of owning Model B for any number of years.

(c) Make a table of the total cost of owning each model from 1 year to 6 years, in 1-year increments.

(d) If Ben expects to keep the car for 3 years, which model is more economical? What if he expects to keep it for 5 years?

THINKING ABOUT THE PROBLEM

Model A has a smaller initial price and costs less in insurance per year but is more costly to operate (uses more gas) than Model B. Model B has a larger initial price and costs more to insure but is cheaper to operate (uses less gas) than Model A. If Ben drives a lot, then what he will save in gas with Model B could make up for the initial cost of buying the car and the higher yearly insurance premiums. So how many years of driving does it take before the gas savings make up for the initial higher price? To find out, we must write formulas for the total cost for each car:

$$\text{cost} = \text{price} + \text{insurance cost} + \text{gas cost}$$

The insurance costs and gas costs depend on the number of years Ben drives the car.

SOLUTION The cost of operating each model depends on the number of years of ownership. So let

$$n = \text{number of years Ben expects to own the car}$$

(a) For Model A we have the following:

In Words	In Algebra
Price of car	12,500
Insurance cost for n years	$350n$
Cost of gas per year	$(36{,}000/25) \times \$4.00 = \$5760$
Cost of gas for n years	$5760n$

Let C represent the cost of owning model A for n years. Then

$$\boxed{\text{cost of ownership}} = \boxed{\text{initial cost}} + \boxed{\text{insurance cost}} + \boxed{\text{gas cost}}$$

$$C = 12{,}500 + 350n + 5760n$$

$$C = 12{,}500 + 5760n$$

(b) For Model B we have the following:

In Words	**In Algebra**
Price of car	21,000
Insurance cost for n years	$425n$
Cost of gas per year	$(36{,}000/48) \times \$4.00 = \$3000$
Cost of gas for n years	$3000n$

Let C represent the cost of owning model B for n years. Then

cost of ownership	=	initial cost	+	insurance cost	+	gas cost

$$C = 21{,}000 + 425n + 3000n$$

$$C = 21{,}000 + 3425n$$

(c) If Ben keeps the car for 2 years, the cost of ownership can be calculated from the formulas we found by substituting 2 for n:

For Model A: $C = 12{,}500 + 5760(2) = 24{,}020$

For Model B: $C = 21{,}000 + 3425(2) = 27{,}850$

The other entries in the table are calculated similarly:

Years	Cost of ownership Model A	Cost of ownership Model B
1	18,260	24,425
2	24,020	27,850
3	29,780	31,275
4	35,540	34,700
5	41,300	38,125
6	47,060	41,550

(d) If Ben intends to keep the car 3 years, then Model A is a better buy (see the table), but if he intends to keep the car 5 years, Model B is the better buy. ■

EXAMPLE 2 | Equal Ownership Cost

Find the number of years of ownership for which the cost to Ben (from Example 1) of owning Model A equals the cost of owning Model B.

THINKING ABOUT THE PROBLEM

We see from the table that the cost of owning Model A starts lower but then exceeds that for Model B. We want to find the value of n for which the two costs are equal.

SOLUTION We equate the cost of owning Model A to that of Model B and solve for n.

$$12{,}500 + 5760n = 21{,}000 + 3425n \quad \text{Set the two costs equal}$$

$$2335n = 8500 \quad \text{Subtract 12,500 and } 3425n$$

$$n \approx 3.64 \quad \text{Divide by 2335}$$

If Ben keeps the car for about 3.64 years, the cost of owning *either model* would be the same. ■

EXAMPLE 3 | Dividing Assets Fairly

When high-tech Company A goes bankrupt, it owes \$120 million to Company B and \$480 million to Company C. Unfortunately, Company A has only \$300 million in assets. How should the court divide these assets between Companies B and C? Explore the following methods, and determine which are fair.

(a) Companies B and C divide the assets equally.

(b) The two companies share the losses equally.

(c) The two companies get an amount that is proportional to the amount they are owed.

THINKING ABOUT THE PROBLEM

It might seem fair for Companies B and C to divide the assets equally between them. Or it might seem fair that they share the loss equally between them. To be certain of the fairness of each plan, we should calculate how much each company loses under each plan.

SOLUTION

(a) Under this method, Company B gets \$150 million and Company C gets \$150 million. Because B is owed only \$120 million, it will get \$30 million more than it is owed. This doesn't seem fair to C, which will still lose \$330 million.

(b) We want Companies B and C to each lose the same amount. Let x be the amount of money Company B gets. Then Company C would get the rest $(300 - x)$. We can organize the information as follows.

In Words	In Algebra
Amount B gets	x
Amount C gets	$300 - x$
Amount B loses	$120 - x$
Amount C loses	$480 - (300 - x) = 180 + x$

Because we want Companies B and C to lose equal amounts, we must have

$$180 + x = 120 - x \qquad \text{Amounts B and C lose are equal}$$

$$2x = -60 \qquad \text{Add } x\text{, subtract 180}$$

$$x = -30 \qquad \text{Divide by 2}$$

Thus Company B gets -30 million dollars. The negative sign means that B must give up an additional \$30 million and pay it to C. So Company C gets all of the \$300 million plus \$30 million from B for a total of \$330 million. Doing this would ensure that the two companies lose the same amount (see *Check Your Answer*). This method is clearly not fair.

CHECK YOUR ANSWER

B loses $120 + 30 = 150$ million.

C loses $480 - 330 = 150$ million.

They lose equal amounts. ✓

(c) The claims total \$120 million + \$480 million = \$600 million. The assets total \$300 million. Because Company B is owed \$120 million out of the total claim of \$600 million, it would get

$$\frac{120 \text{ million}}{600 \text{ million}} \times 300 \text{ million} = \$60 \text{ million}$$

Because Company C is owed 480 million, it would get

$$\frac{480 \text{ million}}{600 \text{ million}} \times 300 \text{ million} = \$240 \text{ million}$$

This seems like the fairest alternative. ■

PROBLEMS

1. Renting Versus Buying a Photocopier A certain office can purchase a photocopier for \$5800 with a maintenance fee of \$25 a month. On the other hand, they can rent the photocopier for \$95 a month (including maintenance). If they purchase the photocopier, each copy would cost 3¢; if they rent, the cost is 6¢ per copy. The office estimates that they make 8000 copies a month.

(a) Find a formula for the cost C of purchasing and using the copier for n months.

(b) Find a formula for the cost C of renting and using the copier for n months.

(c) Make a table of the cost of each method for 1 year to 6 years of use, in 1-year increments.

(d) After how many months of use would the cost be the same for each method?

2. Car Rental A businessman intends to rent a car for a 3-day business trip. The rental is \$65 a day and 15¢ per mile (Plan 1) or \$90 a day with unlimited mileage (Plan 2). He is not sure how many miles he will drive but estimates that it will be between 400 and 800 miles.

(a) For each plan, find a formula for the cost C in terms of the number x of miles driven.

(b) Which rental plan is cheaper if the businessman drives 400 miles? 800 miles?

(c) At what mileage do the two plans cost the same?

3. Cost and Revenue A tire company determines that to manufacture a certain type of tire, it costs \$8000 to set up the production process. Each tire that is produced costs \$22 in material and labor. The company sells this tire to wholesale distributors for \$49 each.

(a) Find a formula for the total cost C of producing x tires.

(b) Find a formula for the revenue R from selling x tires.

profit = revenue − cost

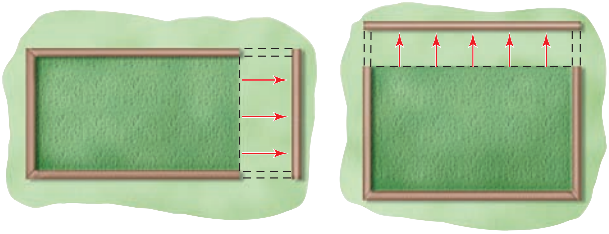
(c) Find a formula for the profit P from selling x tires.

(d) How many tires must the company sell to break even?

4. Enlarging a Field A farmer has a rectangular cow pasture with width 100 ft and length 180 ft. An increase in the number of cows requires the farmer to increase the area of her pasture. She has two options:

Option 1: Increase the length of the field.

Option 2: Increase the width of the field.



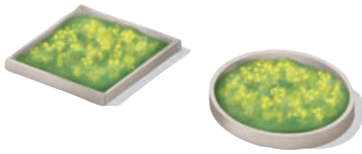
It costs \$10 per foot to install new fence. Moving the old fence costs \$6 per linear foot of fence to be moved.

(a) For each option, find a formula for A, the area gained, in terms of the cost C.

(b) Complete the table for the area gained in terms of the cost for each option.

Cost	Area gain (Option 1)	Area gain (Option 2)
\$1100	2500 ft^2	180 ft^2
\$1200		
\$1500		
\$2000		
\$2500		
\$3000		

(c) If the farmer has \$1200 for this project, which option gives her the greatest gain in area for her money? What if she had \$2000 for the project?



5. Edging a Planter A woman wants to make a small planter and surround it with edging material. She is deciding between two designs.

Design 1: A square planter

Design 2: A circular planter

Edging material costs \$3 a foot for the straight variety, which she would use for Design 1, and \$4 a foot for the flexible variety, which she would use for Design 2.

(a) If she decides on a perimeter of 24 ft, which design would give her the larger planting area?

(b) If she decides to spend \$120 on edging material, which design would give her the larger planting area?

6. Planting Crops A farmer is considering two plans of crop rotation on his 100-acre farm.

Plan A: Plant tomatoes every season.

Plan B: Alternate between soybeans and tomatoes each season.

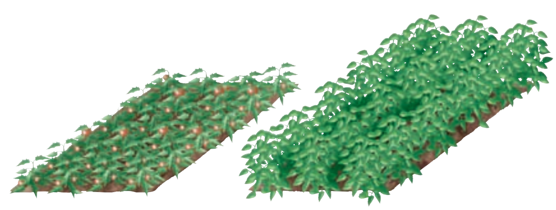
The revenue from tomatoes is \$1600 an acre, and the revenue from soybeans is \$1200 an acre. Tomatoes require fertilizing the land, which costs about \$300 an acre. Soybeans do not require fertilizer; moreover, they add nitrogen to the soil so tomatoes can be planted the following season without fertilizing.

profit = revenue − cost

(a) Find a formula for the profit if Plan A is used for n years.

(b) Find a formula for the profit if Plan B is used for $2n$ years (starting with soybeans).

(c) If the farmer intends to plant these crops for 10 years, which plan is more profitable?



7. Cell Phone Plan Genevieve is mulling over the three cell phone plans shown in the table.

	Minutes included	Monthly cost	Each additional minute
Plan A	500	\$30	\$0.50
Plan B	500	\$40	\$0.30
Plan C	500	\$60	\$0.10

From past experience, Genevieve knows that she will always use more than 500 minutes of cell phone time every month.

(a) Make a table of values that shows the cost of each plan for 500 to 1100 minutes, in 100-minute increments.

(b) Find formulas that give Genevieve's monthly cost for each plan, assuming that she uses x minutes per month (where $x \geq 500$).

(c) What is the charge from each plan when Genevieve uses 550 minutes? 975 minutes? 1200 minutes?

(d) Use your formulas from part (b) to determine the number of usage minutes for which:

(i) Plan A and Plan B give the same cost.

(ii) Plan A and Plan C give the same cost.

8. Profit Sharing To form a new enterprise, Company A invests \$1.4 million and Company B invests \$2.6 million. The enterprise is sold a year later for \$6.4 million. Explore the following methods of dividing the \$6.4 million, and comment on their fairness.

(a) Companies A and B divide the \$6.4 million equally.

(b) Companies A and B get their original investment back and share the profit equally.

(c) Each company gets a fraction of the \$6.4 million proportional to the amount it invested.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CHAPTER 1

EQUATIONS AND GRAPHS

Planning for the Future Governments and businesses are continually planning for the future. Will our freeways be able to handle the traffic ten years from now? How many air-conditioning units should a manufacturer produce for next summer? What will the average global temperature be 20 or 50 years from now? Predicting the future is an uncertain task, but we can give reasonable answers to such questions by examining the relevant available data. A graph of the data may reveal long-term trends that we can use to predict future conditions. For example, available data show a warming trend in global temperature, and significant global warming could have drastic consequences for the survival of many species, including the king penguins pictured here.

In this chapter we begin by reviewing the process of graphing two-variable equations in the coordinate plane. To obtain information from these graphs, we need to solve equations. So we also study the solutions of quadratic and other types of equations. In the *Focus on Modeling* at the end of the chapter, we learn how to find linear trends in data and how to use these trends to make predictions about the future.

1.1 The Coordinate Plane

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Graph points and regions in the coordinate plane ▶ Use the Distance Formula ▶ Use the Midpoint Formula

The *coordinate plane* is the link between algebra and geometry. In the coordinate plane we can draw graphs of algebraic equations. The graphs, in turn, allow us to "see" the relationship between the variables in the equation.

▼ The Coordinate Plane

The Cartesian plane is named in honor of the French mathematician René Descartes (1596–1650), although another Frenchman, Pierre Fermat (1601–1665), also invented the principles of coordinate geometry at the same time. (See their biographies on pages 213 and 107.)

Just as points on a line can be identified with real numbers to form the coordinate line, points in a plane can be identified with ordered pairs of numbers to form the **coordinate plane** or **Cartesian plane**. To do this, we draw two perpendicular real lines that intersect at 0 on each line. Usually, one line is horizontal with positive direction to the right and is called the ***x*-axis**; the other line is vertical with positive direction upward and is called the ***y*-axis**. The point of intersection of the x-axis and the y-axis is the **origin *O***, and the two axes divide the plane into four **quadrants**, labeled I, II, III, and IV in Figure 1. (The points *on* the coordinate axes are not assigned to any quadrant.)

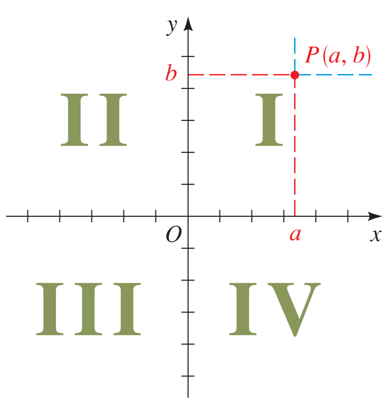


FIGURE 1

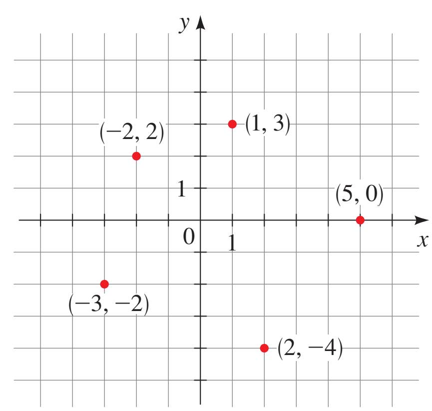


FIGURE 2

Although the notation for a point (a, b) is the same as the notation for an open interval (a, b), the context should make clear which meaning is intended.

Any point P in the coordinate plane can be located by a unique **ordered pair** of numbers (a, b), as shown in Figure 1. The first number a is called the ***x*-coordinate** of P; the second number b is called the ***y*-coordinate** of P. We can think of the coordinates of P as its "address," because they specify its location in the plane. Several points are labeled with their coordinates in Figure 2.

EXAMPLE 1 | Graphing Regions in the Coordinate Plane

Describe and sketch the regions given by each set.

(a) $\{(x, y) \mid x \geq 0\}$ **(b)** $\{(x, y) \mid y = 1\}$ **(c)** $\{(x, y) \mid -1 < y < 1\}$

SOLUTION

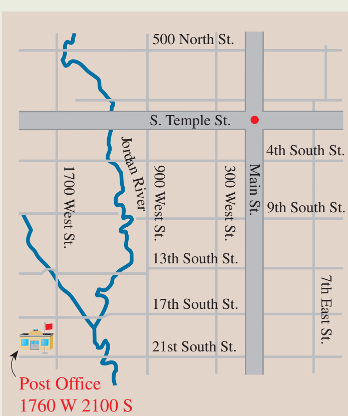
(a) The points whose x-coordinates are 0 or positive lie on the y-axis or to the right of it, as shown in Figure 3(a).

(b) The set of all points with y-coordinate 1 is a horizontal line one unit above the x-axis, as in Figure 3(b).

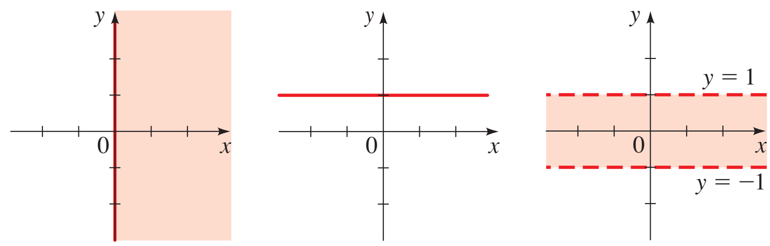
Coordinates as Addresses
The coordinates of a point in the *xy*-plane uniquely determine its location. We can think of the coordinates as the "address" of the point. In Salt Lake City, Utah, the addresses of most buildings are in fact expressed as coordinates. The city is divided into quadrants with Main Street as the vertical (north-south) axis and S. Temple Street as the horizontal (east-west) axis. An address such as

1760 W 2100 S

indicates a location 17.6 blocks west of Main Street and 21 blocks south of S. Temple Street. (This is the address of the main post office in Salt Lake City.) With this logical system it is possible for someone who is unfamiliar with the city to locate any address immediately, as easily as one locates a point in the coordinate plane.



(c) The given region consists of those points in the plane whose y-coordinates lie between -1 and 1. Thus the region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$. These lines are shown as broken lines in Figure 3(c) to indicate that the points on these lines do not lie in the set.



(a) $x \geq 0$ (b) $y = 1$ (c) $-1 < y < 1$

FIGURE 3

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **9** AND **11**

The Distance Formula

We now find a formula for the distance $d(A, B)$ between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane. Recall from Section P.2 that the distance between points a and b on a number line is $d(a, b) = |b - a|$. So from Figure 4 we see that the distance between the points $A(x_1, y_1)$ and $C(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$, and the distance between $B(x_2, y_2)$ and $C(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$.

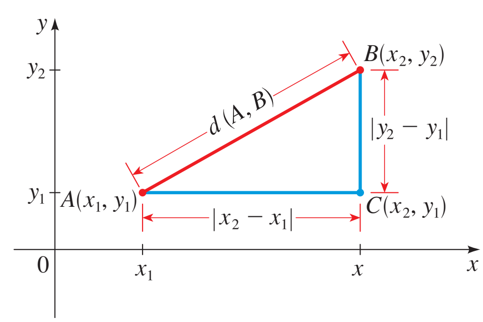


FIGURE 4

Since triangle ABC is a right triangle, the Pythagorean Theorem gives

$$d(A, B) = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

DISTANCE FORMULA

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 2 | Finding the Distance Between Two Points

Find the distance between the points $A(2, 5)$ and $B(4, -1)$.

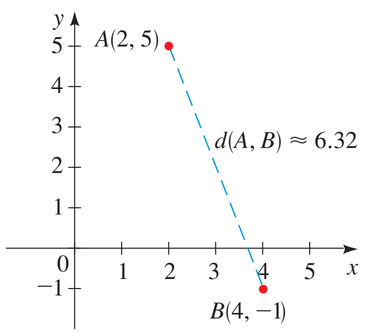


FIGURE 5

SOLUTION Using the Distance Formula, we have

$$\begin{aligned} d(A, B) &= \sqrt{(4-2)^2 + (-1-5)^2} \\ &= \sqrt{2^2 + (-6)^2} \\ &= \sqrt{4 + 36} = \sqrt{40} \approx 6.32 \end{aligned}$$

See Figure 5.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **27(b)**

EXAMPLE 3 | Applying the Distance Formula

Which of the points $P(1, -2)$ or $Q(8, 9)$ is closer to the point $A(5, 3)$?

SOLUTION By the Distance Formula we have

$$d(P, A) = \sqrt{(5-1)^2 + [3-(-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$

$$d(Q, A) = \sqrt{(5-8)^2 + (3-9)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45}$$

This shows that $d(P, A) < d(Q, A)$, so P is closer to A (see Figure 6).

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41**

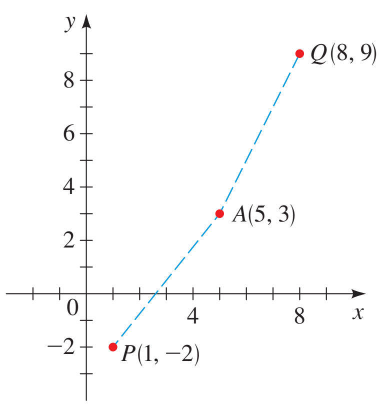


FIGURE 6

The Midpoint Formula

Now let's find the coordinates (x, y) of the midpoint M of the line segment that joins the point $A(x_1, y_1)$ to the point $B(x_2, y_2)$. In Figure 7, notice that triangles APM and MQB are congruent because $d(A, M) = d(M, B)$ and the corresponding angles are equal.

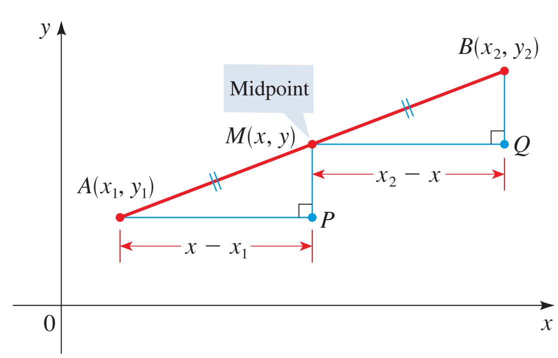


FIGURE 7

It follows that $d(A, P) = d(M, Q)$, so

$$x - x_1 = x_2 - x$$

Solving for x, we get $2x = x_1 + x_2$, so $x = \dfrac{x_1 + x_2}{2}$. Similarly, $y = \dfrac{y_1 + y_2}{2}$.

MIDPOINT FORMULA

The midpoint of the line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

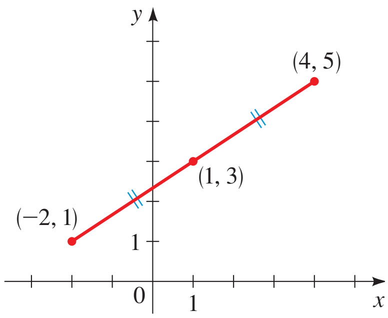


FIGURE 8

EXAMPLE 4 | Finding the Midpoint

Find the midpoint of the line segment that joins $(-2, 1)$ and $(4, 5)$.

SOLUTION By the Midpoint Formula the midpoint is

$$\left(\frac{-2+4}{2}, \frac{1+5}{2}\right) = (1, 3)$$

See Figure 8.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **27(c)** ■

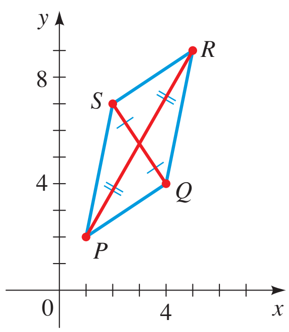


FIGURE 9

EXAMPLE 5 | Applying the Midpoint Formula

Show that the quadrilateral with vertices $P(1, 2)$, $Q(4, 4)$, $R(5, 9)$, and $S(2, 7)$ is a parallelogram by proving that its two diagonals bisect each other.

SOLUTION If the two diagonals have the same midpoint, then they must bisect each other. The midpoint of the diagonal PR is

$$\left(\frac{1+5}{2}, \frac{2+9}{2}\right) = \left(3, \frac{11}{2}\right)$$

and the midpoint of the diagonal QS is

$$\left(\frac{4+2}{2}, \frac{4+7}{2}\right) = \left(3, \frac{11}{2}\right)$$

so each diagonal bisects the other, as shown in Figure 9. (A theorem from elementary geometry states that the quadrilateral is therefore a parallelogram.)

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45** ■

1.1 EXERCISES

CONCEPTS

1. The point that is 2 units to the left of the y-axis and 4 units above the x-axis has coordinates (____, ____).

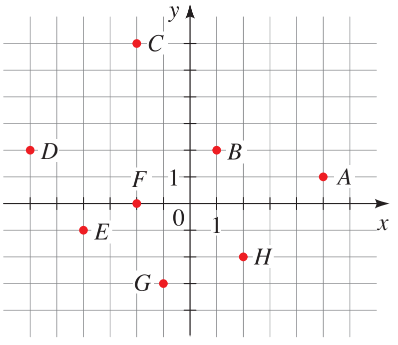
2. If x is positive and y is negative, then the point (x, y) is in Quadrant ________.

3. The distance between the points (a, b) and (c, d) is ________________. So the distance between $(1, 2)$ and $(7, 10)$ is ________.

4. The point midway between (a, b) and (c, d) is ________. So the point midway between $(1, 2)$ and $(7, 10)$ is ________.

SKILLS

5–6 ■ Refer to the following figure.



5. Find the coordinates of the points shown.

6. List the points that lie in Quadrants I and III.

7–8 ■ Plot the given points in a coordinate plane.

7. $(0, 5)$, $(-1, 0)$, $(-1, -2)$, $\left(\frac{1}{2}, \frac{2}{3}\right)$

8. $(-5, 0)$, $(2, 0)$, $(2.6, -1.3)$, $(-2.5, -3.5)$

9–22 ■ Sketch the region given by the set.

9. $\{(x, y) \mid y \geq 0\}$

10. $\{(x, y) \mid x \leq 0\}$

11. $\{(x, y) \mid y = 5\}$

12. $\{(x, y) \mid x = -3\}$

13. $\{(x, y) \mid -1 < x < 1\}$

14. $\{(x, y) \mid 0 \leq y < 3\}$

15. $\{(x, y) \mid xy < 0\}$

16. $\{(x, y) \mid xy > 0\}$

17. $\{(x, y) \mid x \geq 1 \text{ and } y < 3\}$

18. $\{(x, y) \mid -2 < x < 2 \text{ and } y \geq 3\}$

19. $\{(x, y) \mid x < 2 \text{ and } y \geq 1\}$

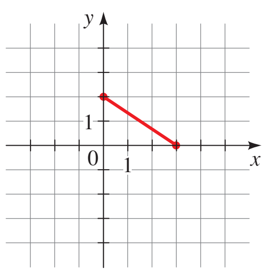
20. $\{(x, y) \mid -1 \leq x \leq 1 \text{ and } y < 4\}$

21. $\{(x, y) \mid -1 < x < 1 \text{ and } -2 < y < 2\}$

22. $\{(x, y) \mid -3 \leq x \leq 3 \text{ and } -1 \leq y \leq 1\}$

23–26 ■ A pair of points is graphed. **(a)** Find the distance between them. **(b)** Find the midpoint of the segment that joins them.

23.



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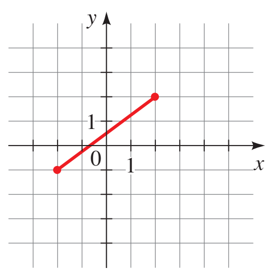

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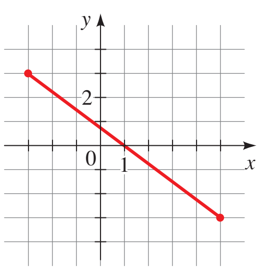

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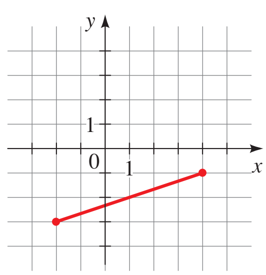


27–36 ■ A pair of points is given. **(a)** Plot the points in a coordinate plane. **(b)** Find the distance between them. **(c)** Find the midpoint of the segment that joins them.

27. $(0, 8), (6, 16)$

28. $(-2, 5), (10, 0)$

29. $(-3, -6), (4, 18)$

30. $(-1, -1), (9, 9)$

31. $(6, -2), (-1, 3)$

32. $(-1, 6), (-1, -3)$

33. $(7, 3), (11, 6)$

34. $(2, 13), (7, 1)$

35. $(3, 4), (-3, -4)$

36. $(5, 0), (0, 6)$

37. Draw the rectangle with vertices $A(1, 3)$, $B(5, 3)$, $C(1, -3)$, and $D(5, -3)$ on a coordinate plane. Find the area of the rectangle.

38. Draw the parallelogram with vertices $A(1, 2)$, $B(5, 2)$, $C(3, 6)$, and $D(7, 6)$ on a coordinate plane. Find the area of the parallelogram.

39. Plot the points $A(1, 0)$, $B(5, 0)$, $C(4, 3)$, and $D(2, 3)$ on a coordinate plane. Draw the segments AB, BC, CD, and DA. What kind of quadrilateral is $ABCD$, and what is its area?

40. Plot the points $P(5, 1)$, $Q(0, 6)$, and $R(-5, 1)$ on a coordinate plane. Where must the point S be located so that the quadrilateral $PQRS$ is a square? Find the area of this square.

41. Which of the points $A(6, 7)$ and $B(-5, 8)$ is closer to the origin?

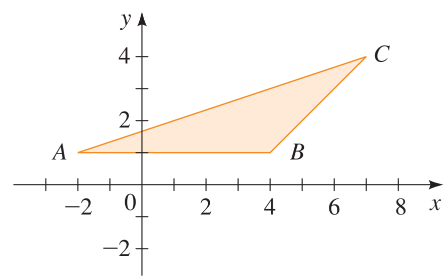
42. Which of the points $C(-6, 3)$ and $D(3, 0)$ is closer to the point $E(-2, 1)$?

43. Which of the points $P(3, 1)$ and $Q(-1, 3)$ is closer to the point $R(-1, -1)$?

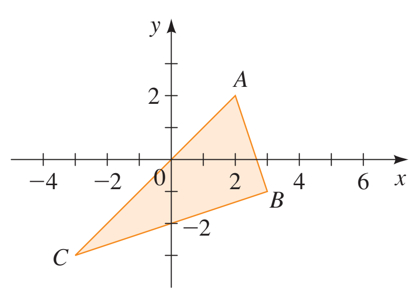
44. **(a)** Show that the points $(7, 3)$ and $(3, 7)$ are the same distance from the origin.
(b) Show that the points (a, b) and (b, a) are the same distance from the origin.

45. Show that the triangle with vertices $A(0, 2)$, $B(-3, -1)$, and $C(-4, 3)$ is isosceles.

46. Find the area of the triangle shown in the figure.



47. Refer to triangle ABC in the figure.
(a) Show that triangle ABC is a right triangle by using the converse of the Pythagorean Theorem (see page 253).
(b) Find the area of triangle ABC.



48. Show that the triangle with vertices $A(6, -7)$, $B(11, -3)$, and $C(2, -2)$ is a right triangle by using the converse of the Pythagorean Theorem. Find the area of the triangle.

49. Show that the points $A(-2, 9)$, $B(4, 6)$, $C(1, 0)$, and $D(-5, 3)$ are the vertices of a square.

50. Show that the points $A(-1, 3)$, $B(3, 11)$, and $C(5, 15)$ are collinear by showing that $d(A, B) + d(B, C) = d(A, C)$.

51. Find a point on the y-axis that is equidistant from the points $(5, -5)$ and $(1, 1)$.

52. Find the lengths of the medians of the triangle with vertices $A(1, 0)$, $B(3, 6)$, and $C(8, 2)$. (A *median* is a line segment from a vertex to the midpoint of the opposite side.)

53. Find the point that is one-fourth of the distance from the point $P(-1, 3)$ to the point $Q(7, 5)$ along the segment PQ.

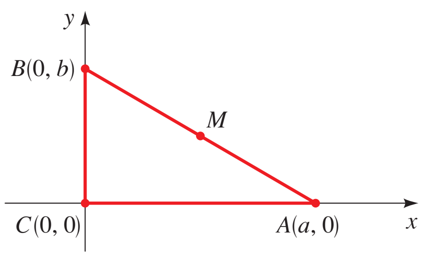
54. Plot the points $P(-2, 1)$ and $Q(12, -1)$ on a coordinate plane. Which (if either) of the points $A(5, -7)$ and $B(6, 7)$ lies on the perpendicular bisector of the segment PQ?

55. Plot the points $P(-1, -4)$, $Q(1, 1)$, and $R(4, 2)$ on a coordinate plane. Where should the point S be located so that the figure $PQRS$ is a parallelogram?

56. If $M(6, 8)$ is the midpoint of the line segment AB, and if A has coordinates $(2, 3)$, find the coordinates of B.

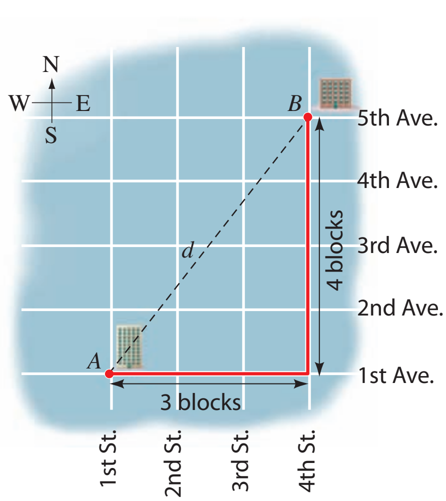
57. **(a)** Sketch the parallelogram with vertices $A(-2, -1)$, $B(4, 2)$, $C(7, 7)$, and $D(1, 4)$.
(b) Find the midpoints of the diagonals of this parallelogram.
(c) From part (b), show that the diagonals bisect each other.

58. The point M in the figure is the midpoint of the line segment AB. Show that M is equidistant from the vertices of triangle ABC.



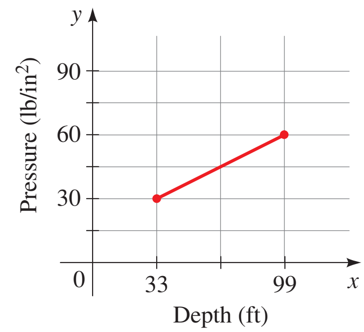
APPLICATIONS

59. Distances in a City A city has streets that run north and south and avenues that run east and west, all equally spaced. Streets and avenues are numbered sequentially, as shown in the figure. The *walking* distance between points A and B is 7 blocks—that is, 3 blocks east and 4 blocks north. To find the *straight-line* distances d, we must use the Distance Formula.
(a) Find the straight-line distance (in blocks) between A and B.
(b) Find the walking distance and the straight-line distance between the corner of 4th St. and 2nd Ave. and the corner of 11th St. and 26th Ave.
(c) What must be true about the points P and Q if the walking distance between P and Q equals the straight-line distance between P and Q?



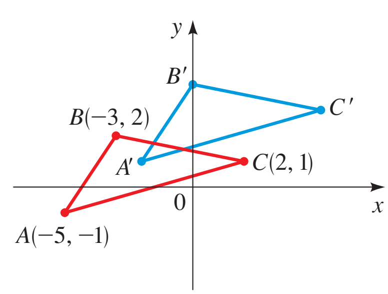
60. Halfway Point Two friends live in the city described in Exercise 59, one at the corner of 3rd St. and 7th Ave. and the other at the corner of 27th St. and 17th Ave. They frequently meet at a coffee shop halfway between their homes.
(a) At what intersection is the coffee shop located?
(b) How far must each of them walk to get to the coffee shop?

61. Pressure and Depth The graph shows the pressure experienced by an ocean diver at two different depths. Find and interpret the midpoint of the line segment shown in the graph.

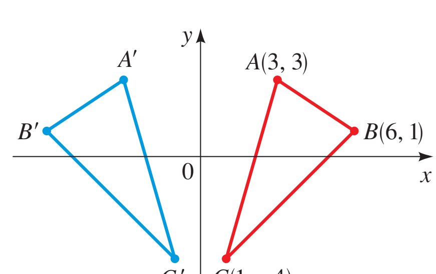


DISCOVERY ■ DISCUSSION ■ WRITING

62. Shifting the Coordinate Plane Suppose that each point in the coordinate plane is shifted 3 units to the right and 2 units upward.
(a) The point $(5, 3)$ is shifted to what new point?
(b) The point (a, b) is shifted to what new point?
(c) What point is shifted to $(3, 4)$?
(d) Triangle ABC in the figure has been shifted to triangle $A'B'C'$. Find the coordinates of the points A', B', and C'.



63. Reflecting in the Coordinate Plane Suppose that the y-axis acts as a mirror that reflects each point to the right of it into a point to the left of it.
(a) The point $(3, 7)$ is reflected to what point?
(b) The point (a, b) is reflected to what point?
(c) What point is reflected to $(-4, -1)$?
(d) Triangle ABC in the figure is reflected to triangle $A'B'C'$. Find the coordinates of the points A', B', and C'.



64. Completing a Line Segment Plot the points $M(6, 8)$ and $A(2, 3)$ on a coordinate plane. If M is the midpoint of the line segment AB, find the coordinates of B. Write a brief description of the steps you took to find B and your reasons for taking them.

65. Completing a Parallelogram Plot the points $P(0, 3)$, $Q(2, 2)$, and $R(5, 3)$ on a coordinate plane. Where should the point S be located so that the figure $PQRS$ is a parallelogram? Write a brief description of the steps you took and your reasons for taking them.

DISCOVERY PROJECT **Visualizing Data**

In this project we discover how graphing can help us find hidden patterns in data. You can find the project at the book companion website: **www.stewartmath.com**

1.2 Graphs of Equations in Two Variables

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Graph equations ▶ Find intercepts ▶ Find equations of circles ▶ Graph circles in a coordinate plane ▶ Determine symmetry properties of an equation

Fundamental Principle of Analytic Geometry

A point (x, y) lies on the graph of an equation if and only if its coordinates satisfy the equation.

An **equation in two variables**, such as $y = x^2 + 1$, expresses a relationship between two quantities. A point (x, y) **satisfies** the equation if it makes the equation true when the values for x and y are substituted into the equation. For example, the point $(3, 10)$ satisfies the equation $y = x^2 + 1$ because $10 = 3^2 + 1$, but the point $(1, 3)$ does not, because $3 \neq 1^2 + 1$.

THE GRAPH OF AN EQUATION

The **graph** of an equation in x and y is the set of all points (x, y) in the coordinate plane that satisfy the equation.

▼ Graphing Equations by Plotting Points

The graph of an equation is a curve, so to graph an equation, we plot as many points as we can, then connect them by a smooth curve.

EXAMPLE 1 | Sketching a Graph by Plotting Points

Sketch the graph of the equation $2x - y = 3$.

SOLUTION We first solve the given equation for y to get

$$y = 2x - 3$$

This helps us to calculate the y-coordinates in the following table:

x	$y = 2x - 3$	(x, y)
-1	-5	$(-1, -5)$
0	-3	$(0, -3)$
1	-1	$(1, -1)$
2	1	$(2, 1)$
3	3	$(3, 3)$
4	5	$(4, 5)$

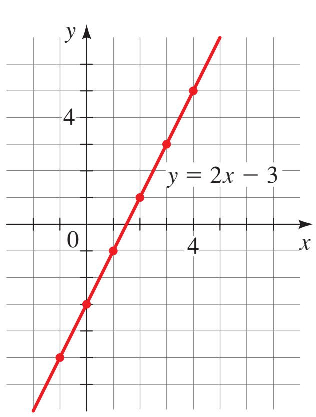


FIGURE 1

Of course, there are infinitely many points on the graph, and it is impossible to plot all of them. But the more points we plot, the better we can imagine what the graph represented by the equation looks like. We plot the points that we found in Figure 1; they appear to lie on a line. So we complete the graph by joining the points by a line. (In Section 1.3 we verify that the graph of an equation of this type is indeed a line.)

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17**

EXAMPLE 2 | Sketching a Graph by Plotting Points

Sketch the graph of the equation $y = x^2 - 2$.

A detailed discussion of parabolas and their geometric properties is presented in Chapter 7.

SOLUTION We find some of the points that satisfy the equation in the table below. In Figure 2 we plot these points and then connect them by a smooth curve. A curve with this shape is called a *parabola*.

x	$y = x^2 - 2$	(x, y)
-3	7	$(-3, 7)$
-2	2	$(-2, 2)$
-1	-1	$(-1, -1)$
0	-2	$(0, -2)$
1	-1	$(1, -1)$
2	2	$(2, 2)$
3	7	$(3, 7)$

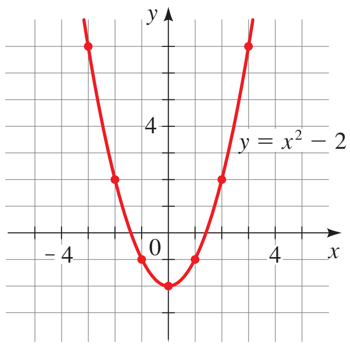


FIGURE 2

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **21**

EXAMPLE 3 | Graphing an Absolute Value Equation

Sketch the graph of the equation $y = |x|$.

SOLUTION We make a table of values:

x	$y = \|x\|$	(x, y)
-3	3	$(-3, 3)$
-2	2	$(-2, 2)$
-1	1	$(-1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	2	$(2, 2)$
3	3	$(3, 3)$

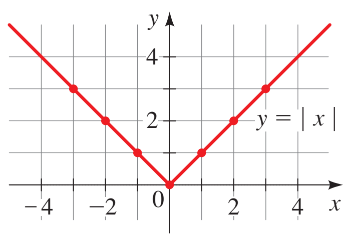


FIGURE 3

In Figure 3 we plot these points and use them to sketch the graph of the equation.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **31**

We can use a graphing calculator to graph equations. A graphing calculator draws the graph of an equation by plotting points, just as we would do by hand.

See Appendix B, *Graphing with a Graphing Calculator,* for general guidelines on using a graphing calculator. See Appendix C, *Using the TI-83/84 Graphing Calculator,* for specific graphing instructions.

EXAMPLE 4 | Graphing an Equation with a Graphing Calculator

Use a graphing calculator to graph the following equation in the viewing rectangle $[-5, 5]$ by $[-1, 2]$:

$$y = \frac{1}{1 + x^2}$$

SOLUTION The graph is shown in Figure 4.

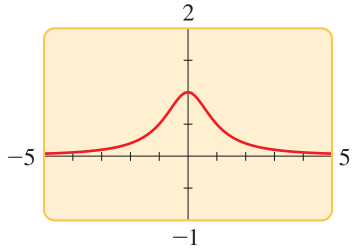


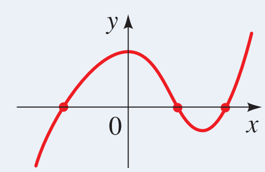
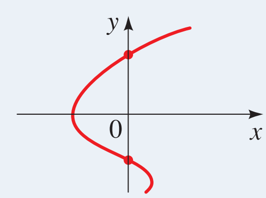
FIGURE 4 Graph of $y = \dfrac{1}{1 + x^2}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **43**

Intercepts

The x-coordinates of the points where a graph intersects the x-axis are called the ***x*-intercepts** of the graph and are obtained by setting $y = 0$ in the equation of the graph. The y-coordinates of the points where a graph intersects the y-axis are called the ***y*-intercepts** of the graph and are obtained by setting $x = 0$ in the equation of the graph.

DEFINITION OF INTERCEPTS

Intercepts	How to find them	Where they are on the graph
***x*-intercepts:** The x-coordinates of points where the graph of an equation intersects the x-axis	Set $y = 0$ and solve for x	
***y*-intercepts:** The y-coordinates of points where the graph of an equation intersects the y-axis	Set $x = 0$ and solve for y	

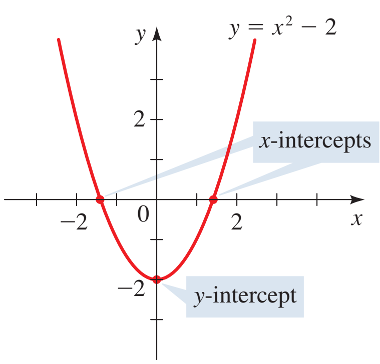


FIGURE 5

EXAMPLE 5 | Finding Intercepts

Find the x- and y-intercepts of the graph of the equation $y = x^2 - 2$.

SOLUTION To find the x-intercepts, we set $y = 0$ and solve for x. Thus

$$\begin{aligned} 0 &= x^2 - 2 && \text{Set } y = 0 \\ x^2 &= 2 && \text{Add 2 to each side} \\ x &= \pm\sqrt{2} && \text{Take the square root} \end{aligned}$$

The x-intercepts are $\sqrt{2}$ and $-\sqrt{2}$.

To find the y-intercepts, we set $x = 0$ and solve for y. Thus

$$\begin{aligned} y &= 0^2 - 2 && \text{Set } x = 0 \\ y &= -2 \end{aligned}$$

The y-intercept is -2.

The graph of this equation was sketched in Example 2. It is repeated in Figure 5 with the x- and y-intercepts labeled.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **47**

EXAMPLE 6 | Finding Intercepts

Find the x- and y-intercepts of the graph of the following equation.

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

SOLUTION To find the x-intercepts, we set $y = 0$ and solve for x:

$$\frac{x^2}{9} = 1 \qquad \text{Set } y = 0$$

$$x^2 = 9 \qquad \text{Multiply by 9}$$

$$x = \pm 3 \qquad \text{Solve for } x$$

So the x-intercepts are 3 and -3. To find the y-intercepts we set $x = 0$ and solve for y:

$$\frac{y^2}{4} = 1 \qquad \text{Set } x = 0$$

$$y^2 = 4 \qquad \text{Multiply by 4}$$

$$y = \pm 2 \qquad \text{Solve for } y$$

So the y-intercepts are 2 and -2. A graph of the equation is shown in Figure 6. The shape of the graph is an ellipse. Ellipses are studied in more detail in Section 7.2.

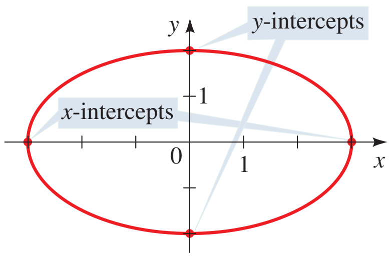


FIGURE 6 Graph of $\frac{x^2}{9} + \frac{y^2}{4} = 1$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **53**

EXAMPLE 7 | Finding Intercepts Graphically

Consider the equation $y = x^3 + 3x^2 - x - 3$.

(a) Graph the equation in the viewing rectangle $[-5, 3]$ by $[-5, 5]$.

(b) Find the x- and y-intercepts from the graph.

(c) Verify your answers to part (b) algebraically.

SOLUTION

(a) The graph is shown in Figure 7.

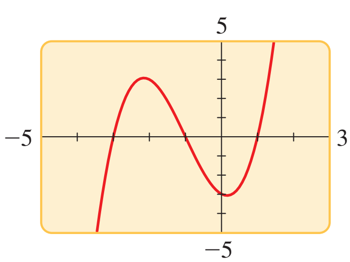


FIGURE 7 Graph of $y = x^3 + 3x^2 - x - 3$

(b) From the graph we see that there are three x-intercepts: -3, -1, and 1. There is one y-intercept: -3.

(c) Setting $x = 0$ in the equation we get $y = -3$, so -3 is a y-intercept. Setting $x = -3$ in the equation, we get $y = 0$, so -3 is an x-intercept. We can similarly verify that -1, and 1 are x-intercepts:

$$y = (-3)^3 + 3(-3)^2 - (-3) - 3 = 0 \qquad \text{Set } x = -3$$

$$y = (-1)^3 + 3(-1)^2 - (-1) - 3 = 0 \qquad \text{Set } x = -1$$

$$y = (1)^3 + 3(1)^2 - (1) - 3 = 0 \qquad \text{Set } x = 1$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **59**

Circles

So far, we have discussed how to find the graph of an equation in x and y. The converse problem is to find an equation of a graph, that is, an equation that represents a given curve in the xy-plane. Such an equation is satisfied by the coordinates of the points on the curve and by no other point. This is the other half of the fundamental principle of analytic geometry as formulated by Descartes and Fermat. The idea is that if a geometric curve can be represented by an algebraic equation, then the rules of algebra can be used to analyze the curve.

As an example of this type of problem, let's find the equation of a circle with radius r and center (h, k). By definition, the circle is the set of all points $P(x, y)$ whose distance from the center $C(h, k)$ is r (see Figure 8). Thus P is on the circle if and only if $d(P, C) = r$. From the Distance Formula we have

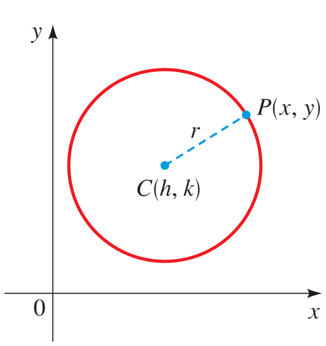


FIGURE 8

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \text{Square each side}$$

This is the desired equation.

EQUATION OF A CIRCLE

An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

This is called the **standard form** for the equation of the circle. If the center of the circle is the origin $(0, 0)$, then the equation is

$$x^2 + y^2 = r^2$$

EXAMPLE 8 | Graphing a Circle

Graph each equation.

(a) $x^2 + y^2 = 25$ **(b)** $(x - 2)^2 + (y + 1)^2 = 25$

SOLUTION

(a) Rewriting the equation as $x^2 + y^2 = 5^2$, we see that this is an equation of the circle of radius 5 centered at the origin. Its graph is shown in Figure 9.

(b) Rewriting the equation as $(x - 2)^2 + (y + 1)^2 = 5^2$, we see that this is an equation of the circle of radius 5 centered at $(2, -1)$. Its graph is shown in Figure 10.

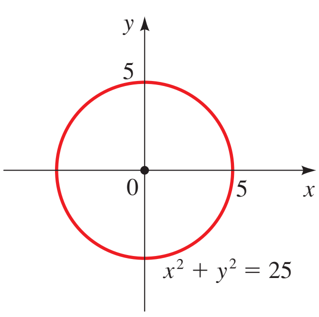


FIGURE 9

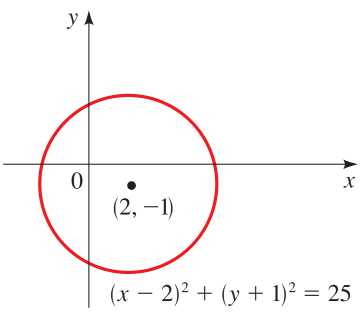


FIGURE 10

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **65** AND **67**

EXAMPLE 9 | Finding an Equation of a Circle

(a) Find an equation of the circle with radius 3 and center $(2, -5)$.

(b) Find an equation of the circle that has the points $P(1, 8)$ and $Q(5, -6)$ as the endpoints of a diameter.

SOLUTION

(a) Using the equation of a circle with $r = 3$, $h = 2$, and $k = -5$, we obtain

$$(x - 2)^2 + (y + 5)^2 = 9$$

The graph is shown in Figure 11.

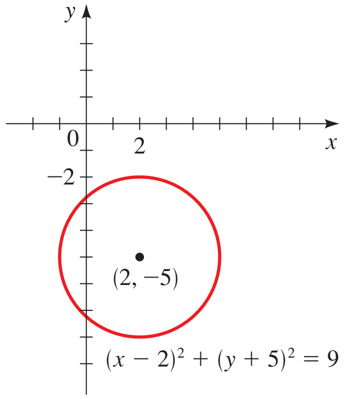
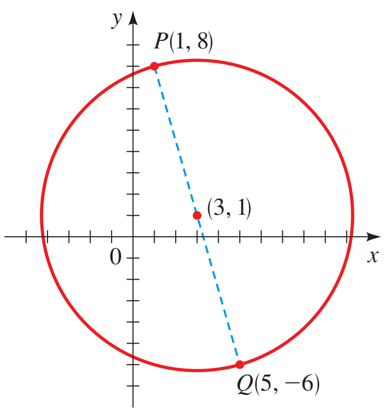


FIGURE 11

(b) We first observe that the center is the midpoint of the diameter PQ, so by the Midpoint Formula the center is

$$\left(\frac{1 + 5}{2}, \frac{8 - 6}{2}\right) = (3, 1)$$



$(x-3)^2 + (y-1)^2 = 53$

FIGURE 12

The radius r is the distance from P to the center, so by the Distance Formula

$$r^2 = (3-1)^2 + (1-8)^2 = 2^2 + (-7)^2 = 53$$

Therefore the equation of the circle is

$$(x-3)^2 + (y-1)^2 = 53$$

The graph is shown in Figure 12.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **71** AND **75** ■

Let's expand the equation of the circle in the preceding example.

$$(x-3)^2 + (y-1)^2 = 53 \qquad \text{Standard form}$$

$$x^2 - 6x + 9 + y^2 - 2y + 1 = 53 \qquad \text{Expand the squares}$$

$$x^2 - 6x + y^2 - 2y = 43 \qquad \text{Subtract 10 to get expanded form}$$

Suppose we are given the equation of a circle in expanded form. Then to find its center and radius, we must put the equation back in standard form. That means that we must reverse the steps in the preceding calculation, and to do that we need to know what to add to an expression like $x^2 - 6x$ to make it a perfect square—that is, we need to "complete the square." To **complete the square**, we must add the square of half the coefficient of x. For example, to complete the square for $x^2 - 6x$, we add the square of half of -6:

Completing the square is used in many contexts in algebra. In Section 1.6 we use completing the square to solve quadratic equations.

$$x^2 - 6x + \left(\tfrac{1}{2}(-6)\right)^2 = x^2 - 6x + 9 = (x-3)^2$$

In general, to make $X^2 + bX$ a perfect square, add $(b/2)^2$.

EXAMPLE 10 | Identifying an Equation of a Circle

Show that the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ represents a circle, and find the center and radius of the circle.

SOLUTION We first group the x-terms and y-terms. Then we complete the square within each grouping. That is, we complete the square for $x^2 + 2x$ by adding $\left(\frac{1}{2}\cdot 2\right)^2 = 1$, and we complete the square for $y^2 - 6y$ by adding $\left[\frac{1}{2}\cdot(-6)\right]^2 = 9$:

$$(x^2 + 2x \qquad) + (y^2 - 6y \qquad) = -7 \qquad \text{Group terms}$$

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9 \qquad \text{Complete the square by adding 1 and 9 to each side}$$

$$(x+1)^2 + (y-3)^2 = 3 \qquad \text{Factor and simplify}$$

We must add the same numbers to *each side* to maintain equality.

Comparing this equation with the standard equation of a circle, we see that $h = -1$, $k = 3$, and $r = \sqrt{3}$, so the given equation represents a circle with center $(-1, 3)$ and radius $\sqrt{3}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **81** ■

▼ Symmetry

Figure 13 shows the graph of $y = x^2$. Notice that the part of the graph to the left of the y-axis is the mirror image of the part to the right of the y-axis. The reason is that if the point (x, y) is on the graph, then so is $(-x, y)$, and these points are reflections of each other about the y-axis. In this situation we say that the graph is **symmetric with respect to the *y*-axis**. Similarly, we say that a graph is **symmetric with respect to the *x*-axis** if whenever the point (x, y) is on the graph, then so is $(x, -y)$. A graph is **symmetric with respect to the origin** if whenever (x, y) is on the graph, so is $(-x, -y)$.

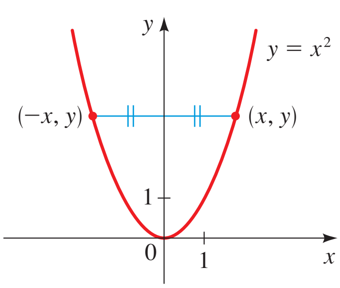
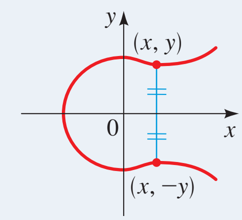
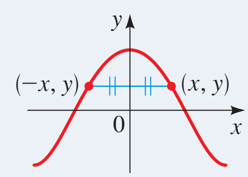
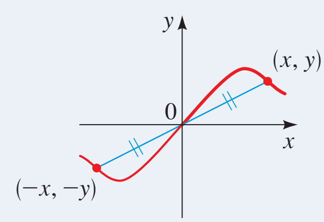


FIGURE 13

DEFINITION OF SYMMETRY

Type of symmetry	How to test an equation for symmetry	What the graph looks like (figures in this section)	Geometric meaning
Symmetry with respect to the x-axis	Replace y by $-y$. The resulting equation is equivalent to the original one.	 (Figures 6, 9, 14, 15)	Graph is unchanged when reflected in the x-axis
Symmetry with respect to the y-axis	Replace x by $-x$. The resulting equation is equivalent to the original one.	 (Figures 2, 3, 5, 9, 13, 15)	Graph is unchanged when reflected in the y-axis
Symmetry with respect to the origin	Replace x by $-x$ and y by $-y$. The resulting equation is equivalent to the original one.	 (Figures 6, 9, 15)	Graph is unchanged when rotated 180° about the origin

The remaining examples in this section show how symmetry helps us to sketch the graphs of equations.

EXAMPLE 11 | Using Symmetry to Sketch a Graph

Test the equation $x = y^2$ for symmetry, and sketch the graph.

SOLUTION If y is replaced by $-y$ in the equation $x = y^2$, we get

$$x = (-y)^2 \qquad \text{Replace } y \text{ by } -y$$

$$x = y^2 \qquad \text{Simplify}$$

so the equation is equivalent to the original one. Therefore the graph is symmetric about the x-axis. But changing x to $-x$ gives the equation $-x = y^2$, which is not the same as the original equation, so the graph is not symmetric about the y-axis.

We use the symmetry about the x-axis to sketch the graph by first plotting points just for $y > 0$ and then reflecting the graph in the x-axis, as shown in Figure 14.

y	$x = y^2$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	4	(4, 2)
3	9	(9, 3)

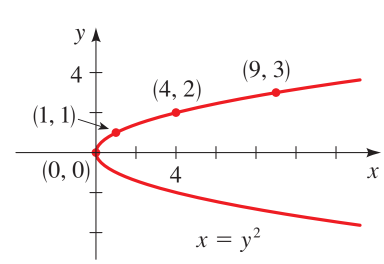


FIGURE 14

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **93** AND **99**

EXAMPLE 12 | Testing an Equation for Symmetry

Test the equation $y = x^3 - 9x$ for symmetry.

SOLUTION If we replace x by $-x$ and y by $-y$ in the equation, we get

$$-y = (-x)^3 - 9(-x) \qquad \text{Replace } x \text{ by } -x \text{ and } y \text{ by } -y$$

$$-y = -x^3 + 9x \qquad \text{Simplify}$$

$$y = x^3 - 9x \qquad \text{Multiply by } -1$$

so the equation is equivalent to the original one. This means that the graph is symmetric with respect to the origin.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **95** ■

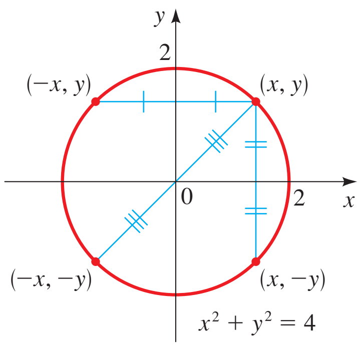


FIGURE 15

EXAMPLE 13 | A Circle That Has All Three Types of Symmetry

Test the equation of the circle $x^2 + y^2 = 4$ for symmetry.

SOLUTION The equation $x^2 + y^2 = 4$ is equivalent to the original one when x is replaced by $-x$ and y is replaced by $-y$, since $(-x)^2 = x^2$ and $(-y)^2 = y^2$, so the circle exhibits all three types of symmetry. It is symmetric with respect to the x-axis, the y-axis, and the origin, as shown in Figure 15.

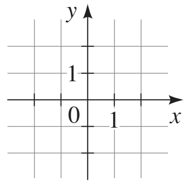
PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **97** ■

1.2 EXERCISES

CONCEPTS

1. If the point (2, 3) is on the graph of an equation in x and y, then the equation is satisfied when we replace x by ________ and y by ________. Is the point (2, 3) on the graph of the equation $2y = x + 1$? Complete the table, and sketch a graph.

x	y	(x, y)
-2		
-1		
0		
1		
2		



2. To find the x-intercept(s) of the graph of an equation, we set ________ equal to 0 and solve for ________. So the x-intercept of $2y = x + 1$ is ________.

3. To find the y-intercept(s) of the graph of an equation, we set ________ equal to 0 and solve for ________. So the y-intercept of $2y = x + 1$ is ________.

4. The graph of the equation $(x - 1)^2 + (y - 2)^2 = 9$ is a circle with center (____, ____) and radius ________.

5. **(a)** If a graph is symmetric with respect to the x-axis and (a, b) is on the graph, then (____, ____) is also on the graph.

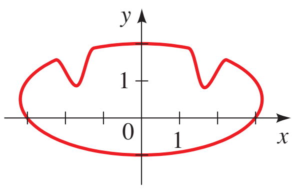
   (b) If a graph is symmetric with respect to the y-axis and (a, b) is on the graph, then (____, ____) is also on the graph.

   (c) If a graph is symmetric about the origin and (a, b) is on the graph, then (____, ____) is also on the graph.

6. The graph of an equation is shown below.

   (a) The x-intercept(s) are ________, and the y-intercept(s) are ________.

   (b) The graph is symmetric about the ________ (x-axis/y-axis/origin).



SKILLS

7–12 ■ Determine whether the given points are on the graph of the equation.

7. $y = 3 - 4x$; $(0, 3), (4, 0), (1, -1)$
8. $y = \sqrt{1 - x}$; $(2, 1), (-3, 2), (0, 1)$
9. $x - 2y - 1 = 0$; $(0, 0), (1, 0), (-1, -1)$
10. $y(x^2 + 1) = 1$; $(1, 1), (1, \frac{1}{2}), (-1, \frac{1}{2})$

11. $x^2 + 2xy + y^2 = 1$; $(0, 1)$, $(2, -1)$, $(-2, 3)$

12. $x^2 + y^2 = 1$; $(0, 1)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

13–38 ■ Make a table of values, and sketch the graph of the equation.

13. $y = -x$ **14.** $y = 2x$

15. $y = x - 3$ **16.** $y = -2x + 1$

17. $2x - y = 6$ **18.** $x + y = 3$

19. $y = 1 - x^2$ **20.** $y = x^2 + 2$

21. $y = 2x^2 - 1$ **22.** $y = 3 - 2x^2$

23. $9y = x^2$ **24.** $4y = -x^2$

25. $x + y^2 = 4$ **26.** $xy = 2$

27. $y = \sqrt{x}$ **28.** $y = 2 + \sqrt{x}$

29. $y = \sqrt{4 - x^2}$ **30.** $y = -\sqrt{4 - x^2}$

31. $y = -|x|$ **32.** $x = |y|$

33. $y = 4 - |x|$ **34.** $y = |4 - x|$

35. $x = y^3$ **36.** $y = x^3 - 1$

37. $y = x^4$ **38.** $y = 16 - x^4$

39–44 ■ Use a graphing calculator to graph the equation in the given viewing rectangle.

39. $y = 0.01x^3 - x^2 + 5$; $[-100, 150]$ by $[-2000, 2000]$

40. $y = 0.03x^2 + 1.7x - 3$; $[-100, 50]$ by $[-50, 100]$

41. $y = \sqrt{12x - 17}$; $[-1, 10]$ by $[-1, 20]$

42. $y = \sqrt[4]{256 - x^2}$; $[-20, 20]$ by $[-2, 6]$

43. $y = \dfrac{x}{x^2 + 25}$; $[-50, 50]$ by $[-0.2, 0.2]$

44. $y = x^4 - 4x^3$; $[-4, 6]$ by $[-50, 100]$

45–54 ■ Find the x- and y-intercepts of the graph of the equation.

45. $y = x - 3$ **46.** $3x + 5y = 5$

47. $y = x^2 - 9$ **48.** $x^2 + y^2 = 4$

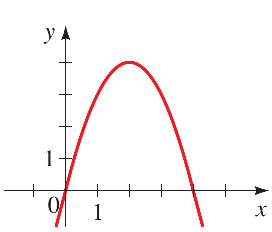
49. $y - 2xy + 2x = 1$ **50.** $x^2 - xy + y = 1$

51. $y = \sqrt{x + 1}$ **52.** $xy = 5$

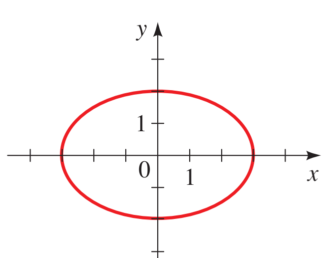
53. $25x^2 + 4y^2 = 100$ **54.** $4x^2 - 9y^2 = 36$

55–58 ■ An equation and its graph are given. Find the x- and y-intercepts.

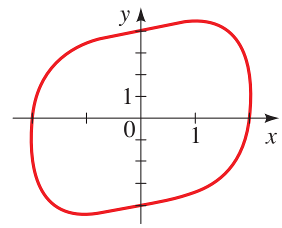
55. $y = 4x - x^2$



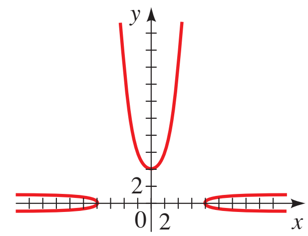
56. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$



57. $x^4 + y^2 - xy = 16$



58. $x^2 + y^3 - x^2y^2 = 64$



59–64 ■ An equation is given. **(a)** Use a graphing calculator to graph the equation in the given viewing rectangle. **(b)** Find the x- and y-intercepts from the graph. **(c)** Verify your answers to part (b) algebraically (from the equation).

59. $y = x^3 - x^2$; $[-2, 2]$ by $[-1, 1]$

60. $y = x^4 - 2x^3$; $[-2, 3]$ by $[-3, 3]$

61. $y = -\dfrac{2}{x^2 + 1}$; $[-5, 5]$ by $[-3, 1]$

62. $y = \dfrac{x}{x^2 + 1}$; $[-5, 5]$ by $[-2, 2]$

63. $y = \sqrt[3]{x}$; $[-5, 5]$ by $[-2, 2]$

64. $y = \sqrt[3]{1 - x^2}$; $[-5, 5]$ by $[-5, 3]$

65–70 ■ Find the center and radius of the circle, and sketch its graph.

65. $x^2 + y^2 = 9$ **66.** $x^2 + y^2 = 5$

67. $(x - 3)^2 + y^2 = 16$ **68.** $x^2 + (y - 2)^2 = 4$

69. $(x + 3)^2 + (y - 4)^2 = 25$ **70.** $(x + 1)^2 + (y + 2)^2 = 36$

71–78 ■ Find an equation of the circle that satisfies the given conditions.

71. Center $(2, -1)$; radius 3

72. Center $(-1, -4)$; radius 8

73. Center at the origin; passes through $(4, 7)$

74. Center $(-1, 5)$; passes through $(-4, -6)$

75. Endpoints of a diameter are $P(-1, 1)$ and $Q(5, 9)$

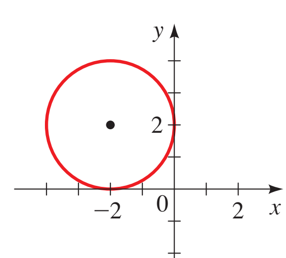
76. Endpoints of a diameter are $P(-1, 3)$ and $Q(7, -5)$

77. Center $(7, -3)$; tangent to the x-axis

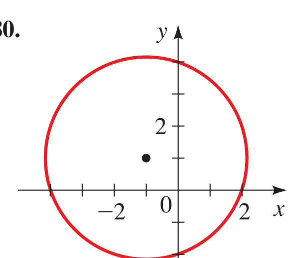
78. Circle lies in the first quadrant, tangent to both x- and y-axes; radius 5

79–80 ■ Find the equation of the circle shown in the figure.

79.



80.



81–88 ■ Show that the equation represents a circle, and find the center and radius of the circle.

81. $x^2 + y^2 - 2x + 4y + 1 = 0$

82. $x^2 + y^2 - 2x - 2y = 2$

83. $x^2 + y^2 - 4x + 10y + 13 = 0$

84. $x^2 + y^2 + 6y + 2 = 0$

85. $x^2 + y^2 + x = 0$

86. $x^2 + y^2 + 2x + y + 1 = 0$

87. $x^2 + y^2 - \frac{1}{2}x + \frac{1}{2}y = \frac{1}{8}$

88. $x^2 + y^2 + \frac{1}{2}x + 2y + \frac{1}{16} = 0$

89–92 ■ Sketch the graph of the equation.

89. $x^2 + y^2 + 4x - 10y = 21$

90. $4x^2 + 4y^2 + 2x = 0$

91. $x^2 + y^2 + 6x - 12y + 45 = 0$

92. $x^2 + y^2 - 16x + 12y + 200 = 0$

93–88 ■ Test the equation for symmetry.

93. $y = x^4 + x^2$

94. $x = y^4 - y^2$

95. $y = x^3 + 10x$

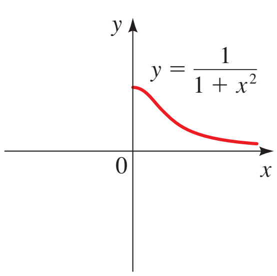
96. $y = x^2 + |x|$

97. $x^4y^4 + x^2y^2 = 1$

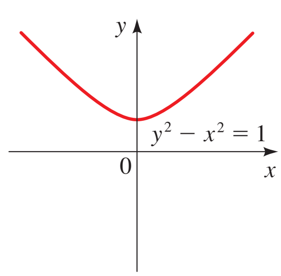
98. $x^2y^2 + xy = 1$

99–102 ■ Complete the graph using the given symmetry property.

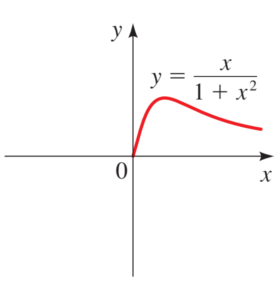
99. Symmetric with respect to the y-axis



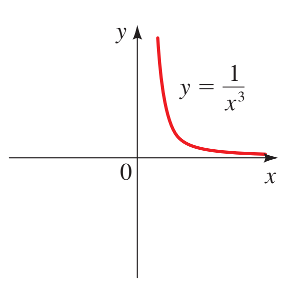
100. Symmetric with respect to the x-axis



101. Symmetric with respect to the origin



102. Symmetric with respect to the origin



APPLICATIONS

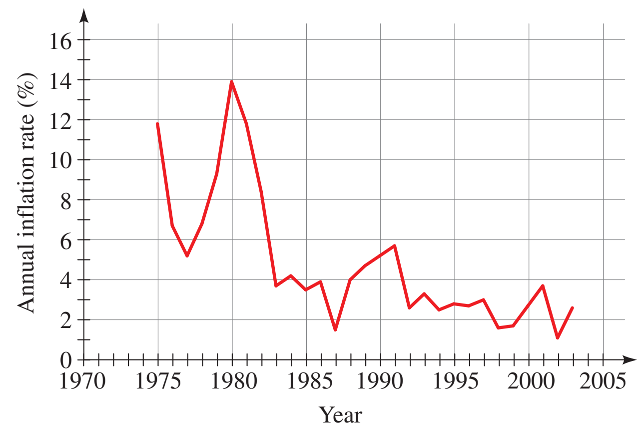
103. U.S. Inflation Rates The following graph shows the annual inflation rate in the United States from 1975 to 2003.

(a) Estimate the inflation rates in 1980, 1991, and 1999 to the nearest percent.

(b) For which years in this period did the inflation rate exceed 6%?

(c) Did the inflation rate generally increase or decrease in the years from 1980 to 1985? What about from 1987 to 1992?

(d) Estimate the highest and lowest inflation rates in this time period to the nearest percent.

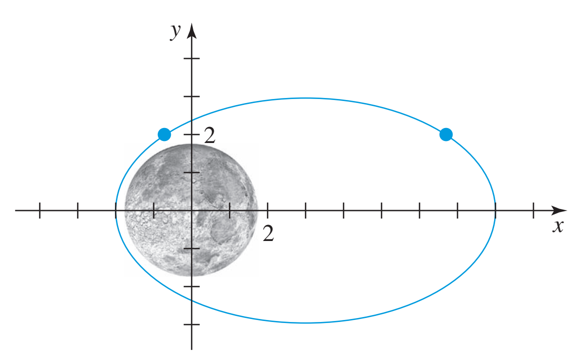


104. Orbit of a Satellite A satellite is in orbit around the moon. A coordinate plane containing the orbit is set up with the center of the moon at the origin, as shown in the graph below, with distances measured in megameters (Mm). The equation of the satellite's orbit is

$$\frac{(x-3)^2}{25} + \frac{y^2}{16} = 1$$

(a) From the graph, determine the closest to and the farthest from the center of the moon that the satellite gets.

(b) There are two points in the orbit with y-coordinates 2. Find the x-coordinates of these points, and determine their distances to the center of the moon.



DISCOVERY ■ DISCUSSION ■ WRITING

105. Circle, Point, or Empty Set? Complete the squares in the general equation $x^2 + ax + y^2 + by + c = 0$, and simplify the result as much as possible. Under what conditions on the coefficients a, b, and c does this equation represent a circle? A single point? The empty set? In the case in which the equation does represent a circle, find its center and radius.

106. Do the Circles Intersect?

(a) Find the radius of each circle in the pair and the distance between their centers; then use this information to determine whether the circles intersect.

(i) $(x-2)^2 + (y-1)^2 = 9$;
$(x-6)^2 + (y-4)^2 = 16$

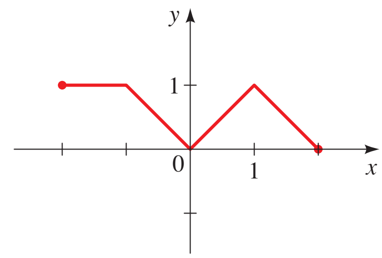
(ii) $x^2 + (y-2)^2 = 4$;
$(x-5)^2 + (y-14)^2 = 9$

(iii) $(x-3)^2 + (y+1)^2 = 1$;
$(x-2)^2 + (y-2)^2 = 25$

(b) How can you tell, just by knowing the radii of two circles and the distance between their centers, whether the circles intersect? Write a short paragraph describing how you would decide this, and draw graphs to illustrate your answer.

107. Making a Graph Symmetric The graph shown in the figure is not symmetric about the x-axis, the y-axis, or the origin. Add more line segments to the graph so that it exhibits the indicated symmetry. In each case, add as little as possible.

(a) Symmetry about the x-axis
(b) Symmetry about the y-axis
(c) Symmetry about the origin



1.3 LINES

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find the slope of a line ▶ Find the equation of a line given a point and the slope ▶ Find the equation of a line given the slope and y-intercept ▶ Find equations of horizontal and vertical lines ▶ Graph equations of lines ▶ Find equations for parallel and perpendicular lines ▶ Make a linear model: interpret slope as rate of change

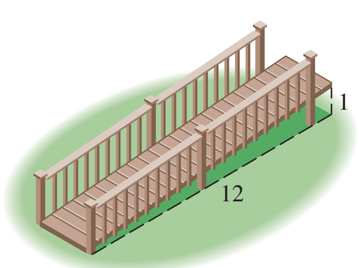
In this section we find equations for straight lines lying in a coordinate plane. The equations will depend on how the line is inclined, so we begin by discussing the concept of slope.

▼ The Slope of a Line

We first need a way to measure the "steepness" of a line, or how quickly it rises (or falls) as we move from left to right. We define *run* to be the distance we move to the right and *rise* to be the corresponding distance that the line rises (or falls). The *slope* of a line is the ratio of rise to run:

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

Figure 1 shows situations in which slope is important. Carpenters use the term *pitch* for the slope of a roof or a staircase; the term *grade* is used for the slope of a road.



Slope of a ramp
Slope $= \frac{1}{12}$

Pitch of a roof
Slope $= \frac{1}{3}$

Grade of a road
Slope $= \frac{8}{100}$

FIGURE 1

If a line lies in a coordinate plane, then the **run** is the change in the x-coordinate and the **rise** is the corresponding change in the y-coordinate between any two points on the line (see Figure 2). This gives us the following definition of slope.

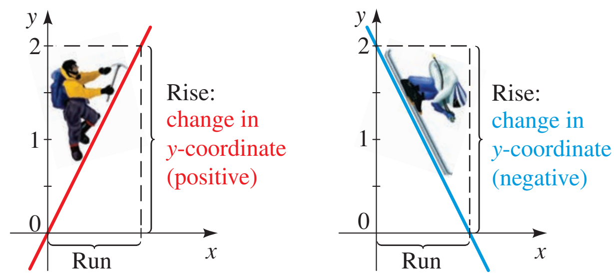


FIGURE 2

SLOPE OF A LINE

The **slope** m of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

The slope is independent of which two points are chosen on the line. We can see that this is true from the similar triangles in Figure 3:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2' - y_1'}{x_2' - x_1'}$$

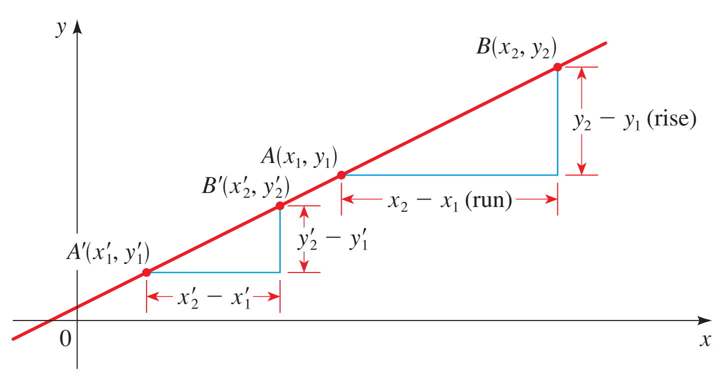


FIGURE 3

Figure 4 shows several lines labeled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. The steepest lines are those for which the absolute value of the slope is the largest; a horizontal line has slope zero.

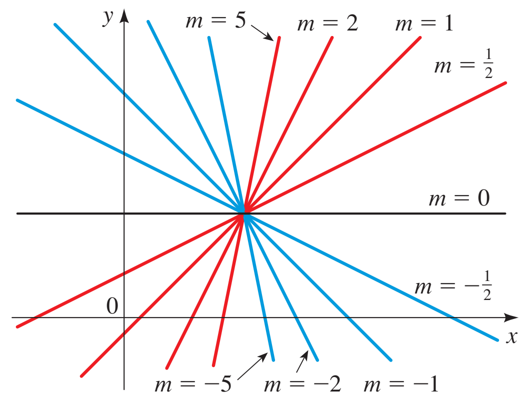


FIGURE 4 Lines with various slopes

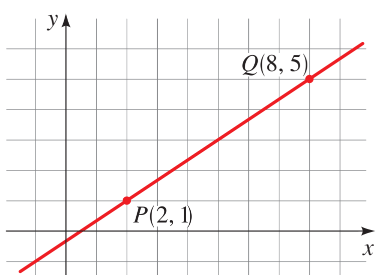


FIGURE 5

EXAMPLE 1 | Finding the Slope of a Line Through Two Points

Find the slope of the line that passes through the points $P(2, 1)$ and $Q(8, 5)$.

SOLUTION Since any two different points determine a line, only one line passes through these two points. From the definition the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{8 - 2} = \frac{4}{6} = \frac{2}{3}$$

This says that for every 3 units we move to the right, the line rises 2 units. The line is drawn in Figure 5.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **9** ■

▼ Point-Slope Form of the Equation of a Line

Now let's find the equation of the line that passes through a given point $P(x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through P_1 and P is equal to m (see Figure 6), that is,

$$\frac{y - y_1}{x - x_1} = m$$

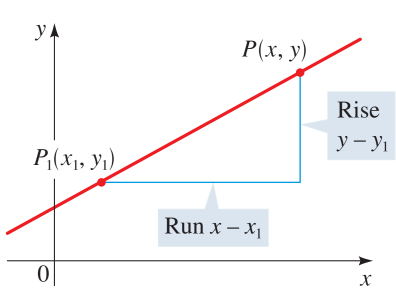


FIGURE 6

This equation can be rewritten in the form $y - y_1 = m(x - x_1)$; note that the equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore it is an equation of the given line. It is called the **point-slope form** of the equation of a line.

POINT-SLOPE FORM OF THE EQUATION OF A LINE

An equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1)$$

EXAMPLE 2 | Finding the Equation of a Line with Given Point and Slope

(a) Find an equation of the line through $(1, -3)$ with slope $-\frac{1}{2}$.

(b) Sketch the line.

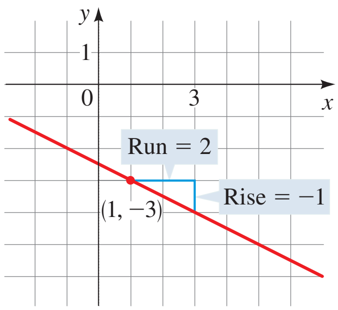


FIGURE 7

SOLUTION

(a) Using the point-slope form with $m = -\frac{1}{2}$, $x_1 = 1$, and $y_1 = -3$, we obtain an equation of the line as

$$y + 3 = -\tfrac{1}{2}(x - 1) \qquad \text{Slope } m = -\tfrac{1}{2}\text{, point } (1, -3)$$

$$2y + 6 = -x + 1 \qquad \text{Multiply by 2}$$

$$x + 2y + 5 = 0 \qquad \text{Rearrange}$$

(b) The fact that the slope is $-\frac{1}{2}$ tells us that when we move to the right 2 units, the line drops 1 unit. This enables us to sketch the line in Figure 7.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **23**

EXAMPLE 3 | Finding the Equation of a Line Through Two Given Points

Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

SOLUTION The slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{6}{4} = -\frac{3}{2}$$

We can use *either* point, $(-1, 2)$ *or* $(3, -4)$, in the point-slope equation. We will end up with the same final answer.

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\tfrac{3}{2}(x + 1) \qquad \text{Slope } m = -\tfrac{3}{2}\text{, point } (-1, 2)$$

$$2y - 4 = -3x - 3 \qquad \text{Multiply by 2}$$

$$3x + 2y - 1 = 0 \qquad \text{Rearrange}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **27**

▼ Slope-Intercept Form of the Equation of a Line

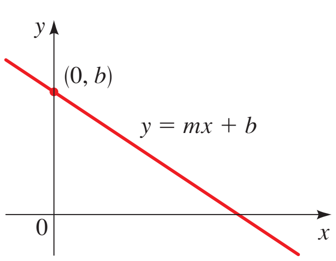


FIGURE 8

Suppose a nonvertical line has slope m and y-intercept b (see Figure 8). This means that the line intersects the y-axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x = 0$ and $y = b$, becomes

$$y - b = m(x - 0)$$

This simplifies to $y = mx + b$, which is called the **slope-intercept form** of the equation of a line.

SLOPE-INTERCEPT FORM OF THE EQUATION OF A LINE

An equation of the line that has slope m and y-intercept b is

$$y = mx + b$$

EXAMPLE 4 | Lines in Slope-Intercept Form

(a) Find the equation of the line with slope 3 and y-intercept -2.

(b) Find the slope and y-intercept of the line $3y - 2x = 1$.

SOLUTION

(a) Since $m = 3$ and $b = -2$, from the slope-intercept form of the equation of a line we get

$$y = 3x - 2$$

(b) We first write the equation in the form $y = mx + b$:

$$3y - 2x = 1$$

$$3y = 2x + 1 \qquad \text{Add } 2x$$

$$y = \tfrac{2}{3}x + \tfrac{1}{3} \qquad \text{Divide by 3}$$

Slope y-intercept

$y = \frac{2}{3}x + \frac{1}{3}$

From the slope-intercept form of the equation of a line, we see that the slope is $m = \frac{2}{3}$ and the y-intercept is $b = \frac{1}{3}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **21** AND **61**

Vertical and Horizontal Lines

If a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where b is the y-intercept (see Figure 9). A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the line consists of all points whose x-coordinate is a.

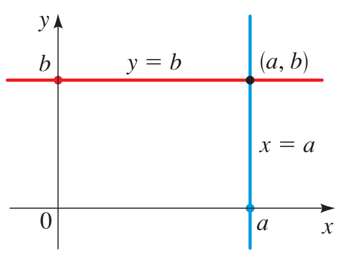


FIGURE 9

VERTICAL AND HORIZONTAL LINES

An equation of the vertical line through (a, b) is $x = a$. The slope of a vertical line is undefined.

An equation of the horizontal line through (a, b) is $y = b$. The slope of a horizontal line is 0.

EXAMPLE 5 | Vertical and Horizontal Lines

(a) An equation for the vertical line through $(3, 5)$ is $x = 3$.

(b) The graph of the equation $x = 3$ is a vertical line with x-intercept 3.

(c) An equation for the horizontal line through $(8, -2)$ is $y = -2$.

(d) The graph of the equation $y = -2$ is a horizontal line with y-intercept -2.

The lines are graphed in Figure 10.

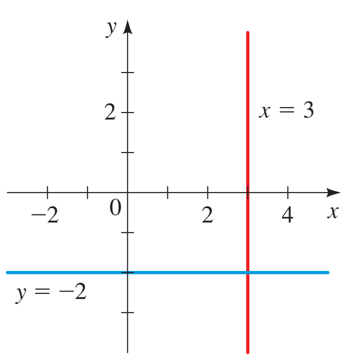


FIGURE 10

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **33, 35, 65,** AND **67**

General Equation of a Line

A **linear equation** is an equation of the form

$$Ax + By + C = 0$$

where A, B, and C are constants and A and B are not both 0. The equation of a line is a linear equation:

- A nonvertical line has the equation $y = mx + b$ or $-mx + y - b = 0$, which is a linear equation with $A = -m$, $B = 1$, and $C = -b$.
- A vertical line has the equation $x = a$ or $x - a = 0$, which is a linear equation with $A = 1$, $B = 0$, and $C = -a$.

Conversely, the graph of a linear equation is a line:

- If $B \neq 0$, the equation becomes

$$y = -\frac{A}{B}x - \frac{C}{B} \qquad \text{Divide by } B$$

  and this is the slope-intercept form of the equation of a line (with $m = -A/B$ and $b = -C/B$).

- If $B = 0$, the equation becomes

$$Ax + C = 0 \qquad \text{Set } B = 0$$

  or $x = -C/A$, which represents a vertical line.

We have proved the following:

GENERAL EQUATION OF A LINE

The graph of every **linear equation**

$$Ax + By + C = 0 \qquad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

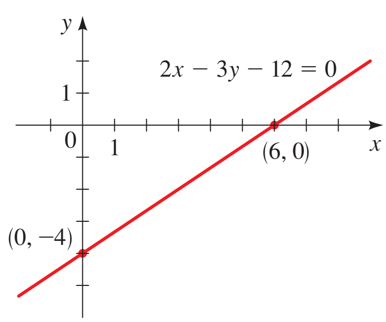


FIGURE 11

EXAMPLE 6 | Graphing a Linear Equation

Sketch the graph of the equation $2x - 3y - 12 = 0$.

SOLUTION 1 Since the equation is linear, its graph is a line. To draw the graph, it is enough to find any two points on the line. The intercepts are the easiest points to find.

x-intercept: Substitute $y = 0$, to get $2x - 12 = 0$, so $x = 6$

y-intercept: Substitute $x = 0$, to get $-3y - 12 = 0$, so $y = -4$

With these points we can sketch the graph in Figure 11.

SOLUTION 2 We write the equation in slope-intercept form:

$$\begin{aligned} 2x - 3y - 12 &= 0 \\ 2x - 3y &= 12 && \text{Add 12} \\ -3y &= -2x + 12 && \text{Subtract } 2x \\ y &= \tfrac{2}{3}x - 4 && \text{Divide by } -3 \end{aligned}$$

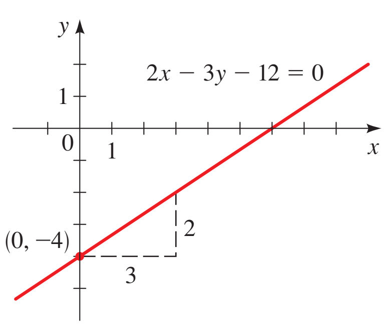


FIGURE 12

This equation is in the form $y = mx + b$, so the slope is $m = \frac{2}{3}$ and the y-intercept is $b = -4$. To sketch the graph, we plot the y-intercept and then move 3 units to the right and 2 units up as shown in Figure 12.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **69**

Parallel and Perpendicular Lines

Since slope measures the steepness of a line, it seems reasonable that parallel lines should have the same slope. In fact, we can prove this.

PARALLEL LINES

Two nonvertical lines are parallel if and only if they have the same slope.

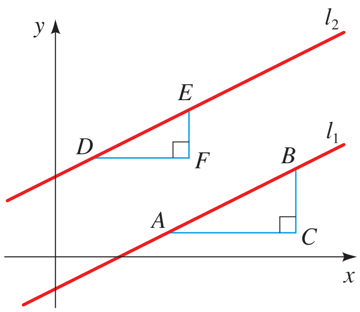


FIGURE 13

PROOF Let the lines l_1 and l_2 in Figure 13 have slopes m_1 and m_2. If the lines are parallel, then the right triangles ABC and DEF are similar, so

$$m_1 = \frac{d(B, C)}{d(A, C)} = \frac{d(E, F)}{d(D, F)} = m_2$$

Conversely, if the slopes are equal, then the triangles will be similar, so $\angle BAC = \angle EDF$ and the lines are parallel. ■

EXAMPLE 7 | Finding the Equation of a Line Parallel to a Given Line

Find an equation of the line through the point $(5, 2)$ that is parallel to the line $4x + 6y + 5 = 0$.

SOLUTION First we write the equation of the given line in slope-intercept form:

$$4x + 6y + 5 = 0$$

$$6y = -4x - 5 \qquad \text{Subtract } 4x + 5$$

$$y = -\tfrac{2}{3}x - \tfrac{5}{6} \qquad \text{Divide by 6}$$

So the line has slope $m = -\frac{2}{3}$. Since the required line is parallel to the given line, it also has slope $m = -\frac{2}{3}$. From the point-slope form of the equation of a line, we get

$$y - 2 = -\tfrac{2}{3}(x - 5) \qquad \text{Slope } m = -\tfrac{2}{3}, \text{ point } (5, 2)$$

$$3y - 6 = -2x + 10 \qquad \text{Multiply by 3}$$

$$2x + 3y - 16 = 0 \qquad \text{Rearrange}$$

Thus the equation of the required line is $2x + 3y - 16 = 0$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41** ■

The condition for perpendicular lines is not as obvious as that for parallel lines.

PERPENDICULAR LINES

Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$, that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).

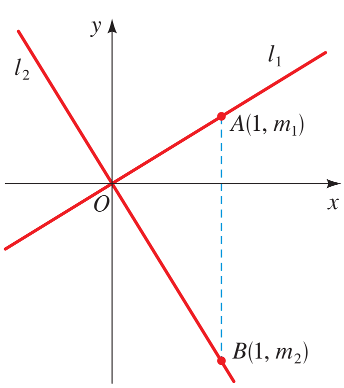


FIGURE 14

PROOF In Figure 14 we show two lines intersecting at the origin. (If the lines intersect at some other point, we consider lines parallel to these that intersect at the origin. These lines have the same slopes as the original lines.)

If the lines l_1 and l_2 have slopes m_1 and m_2, then their equations are $y = m_1 x$ and $y = m_2 x$. Notice that $A(1, m_1)$ lies on l_1 and $B(1, m_2)$ lies on l_2. By the Pythagorean Theorem and its converse (see page 253), $OA \perp OB$ if and only if

$$[d(O, A)]^2 + [d(O, B)]^2 = [d(A, B)]^2$$

By the Distance Formula this becomes

$$
\begin{aligned}
(1^2 + m_1^2) + (1^2 + m_2^2) &= (1 - 1)^2 + (m_2 - m_1)^2 \\
2 + m_1^2 + m_2^2 &= m_2^2 - 2m_1m_2 + m_1^2 \\
2 &= -2m_1m_2 \\
m_1m_2 &= -1
\end{aligned}
$$

■

EXAMPLE 8 | Perpendicular Lines

Show that the points $P(3, 3)$, $Q(8, 17)$, and $R(11, 5)$ are the vertices of a right triangle.

SOLUTION The slopes of the lines containing PR and QR are, respectively,

$$m_1 = \frac{5 - 3}{11 - 3} = \frac{1}{4} \qquad \text{and} \qquad m_2 = \frac{5 - 17}{11 - 8} = -4$$

Since $m_1m_2 = -1$, these lines are perpendicular, so PQR is a right triangle. It is sketched in Figure 15.

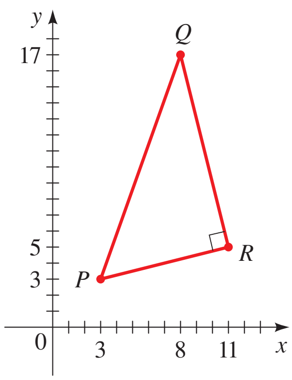


FIGURE 15

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **83** ■

EXAMPLE 9 | Finding an Equation of a Line Perpendicular to a Given Line

Find an equation of the line that is perpendicular to the line $4x + 6y + 5 = 0$ and passes through the origin.

SOLUTION In Example 7 we found that the slope of the line $4x + 6y + 5 = 0$ is $-\frac{2}{3}$. Thus the slope of a perpendicular line is the negative reciprocal, that is, $\frac{3}{2}$. Since the required line passes through $(0, 0)$, the point-slope form gives

$$
\begin{aligned}
y - 0 &= \tfrac{3}{2}(x - 0) \\
y &= \tfrac{3}{2}x
\end{aligned}
$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45** ■

EXAMPLE 10 | Graphing a Family of Lines

Use a graphing calculator to graph the family of lines

$$y = 0.5x + b$$

for $b = -2, -1, 0, 1, 2$. What property do the lines share?

MATHEMATICS IN THE MODERN WORLD

Changing Words, Sound, and Pictures into Numbers

Pictures, sound, and text are routinely transmitted from one place to another via the Internet, fax machines, or modems. How can such things be transmitted through telephone wires? The key to doing this is to change them into numbers or bits (the digits 0 or 1). It's easy to see how to change text to numbers. For example, we could use the correspondence A = 00000001, B = 00000010, C = 00000011, D = 00000100, E = 00000101, and so on. The word "BED" then becomes 000000100000010100000100. By reading the digits in groups of eight, it is possible to translate this number back to the word "BED."

Changing sound to bits is more complicated. A sound wave can be graphed on an oscilloscope or a computer. The graph is then broken down mathematically into simpler components corresponding to the different frequencies of the original sound. (A branch of mathematics called Fourier analysis is used here.) The intensity of each component is a number, and the original sound can be reconstructed from these numbers. For example, music is stored on a CD as a sequence of bits; it may look like 10101000101001010010101010000001011110101000101011.... (One second of music requires 1.5 million bits!) The CD player reconstructs the music from the numbers on the CD.

Changing pictures into numbers involves expressing the color and brightness of each dot (or pixel) as a number. This is done very efficiently by using a branch of mathematics called wavelet theory. The FBI uses wavelets as a compact way to store the millions of fingerprints they need on file.

See Appendix B, *Graphing with a Graphing Calculator,* for guidelines on using a graphing calculator. See Appendix C, *Using the TI-83/84 Graphing Calculator,* for specific graphing instructions.

SOLUTION We use a graphing calculator to graph the lines in the viewing rectangle $[-6, 6]$ by $[-6, 6]$. The graphs are shown in Figure 16. The lines all have the same slope, so they are parallel.

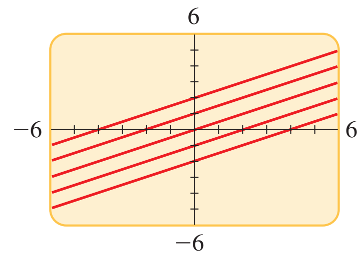


FIGURE 16 $y = 0.5x + b$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **51**

Modeling with Linear Equations: Slope as Rate of Change

When a line is used to model the relationship between two quantities, the slope of the line is the **rate of change** of one quantity with respect to the other. For example, the graph in Figure 17(a) gives the amount of gas in a tank that is being filled. The slope between the indicated points is

$$m = \frac{6 \text{ gallons}}{3 \text{ minutes}} = 2 \text{ gal/min}$$

The slope is the *rate* at which the tank is being filled, 2 gallons per minute. In Figure 17(b) the tank is being drained at the *rate* of 0.03 gallon per minute, and the slope is -0.03.

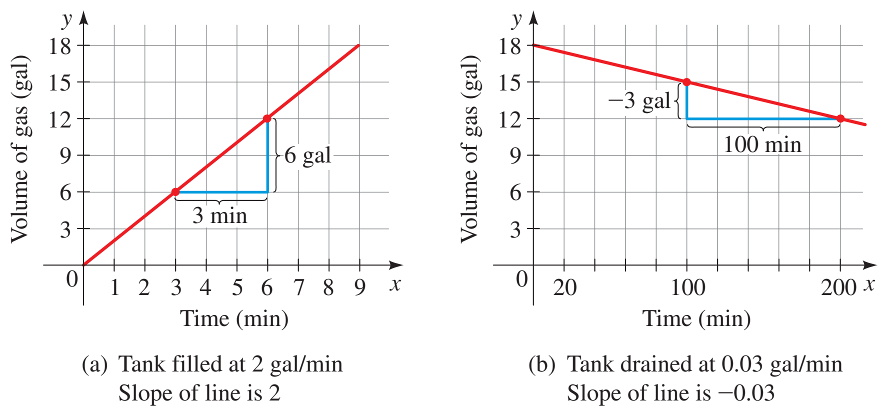


FIGURE 17

The next two examples give other situations in which the slope of a line is a rate of change.

EXAMPLE 11 | Slope as Rate of Change

A dam is built on a river to create a reservoir. The water level w in the reservoir is given by the equation

$$w = 4.5t + 28$$

where t is the number of years since the dam was constructed and w is measured in feet.

(a) Sketch a graph of this equation.

(b) What do the slope and w-intercept of this graph represent?

SOLUTION

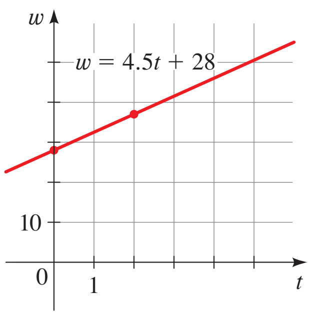


FIGURE 18

(a) This equation is linear, so its graph is a line. Since two points determine a line, we plot two points that lie on the graph and draw a line through them.

When $t = 0$, then $w = 4.5(0) + 28 = 28$, so $(0, 28)$ is on the line.

When $t = 2$, then $w = 4.5(2) + 28 = 37$, so $(2, 37)$ is on the line.

The line that is determined by these points is shown in Figure 18.

(b) The slope is $m = 4.5$; it represents the rate of change of water level with respect to time. This means that the water level *increases* 4.5 ft per year. The w-intercept is 28 and occurs when $t = 0$, so it represents the water level when the dam was constructed.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **95**

EXAMPLE 12 | Fitting a Line to Data

h (km)	T (°C)
0	20
1	10
2	0
3	−10

As dry air moves upward, it expands and cools. The table in the margin shows the air temperature at various heights above the ground on a particular day.

(a) Plot the points given in the table. Do the points appear to lie along a line? Prove your answer.

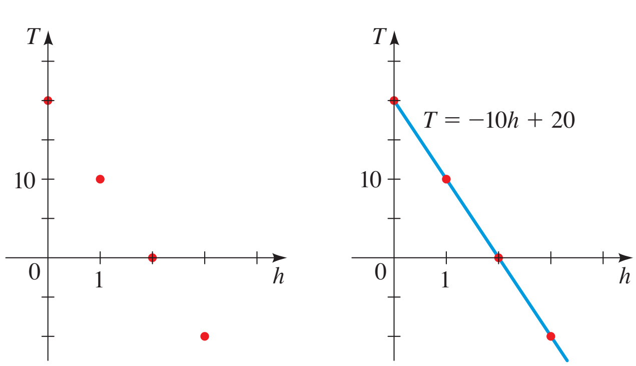
(b) Find the equation of the line that fits the data, and graph the line.

(c) Use the equation you found in part (b) to estimate the temperature 2.5 km above the ground.

(d) What does the slope of the line represent?

SOLUTION

(a) The plot in Figure 19(a) shows that the points appear to lie along a line. To verify this, we check that the slope between any two points in the data is the same. The slope between the points $(0, 20)$ and $(1, 10)$ is -10. Similarly, the slope between $(1, 10)$ and $(2, 0)$ is also -10. You can check that the slope between any two points in the table is -10. Thus the points lie along a line.



(a) Plot of data (b) Line that fits the data

FIGURE 19

(b) To find the equation of the line, we use the slope-intercept form. Let h represent the height, and let T represent the temperature. The slope of the line is -10, and the T-intercept is 20. So the equation of the line is

$$T = -10h + 20$$

A graph of the line together with a plot of the given data is shown in Figure 19(b).

(c) To estimate the temperature 2.5 km above the ground, we replace h by 2.5:

$$T = -10(2.5) + 20 = -5$$

So the temperature is $-5°C$ at 2.5 km above the ground.

(d) The slope is $m = -10°\text{C}/\text{km}$. The slope represents the rate of change of temperature with respect to height above the ground. So the temperature decreases 10°C for each kilometer of height.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **81** AND **103**

1.3 EXERCISES

CONCEPTS

1. We find the "steepness," or slope, of a line passing through two points by dividing the difference in the ____-coordinates of these points by the difference in the ____-coordinates. So the line passing through the points $(0, 1)$ and $(2, 5)$ has slope ________.

2. A line has the equation $y = 3x + 2$.
   (a) This line has slope ________.
   (b) Any line parallel to this line has slope ________.
   (c) Any line perpendicular to this line has slope ________.

3. The point-slope form of the equation of the line with slope 3 passing through the point $(1, 2)$ is ________.

4. The slope of a horizontal line is ________. The equation of the horizontal line passing through $(2, 3)$ is ________.

5. The slope of a vertical line is ________. The equation of the vertical line passing through $(2, 3)$ is ________.

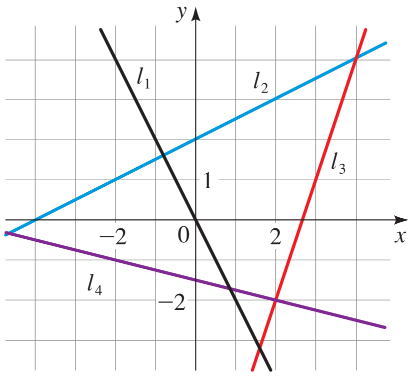
6. For the linear equation $2x + 3y - 12 = 0$, the x-intercept is ________ and the y-intercept is ________. The equation in slope-intercept form is $y =$ ________.

SKILLS

7–14 ■ Find the slope of the line through P and Q.

7. $P(3, 0)$, $Q(0, 4)$
8. $P(0, 1)$, $Q(-3, 0)$
9. $P(-1, 4)$, $Q(5, 2)$
10. $P(1, -1)$, $Q(10, 3)$
11. $P(2, 4)$, $Q(4, 3)$
12. $P(2, -5)$, $Q(-4, 3)$
13. $P(1, -3)$, $Q(-1, 6)$
14. $P(-1, -4)$, $Q(6, 0)$

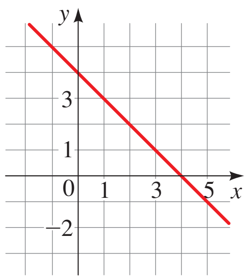
15. Find the slopes of the lines l_1, l_2, l_3, and l_4 in the figure below.



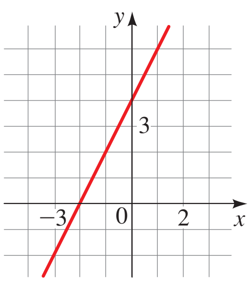
16. (a) Sketch lines through $(0, 0)$ with slopes $1, 0, \frac{1}{2}, 2,$ and -1.
    (b) Sketch lines through $(0, 0)$ with slopes $\frac{1}{3}, \frac{1}{2}, -\frac{1}{3},$ and 3.

17–20 ■ Find an equation for the line whose graph is sketched.

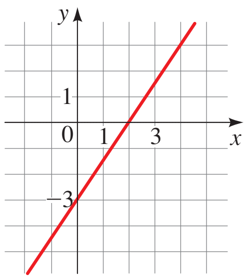
17.



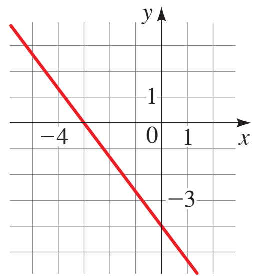
18.



19.



20.



21–48 ■ Find an equation of the line that satisfies the given conditions.

21. Slope 3; y-intercept -2

22. Slope $\frac{2}{5}$; y-intercept 4

23. Through $(2, 3)$; slope 5

24. Through $(-2, 4)$; slope -1

25. Through $(1, 7)$; slope $\frac{2}{3}$

26. Through $(-3, -5)$; slope $-\frac{7}{2}$

27. Through $(2, 1)$ and $(1, 6)$

28. Through $(-1, -2)$ and $(4, 3)$

29. Through $(-2, 5)$ and $(-1, -3)$

30. Through $(1, 7)$ and $(4, 7)$

31. x-intercept 1; y-intercept -3

32. x-intercept -8; y-intercept 6

33. Through $(1, 3)$; slope 0

34. Through $(-1, 4)$; slope undefined

35. Through $(2, -1)$; slope undefined

36. Through $(5, 1)$; slope 0

37. Through $(1, 2)$; parallel to the line $y = 3x - 5$

38. Through $(-3, 2)$; perpendicular to the line $y = -\frac{1}{2}x + 7$

39. Through $(4, 5)$; parallel to the x-axis

40. Through $(4, 5)$; parallel to the y-axis

41. Through $(1, -6)$; parallel to the line $x + 2y = 6$

42. y-intercept 6; parallel to the line $2x + 3y + 4 = 0$

43. Through $(-1, 2)$; parallel to the line $x = 5$

44. Through $(2, 6)$; perpendicular to the line $y = 1$

45. Through $(-1, -2)$; perpendicular to the line $2x + 5y + 8 = 0$

46. Through $(\frac{1}{2}, -\frac{2}{3})$; perpendicular to the line $4x - 8y = 1$

47. Through $(1, 7)$; parallel to the line passing through $(2, 5)$ and $(-2, 1)$

48. Through $(-2, -11)$; perpendicular to the line passing through $(1, 1)$ and $(5, -1)$

49. **(a)** Sketch the line with slope $\frac{3}{2}$ that passes through the point $(-2, 1)$.
(b) Find an equation for this line.

50. **(a)** Sketch the line with slope -2 that passes through the point $(4, -1)$.
(b) Find an equation for this line.

51–54 ■ Use a graphing device to graph the given family of lines in the same viewing rectangle. What do the lines have in common?

51. $y = -2x + b$ for $b = 0, \pm 1, \pm 3, \pm 6$

52. $y = mx - 3$ for $m = 0, \pm 0.25, \pm 0.75, \pm 1.5$

53. $y = m(x - 3)$ for $m = 0, \pm 0.25, \pm 0.75, \pm 1.5$

54. $y = 2 + m(x + 3)$ for $m = 0, \pm 0.5, \pm 1, \pm 2, \pm 6$

55–68 ■ Find the slope and y-intercept of the line, and draw its graph.

55. $y = 3 - x$

56. $y = \frac{2}{3}x - 2$

57. $-2x + y = 7$

58. $\frac{1}{2}x + y = 4$

59. $x + 3y = 0$

60. $2x - 5y = 0$

61. $4x + 5y = 10$

62. $3x - 4y = 12$

63. $-3x - 5y + 30 = 0$

64. $\frac{1}{2}x - \frac{1}{3}y + 1 = 0$

65. $y = 4$

66. $x = -5$

67. $x = 3$

68. $y = -2$

69–74 ■ Find the x- and y-intercepts of the line, and draw its graph.

69. $5x + 2y - 10 = 0$

70. $6x - 7y - 42 = 0$

71. $\frac{1}{2}x - \frac{1}{3}y + 1 = 0$

72. $\frac{1}{3}x - \frac{1}{5}y - 2 = 0$

73. $y = 6x + 4$

74. $y = -4x - 10$

75–80 ■ The equations of two lines are given. Determine whether the lines are parallel, perpendicular, or neither.

75. $y = 2x + 3$; $2y - 4x - 5 = 0$

76. $y = \frac{1}{2}x + 4$; $2x + 4y = 1$

77. $-3x + 4y = 4$; $4x + 3y = 5$

78. $2x - 3y = 10$; $3y - 2x - 7 = 0$

79. $7x - 3y = 2$; $9y + 21x = 1$

80. $6y - 2x = 5$; $2y + 6x = 1$

81–82 ■ A table is given. **(a)** Plot the points in the table. Do the points appear to lie along a line? Prove your answer. **(b)** Find the equation of the line that fits the data, and graph the line.

81.

x	y
0	10
2	16
4	22
6	28

82.

x	y
0	100
5	80
10	60
15	40

83. Use slopes to show that $A(1, 1)$, $B(7, 4)$, $C(5, 10)$, and $D(-1, 7)$ are vertices of a parallelogram.

84. Use slopes to show that $A(-3, -1)$, $B(3, 3)$, and $C(-9, 8)$ are vertices of a right triangle.

85. Use slopes to show that $A(1, 1)$, $B(11, 3)$, $C(10, 8)$, and $D(0, 6)$ are vertices of a rectangle.

86. Use slopes to determine whether the given points are collinear (lie on a line).
(a) $(1, 1), (3, 9), (6, 21)$
(b) $(-1, 3), (1, 7), (4, 15)$

87. Find an equation of the perpendicular bisector of the line segment joining the points $A(1, 4)$ and $B(7, -2)$.

88. Find the area of the triangle formed by the coordinate axes and the line

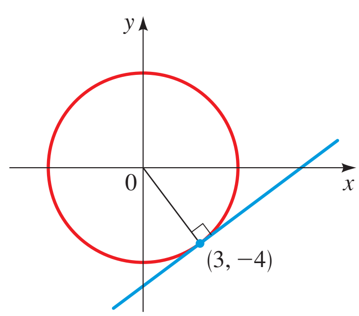
$$2y + 3x - 6 = 0$$

89. (a) Show that if the x- and y-intercepts of a line are nonzero numbers a and b, then the equation of the line can be written in the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

This is called the **two-intercept form** of the equation of a line.
(b) Use part (a) to find an equation of the line whose x-intercept is 6 and whose y-intercept is -8.

90. (a) Find an equation for the line tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$. (See the figure.)
(b) At what other point on the circle will a tangent line be parallel to the tangent line in part (a)?



APPLICATIONS

91. Grade of a Road West of Albuquerque, New Mexico, Route 40 eastbound is straight and makes a steep descent toward the city. The highway has a 6% grade, which means that its slope is $-\frac{6}{100}$. Driving on this road, you notice from elevation signs that you have descended a distance of 1000 ft. What is the change in your horizontal distance?

92. Global Warming Some scientists believe that the average surface temperature of the world has been rising steadily. The average surface temperature can be modeled by

$$T = 0.02t + 15.0$$

where T is temperature in °C and t is years since 1950.
(a) What do the slope and T-intercept represent?
(b) Use the equation to predict the average global surface temperature in 2050.

93. Drug Dosages If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a, pharmacists use the equation

$$c = 0.0417D(a + 1)$$

Suppose the dosage for an adult is 200 mg.
(a) Find the slope. What does it represent?
(b) What is the dosage for a newborn?

94. Flea Market The manager of a weekend flea market knows from past experience that if she charges x dollars for a rental space at the flea market, then the number y of spaces she can rent is given by the equation $y = 200 - 4x$.
(a) Sketch a graph of this linear equation. (Remember that the rental charge per space and the number of spaces rented must both be nonnegative quantities.)
(b) What do the slope, the y-intercept, and the x-intercept of the graph represent?

95. Production Cost A small-appliance manufacturer finds that if he produces x toaster ovens in a month, his production cost is given by the equation

$$y = 6x + 3000$$

(where y is measured in dollars).
(a) Sketch a graph of this linear equation.
(b) What do the slope and y-intercept of the graph represent?

96. Temperature Scales The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the equation $F = \frac{9}{5}C + 32$.
(a) Complete the table to compare the two scales at the given values.
(b) Find the temperature at which the scales agree. [*Hint:* Suppose that a is the temperature at which the scales agree. Set $F = a$ and $C = a$. Then solve for a.]

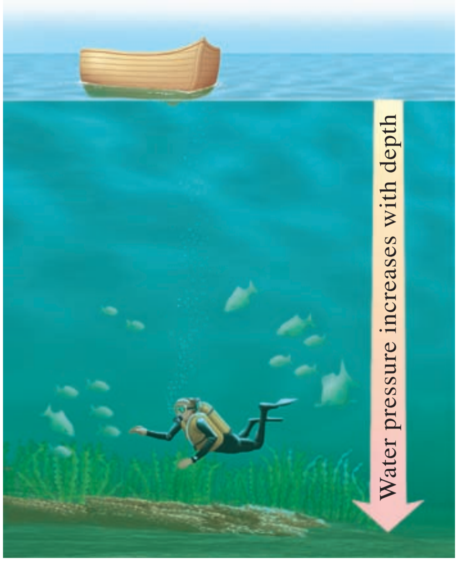
C	F
−30°	
−20°	
−10°	
0°	
	50°
	68°
	86°

97. Crickets and Temperature Biologists have observed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 120 chirps per minute at 70°F and 168 chirps per minute at 80°F.

(a) Find the linear equation that relates the temperature t and the number of chirps per minute n.
(b) If the crickets are chirping at 150 chirps per minute, estimate the temperature.

98. Depreciation A small business buys a computer for \$4000. After 4 years the value of the computer is expected to be \$200. For accounting purposes the business uses *linear depreciation* to assess the value of the computer at a given time. This means that if V is the value of the computer at time t, then a linear equation is used to relate V and t.
(a) Find a linear equation that relates V and t.
(b) Sketch a graph of this linear equation.
(c) What do the slope and V-intercept of the graph represent?
(d) Find the depreciated value of the computer 3 years from the date of purchase.

99. Pressure and Depth At the surface of the ocean the water pressure is the same as the air pressure above the water, 15 lb/in^2. Below the surface the water pressure increases by 4.34 lb/in^2 for every 10 ft of descent.
(a) Find an equation for the relationship between pressure and depth below the ocean surface.
(b) Sketch a graph of this linear equation.
(c) What do the slope and y-intercept of the graph represent?
(d) At what depth is the pressure 100 lb/in^2?



100. Distance, Speed, and Time Jason and Debbie leave Detroit at 2:00 P.M. and drive at a constant speed, traveling west on I-90. They pass Ann Arbor, 40 mi from Detroit, at 2:50 P.M.
(a) Express the distance traveled in terms of the time elapsed.
(b) Draw the graph of the equation in part (a).
(c) What is the slope of this line? What does it represent?

101. Cost of Driving The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May her driving cost was \$380 for 480 mi and in June her cost was \$460 for 800 mi. Assume that there is a linear relationship between the monthly cost C of driving a car and the distance driven d.
(a) Find a linear equation that relates C and d.
(b) Use part (a) to predict the cost of driving 1500 mi per month.
(c) Draw the graph of the linear equation. What does the slope of the line represent?
(d) What does the y-intercept of the graph represent?
(e) Why is a linear relationship a suitable model for this situation?

102. Manufacturing Cost The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.
(a) Assuming that the relationship between cost and the number of chairs produced is linear, find an equation that expresses this relationship. Then graph the equation.
(b) What is the slope of the line in part (a), and what does it represent?
(c) What is the y-intercept of this line, and what does it represent?

103. Boiling Point Most high-altitude hikers know that cooking takes longer at higher elevations. This is because the atmospheric pressure decreases as the elevation increases, causing water to boil at a lower temperature. The table below gives data for the boiling point of water at different elevations above sea level.
(a) Plot the points in the table. Do the points appear to lie along a line? Prove your answer.
(b) Find the equation of the line that fits the data, and graph the line.
(c) Use the equation you found in part (b) to estimate the boiling point at the peak of Mount Kilimanjaro, 19,340 ft above sea level.
(d) What does the slope of the line represent?

h (× 1000 ft)	T (°F)
0	212.0
1	210.2
2	208.4
3	206.6
4	204.8

104. Salary A woman is hired as CEO of a small company and is offered a salary of \$150,000 for the first year. In addition, she is promised regular salary increases. The table below shows her potential salary (in thousands of dollars) for the first few years that she works for the company.
(a) Plot the points in the table. Do the points appear to lie along a line? Prove your answer.
(b) Find the equation of the line that fits the data, and graph the line.
(c) Use the equation you found in part (b) to determine her potential salary for year 9.
(d) What does the slope of the line represent?

Year t	Salary S (× \$1000)
0	150
1	160
2	170
3	180
4	190

DISCOVERY ▪ DISCUSSION ▪ WRITING

105. What Does the Slope Mean? Suppose that the graph of the outdoor temperature over a certain period of time is a line. How is the weather changing if the slope of the line is positive? If it is negative? If it is zero?

106. Collinear Points Suppose you are given the coordinates of three points in the plane and you want to see whether they lie on the same line. How can you do this using slopes? Using the Distance Formula? Can you think of another method?

1.4 Solving Equations Graphically

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Solve Equations Graphically

GET READY **Prepare for this section by reviewing how to solve equations (Section P.8) and how to graph with a graphing calculator (Appendix B and Appendix C).**

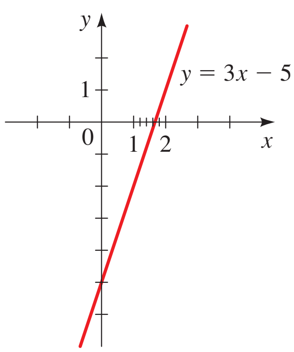


FIGURE 1

In Section P.8 we learned how to solve one-variable equations algebraically. Let's solve the equation

$$0 = 3x - 5$$

To solve this equation **algebraically**, we view x as an *unknown* and then use the rules of algebra to isolate it on one side of the equation. For this equation we add 5 and then divide by 3 to obtain the solution $x = \frac{5}{3}$. To solve this equation **graphically**, we view x as a *variable* and sketch the graph of the equation

$$y = 3x - 5$$

Different values for x give different values for y. Our goal is to find the value of x for which $y = 0$. So the solution is the x-intercept of the graph. From the graph in Figure 1 we see that $y = 0$ when $x \approx 1.7$, so the solution is $x \approx 1.7$.

The two methods for solving equations are summarized below.

SOLVING AN EQUATION

Algebraic Method

Use the rules of algebra to isolate the unknown x on one side of the equation.

Example: $2x = 6 - x$

$$3x = 6 \quad \text{Add } x$$

$$x = 2 \quad \text{Divide by 3}$$

The solution is $x = 2$.

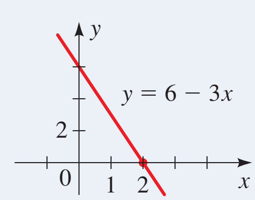
Graphical Method

Move all terms to one side, and set the result equal to y. Sketch the graph to find the value of x where $y = 0$.

Example: $2x = 6 - x$

$$0 = 6 - 3x$$

Set $y = 6 - 3x$ and graph.



From the graph, the solution is $x \approx 2$.

The advantage of the algebraic method is that it gives exact answers. Also, the process of unraveling the equation to arrive at the answer helps us to understand the algebraic structure of the equation. On the other hand, for many equations it is difficult or impossible to isolate x.

The graphical method gives a numerical approximation to the answer. This is an advantage when a numerical answer is desired. (For example, an engineer might find an answer expressed as $x \approx 2.6$ more immediately useful than $x = \sqrt{7}$.) Also, graphing an equation helps us to visualize how the solution is related to other values of the variable.

The *Discovery Project* referenced on page 297 describes a numerical method for solving equations.

▼ Solving Equations Graphically

To solve a one-variable equation graphically, we need to use a graphing calculator. We use the calculator to get a graph of the corresponding two-variable equation. The following examples illustrate the method.

EXAMPLE 1 | Solving an Equation Algebraically and Graphically

Solve the equation algebraically and graphically: $x^2 - 7 = 0$

SOLUTION 1: Algebraic

We isolate x^2 on one side of the equal sign and take square roots.

$$\begin{aligned} x^2 - 7 &= 0 \\ x^2 &= 7 && \text{Add 7} \\ x &= \pm\sqrt{7} && \text{Take square roots} \end{aligned}$$

The solutions are $x = \sqrt{7}$ and $x = -\sqrt{7}$

SOLUTION 2: Graphical

We graph the equation $y = x^2 - 7$ and determine the x-intercepts from the graph. From Figure 2 we see that the graph crosses the x-axis at $x \approx 2.6$ and $x \approx -2.6$.

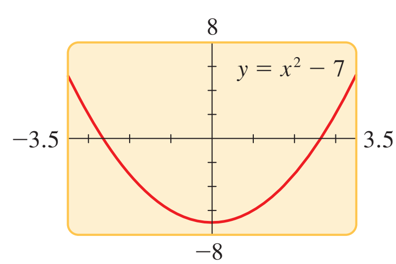


FIGURE 2

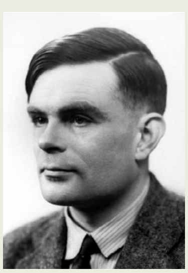
PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **7** ■

EXAMPLE 2 | Another Graphical Method

Solve the equation algebraically and graphically: $5 - 3x = 8x - 20$

SOLUTION 1: Algebraic

$$\begin{aligned} 5 - 3x &= 8x - 20 \\ -3x &= 8x - 25 && \text{Subtract 5} \\ -11x &= -25 && \text{Subtract } 8x \\ x &= \frac{-25}{-11} = 2\tfrac{3}{11} && \text{Divide by } -11 \text{ and simplify} \end{aligned}$$



The Granger Collection, New York

ALAN TURING (1912–1954) was at the center of two pivotal events of the 20th century: World War II and the invention of computers. At the age of 23 Turing made his mark on mathematics by solving an important problem in the foundations of mathematics that had been posed by David Hilbert at the 1928 International Congress of Mathematicians (see page 502). In this research he invented a theoretical machine, now called a Turing machine, which was the inspiration for modern digital computers. During World War II Turing was in charge of the British effort to decipher secret German codes. His complete success in this endeavor played a decisive role in the Allies' victory. To carry out the numerous logical steps that are required to break a coded message, Turing developed decision procedures similar to modern computer programs. After the war he helped to develop the first electronic computers in Britain. He also did pioneering work on artificial intelligence and computer models of biological processes. At the age of 42 Turing died of poisoning after eating an apple that had mysteriously been laced with cyanide.

SOLUTION 2: Graphical

We could move all terms to one side of the equal sign, set the result equal to y, and graph the resulting equation. But to avoid all this algebra, we graph two equations instead:

$$y_1 = 5 - 3x \qquad \text{and} \qquad y_2 = 8x - 20$$

See Appendix C, *Using a TI-83/84 Graphing Calculator,* for specific instructions.

The solution of the original equation will be the value of x that makes y_1 equal to y_2; that is, the solution is the x-coordinate of the intersection point of the two graphs. Using the TRACE feature or the `intersect` command on a graphing calculator, we see from Figure 3 that the solution is $x \approx 2.27$.

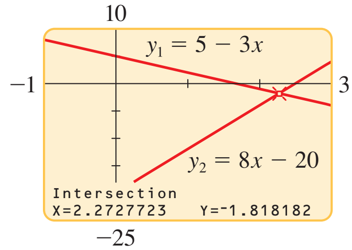


FIGURE 3

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **3**

In the next example we use the graphical method to solve an equation that is extremely difficult to solve algebraically.

EXAMPLE 3 | Solving an Equation in an Interval

Solve the equation

$$x^3 - 6x^2 + 9x = \sqrt{x}$$

in the interval $[1, 6]$.

SOLUTION We are asked to find all solutions x that satisfy $1 \le x \le 6$, so we will graph the equation in a viewing rectangle for which the x-values are restricted to this interval:

$$x^3 - 6x^2 + 9x = \sqrt{x}$$

$$x^3 - 6x^2 + 9x - \sqrt{x} = 0 \qquad \text{Subtract } \sqrt{x}$$

We can also use the `zero` command to find the solutions, as shown in Figures 4(a) and 4(b).

Figure 4 shows the graph of the equation $y = x^3 - 6x^2 + 9x - \sqrt{x}$ in the viewing rectangle $[1, 6]$ by $[-5, 5]$. There are two x-intercepts in this viewing rectangle; zooming in, we see that the solutions are $x \approx 2.18$ and $x \approx 3.72$.

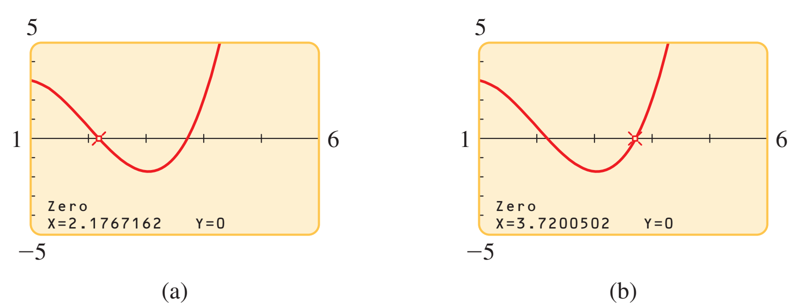


FIGURE 4

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **15**

The equation in Example 3 actually has four solutions. You are asked to find the other two in Exercise 32.

EXAMPLE 4 | Intensity of Light

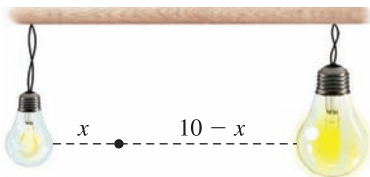


FIGURE 5

Two light sources are 10 m apart. One is three times as intense as the other. The light intensity L (in lux) at a point x meters from the weaker source is given by

$$L = \frac{10}{x^2} + \frac{30}{(10 - x)^2}$$

(See Figure 5.) Find the points at which the light intensity is 4 lux.

SOLUTION We need to solve the equation

$$4 = \frac{10}{x^2} + \frac{30}{(10 - x)^2}$$

The graphs of

$$y_1 = 4 \qquad \text{and} \qquad y_2 = \frac{10}{x^2} + \frac{30}{(10 - x)^2}$$

are shown in Figure 6. Zooming in (or using the `intersect` command), we find two solutions, $x \approx 1.67431$ and $x \approx 7.1927193$. So the light intensity is 4 lux at the points that are 1.67 m and 7.19 m from the weaker source.

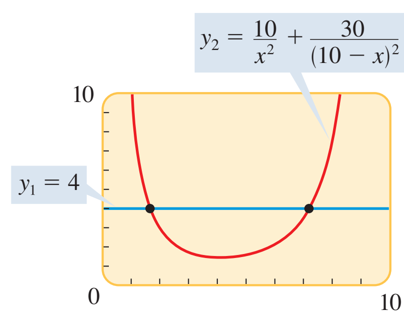
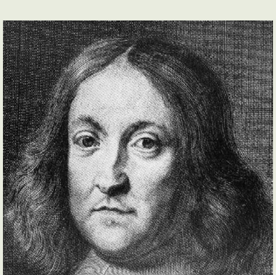


FIGURE 6

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **33** ■



PIERRE DE FERMAT (1601–1665) was a French lawyer who became interested in mathematics at the age of 30. Because of his job as a magistrate, Fermat had little time to write complete proofs of his discoveries and often wrote them in the margin of whatever book he was reading at the time. After his death, his copy of Diophantus' *Arithmetica* (see page 47) was found to contain a particularly tantalizing comment. Where Diophantus discusses the solutions of $x^2 + y^2 = z^2$ (for example, $x = 3$, $y = 4$, and $z = 5$), Fermat states in the margin that for $n \geq 3$ there are no natural number solutions to the equation $x^n + y^n = z^n$. In other words, it's impossible for a cube to equal the sum of two cubes, a fourth power to equal the sum of two fourth powers, and so on. Fermat writes "I have discovered a truly wonderful proof for this but the margin is too small to contain it." All the other margin comments in Fermat's copy of *Arithmetica* have been proved. This one, however, remained unproved, and it came to be known as "Fermat's Last Theorem."

In 1994, Andrew Wiles of Princeton University announced a proof of Fermat's Last Theorem, an astounding 350 years after it was conjectured. His proof is one of the most widely reported mathematical results in the popular press.

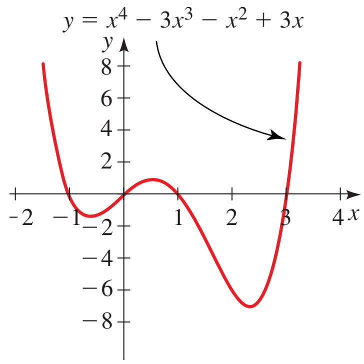
1.4 EXERCISES

CONCEPTS

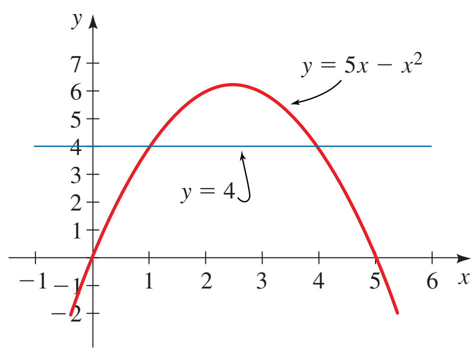
1. A graph of $y = x^4 - 3x^3 - x^2 + 3x$ is shown in the figure below.

(a) The solutions of the equation $x^4 - 3x^3 - x^2 + 3x = 0$ are the ________-intercepts of the graph of $y = x^4 - 3x^3 - x^2 + 3x$.

(b) Use the graph to find the solutions of the equation $x^4 - 3x^3 - x^2 + 3x = 0$.



2. The figure shows the graphs of $y = 5x - x^2$ and $y = 4$. Use the graphs to find the solutions of the equation $5x - x^2 = 4$.



SKILLS

3–14 ■ Solve the equation both algebraically and graphically.

3. $x - 4 = 5x + 12$

4. $\frac{1}{2}x - 3 = 6 + 2x$

5. $\dfrac{2}{x} + \dfrac{1}{2x} = 7$

6. $\dfrac{4}{x+2} - \dfrac{6}{2x} = \dfrac{5}{2x+4}$

7. $x^2 - 32 = 0$

8. $x^3 + 16 = 0$

9. $x^2 + 9 = 0$

10. $x^2 + 3 = 2x$

11. $16x^4 = 625$

12. $2x^5 - 243 = 0$

13. $(x - 5)^4 - 80 = 0$

14. $6(x + 2)^5 = 64$

15–26 ■ Solve the equation graphically in the given interval. State each answer rounded to two decimals.

15. $x^2 - 7x + 12 = 0$; $[0, 6]$

16. $x^2 - 0.75x + 0.125 = 0$; $[-2, 2]$

17. $x^3 - 6x^2 + 11x - 6 = 0$; $[-1, 4]$

18. $16x^3 + 16x^2 = x + 1$; $[-2, 2]$

19. $\sqrt{5 - x} + 1 = x - 2$; $[2, 5]$

20. $2x + \sqrt{x + 1} = 8$; $[1, 4]$

21. $x - \sqrt{x + 1} = 0$; $[-1, 5]$

22. $1 + \sqrt{x} = \sqrt{1 + x^2}$; $[-1, 5]$

23. $x - 5\sqrt{x} + 6 = 0$; $[2, 11]$

24. $\dfrac{1}{x^3} + \dfrac{4}{x^2} + \dfrac{4}{x} = 0$; $[-1, -0.25]$

25. $x^{1/3} - x = 0$; $[-3, 3]$

26. $x^{1/2} + x^{1/3} - x = 0$; $[-1, 5]$

27–30 ■ Find all real solutions of the equation, rounded to two decimals.

27. $x^3 - 2x^2 - x - 1 = 0$

28. $x^4 = 16 - x^3$

29. $x(x - 1)(x + 2) = \frac{1}{6}x$

30. $x^4 - 8x^2 + 2 = 0$

31. In Example 2 we solved the equation $5 - 3x = 8x - 20$ by drawing graphs of two equations. Solve the equation by drawing a graph of only one equation (as in Example 1). Compare your answer to the one obtained in Example 2.

32. In Example 3 we found two solutions of the equation $x^3 - 6x^2 + 9x = \sqrt{x}$, the solutions that lie between 1 and 6. Find two more solutions, rounded to two decimals.

APPLICATIONS

33. Estimating Profit An appliance manufacturer estimates that the profit y (in dollars) generated by producing x cooktops per month is given by the equation

$$y = 10x + 0.5x^2 - 0.001x^3 - 5000$$

where $0 \le x \le 450$.

(a) Graph the equation.

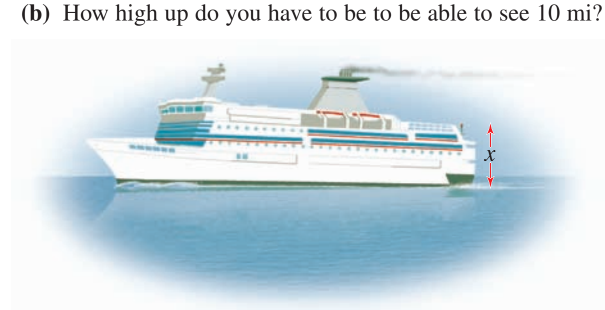
(b) How many cooktops must be produced to begin generating a profit?

34. How Far Can You See? If you stand on a ship in a calm sea, then your height x (in feet) above sea level is related to the farthest distance y (in miles) that you can see by the equation

$$y = \sqrt{1.5x + \left(\frac{x}{5280}\right)^2}$$

(a) Graph the equation for $0 \le x \le 100$.

(b) How high up do you have to be to be able to see 10 mi?



DISCOVERY ■ DISCUSSION ■ WRITING

35. Algebraic and Graphical Solution Methods Write a short essay comparing the algebraic and graphical methods for solving equations. Make up your own examples to illustrate the advantages and disadvantages of each method.

1.5 Modeling with Equations

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Make equations that model real-world situations ▶ Solve problems about interest ▶ Solve problems about areas and lengths ▶ Solve problems about mixtures and concentrations ▶ Solve problems about the time needed to do a job ▶ Solve problems about distance, speed, and time

Many problems in the sciences, economics, finance, medicine, and numerous other fields can be translated into algebra problems; this is one reason that algebra is so useful. In this section we use equations as mathematical models to solve real-life problems.

▼ Making and Using Models

We will use the following guidelines to help us set up equations that model situations described in words. To show how the guidelines can help you to set up equations, we note them as we work each example in this section.

GUIDELINES FOR MODELING WITH EQUATIONS

1. **Identify the Variable.** Identify the quantity that the problem asks you to find. This quantity can usually be determined by a careful reading of the question that is posed at the end of the problem. Then **introduce notation** for the variable (call it x or some other letter).
2. **Translate from Words to Algebra.** Read each sentence in the problem again, and express all the quantities mentioned in the problem in terms of the variable you defined in Step 1. To organize this information, it is sometimes helpful to **draw a diagram** or **make a table**.
3. **Set Up the Model.** Find the crucial fact in the problem that gives a relationship between the expressions you listed in Step 2. **Set up an equation** (or **model**) that expresses this relationship.
4. **Solve the Equation and Check Your Answer.** Solve the equation, check your answer, and express it as a sentence that answers the question posed in the problem.

The following example illustrates how these guidelines are used to translate a "word problem" into the language of algebra.

EXAMPLE 1 | Renting a Car

A car rental company charges \$30 a day and 15¢ a mile for renting a car. Helen rents a car for two days, and her bill comes to \$108. How many miles did she drive?

SOLUTION 1: Algebraic

Identify the variable. We are asked to find the number of miles Helen has driven. So we let

$$x = \text{number of miles driven}$$

Translate from words to algebra. Now we translate all the information given in the problem into the language of algebra:

In Words	In Algebra
Number of miles driven	x
Mileage cost (at \$0.15 per mile)	$0.15x$
Daily cost (at \$30 per day)	$2(30)$

Set up the model. Now we set up the model:

$$\boxed{\text{mileage cost}} + \boxed{\text{daily cost}} = \boxed{\text{total cost}}$$

$$0.15x + 2(30) = 108$$

Solve. Now we solve for x:

$$0.15x = 48 \qquad \text{Subtract 60}$$

$$x = \frac{48}{0.15} \qquad \text{Divide by 0.15}$$

$$x = 320 \qquad \text{Calculator}$$

CHECK YOUR ANSWER

total cost $=$ mileage cost $+$ daily cost

$= 0.15(320) + 2(30)$

$= 108$ ✓

Helen drove her rental car 320 miles.

SOLUTION 2: Graphical

Let x be as in Solution 1, and let y be the rental cost for driving x miles over two days. So

$$y = 0.15x + 60$$

We want to find the value of x for which $y = 108$. The graphs of $y = 0.15x + 60$ and $y = 108$ are shown in Figure 1. From the figure we see that the graphs intersect where $x = 320$. So Helen drove her rental car 320 miles.

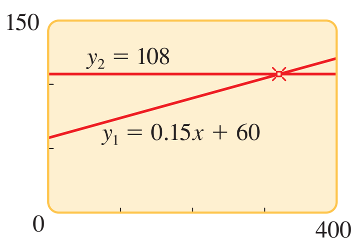


FIGURE 1

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **21**

In the examples and exercises that follow, we construct equations that model problems in many different real-life situations. We will solve the equations that arise algebraically, but of course they can be solved graphically as well.

Problems About Interest

When you borrow money from a bank or when a bank "borrows" your money by keeping it for you in a savings account, the borrower in each case must pay for the privilege of using the money. The fee that is paid is called **interest**. The most basic type of interest is **simple interest**, which is just an annual percentage of the total amount borrowed or deposited. The amount of a loan or deposit is called the **principal** P. The annual percentage paid for the use of this money is the **interest rate** r. We will use the variable t to stand for the number of years that the money is on deposit and the variable I to stand for the total interest earned. The following **simple interest formula** gives the amount of interest I earned when a principal P is deposited for t years at an interest rate r.

$$\boxed{I = Prt}$$

When using this formula, remember to convert r from a percentage to a decimal. For example, in decimal form, 5% is 0.05. So at an interest rate of 5%, the interest paid on a \$1000 deposit over a 3-year period is $I = Prt = 1000(0.05)(3) = \$150$.

BHASKARA (born 1114) was an Indian mathematician, astronomer, and astrologer. Among his many accomplishments was an ingenious proof of the Pythagorean Theorem. (See *Focus on Problem Solving* 5, Problem 12, at the book companion website **www.stewartmath.com**.) His important mathematical book *Lilavati* [*The Beautiful*] consists of algebra problems posed in the form of stories to his daughter Lilavati. Many of the problems begin "Oh beautiful maiden, suppose . . ." The story is told that using astrology, Bhaskara had determined that great misfortune would befall his daughter if she married at any time other than at a certain hour of a certain day. On her wedding day, as she was anxiously watching the water clock, a pearl fell unnoticed from her headdress. It stopped the flow of water in the clock, causing her to miss the opportune moment for marriage. Bhaskara's *Lilavati* was written to console her.

EXAMPLE 2 | Interest on an Investment

Mary inherits \$100,000 and invests it in two certificates of deposit. One certificate pays 6% and the other pays $4\frac{1}{2}\%$ simple interest annually. If Mary's total interest is \$5025 per year, how much money is invested at each rate?

SOLUTION Identify the variable. The problem asks for the amount she has invested at each rate. So we let

$$x = \text{the amount invested at } 6\%$$

Translate from words to algebra. Since Mary's total inheritance is \$100,000, it follows that she invested $100{,}000 - x$ at $4\frac{1}{2}\%$. We translate all the information given into the language of algebra:

In Words	In Algebra
Amount invested at 6%	x
Amount invested at $4\frac{1}{2}\%$	$100{,}000 - x$
Interest earned at 6%	$0.06x$
Interest earned at $4\frac{1}{2}\%$	$0.045(100{,}000 - x)$

Set up the model. We use the fact that Mary's total interest is \$5025 to set up the model:

$$\boxed{\text{interest at } 6\%} + \boxed{\text{interest at } 4\tfrac{1}{2}\%} = \boxed{\text{total interest}}$$

$$0.06x + 0.045(100{,}000 - x) = 5025$$

Solve. Now we solve for x:

$$0.06x + 4500 - 0.045x = 5025 \qquad \text{Multiply}$$

$$0.015x + 4500 = 5025 \qquad \text{Combine the } x\text{-terms}$$

$$0.015x = 525 \qquad \text{Subtract 4500}$$

$$x = \frac{525}{0.015} = 35{,}000 \qquad \text{Divide by 0.015}$$

So Mary has invested \$35,000 at 6% and the remaining \$65,000 at $4\frac{1}{2}\%$.

CHECK YOUR ANSWER

$$\begin{aligned} \text{total interest} &= 6\% \text{ of } \$35{,}000 + 4\tfrac{1}{2}\% \text{ of } \$65{,}000 \\ &= \$2100 + \$2925 = \$5025 \quad ✓ \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **25** ■

Problems About Area or Length

When we use algebra to model a physical situation, we must sometimes use basic formulas from geometry. For example, we may need a formula for an area or a perimeter, or the formula that relates the sides of similar triangles, or the Pythagorean Theorem. Most of these formulas are listed in the inside back cover of this book. The next two examples use these geometric formulas to solve some real-world problems.

EXAMPLE 3 | Dimensions of a Garden

A square garden has a walkway 3 ft wide around its outer edge, as shown in Figure 2 on following page. If the area of the entire garden, including the walkway, is 18,000 ft^2, what are the dimensions of the planted area?

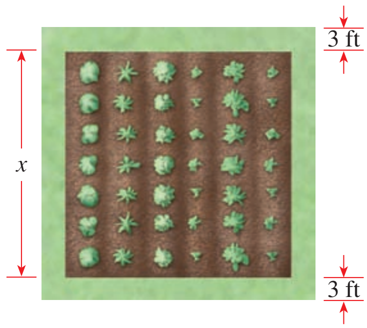


FIGURE 2

SOLUTION Identify the variable. We are asked to find the length and width of the planted area. So we let

$$x = \text{the length of the planted area}$$

Translate from words to algebra. Next, translate the information from Figure 2 into the language of algebra:

In Words	In Algebra
Length of planted area	x
Length of entire garden	$x + 6$
Area of entire garden	$(x + 6)^2$

Set up the model. We now set up the model:

$$\text{area of entire garden} = 18{,}000 \text{ ft}^2$$

$$(x + 6)^2 = 18{,}000$$

Solve. Now we solve for x:

$$x + 6 = \sqrt{18{,}000} \qquad \text{Take square roots}$$

$$x = \sqrt{18{,}000} - 6 \qquad \text{Subtract 6}$$

$$x \approx 128$$

The planted area of the garden is about 128 ft by 128 ft.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **43**

EXAMPLE 4 | Determining the Height of a Building Using Similar Triangles

A man who is 6 ft tall wishes to find the height of a certain four-story building. He measures its shadow and finds it to be 28 ft long, while his own shadow is $3\frac{1}{2}$ ft long. How tall is the building?

SOLUTION Identify the variable. The problem asks for the height of the building. So let

$$h = \text{the height of the building}$$

Translate from words to algebra. We use the fact that the triangles in Figure 3 on the next page are similar. Recall that for any pair of similar triangles the ratios of corresponding sides are equal. Now we translate these observations into the language of algebra:

In Words	In Algebra
Height of building	h
Ratio of height to base in large triangle	$\frac{h}{28}$
Ratio of height to base in small triangle	$\frac{6}{3.5}$

Set up the model. Since the large and small triangles are similar, we get the equation

$$\boxed{\text{ratio of height to base in large triangle}} = \boxed{\text{ratio of height to base in small triangle}}$$

$$\frac{h}{28} = \frac{6}{3.5}$$

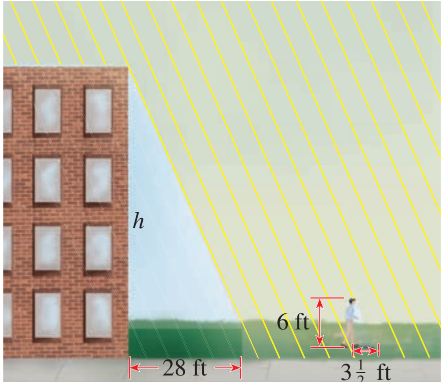


FIGURE 3

Solve. Now we solve for h:

$$h = \frac{6 \cdot 28}{3.5} = 48 \qquad \text{Multiply by 28}$$

So the building is 48 ft tall.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45**

Problems About Mixtures

Many real-world problems involve mixing different types of substances. For example, construction workers may mix cement, gravel, and sand; fruit juice from concentrate may involve mixing different types of juices. Problems involving mixtures and concentrations make use of the fact that if an amount x of a substance is dissolved in a solution with volume V, then the concentration C of the substance is given by

$$C = \frac{x}{V}$$

So if 10 g of sugar is dissolved in 5 L of water, then the sugar concentration is $C = 10/5 = 2$ g/L. Solving a mixture problem usually requires us to analyze the amount x of the substance that is in the solution. When we solve for x in this equation, we see that $x = CV$. Note that in many mixture problems the concentration C is expressed as a percentage, as in the next example.

EXAMPLE 5 | Mixtures and Concentration

A manufacturer of soft drinks advertises its orange soda as "naturally flavored," although it contains only 5% orange juice. A new federal regulation stipulates that to be called "natural," a drink must contain at least 10% fruit juice. How much pure orange juice must this manufacturer add to 900 gal of orange soda to conform to the new regulation?

SOLUTION Identify the variable. The problem asks for the amount of pure orange juice to be added. So let

x = the amount (in gallons) of pure orange juice to be added

Translate from words to algebra. In any problem of this type—in which two different substances are to be mixed—drawing a diagram helps us to organize the given information (see Figure 4).

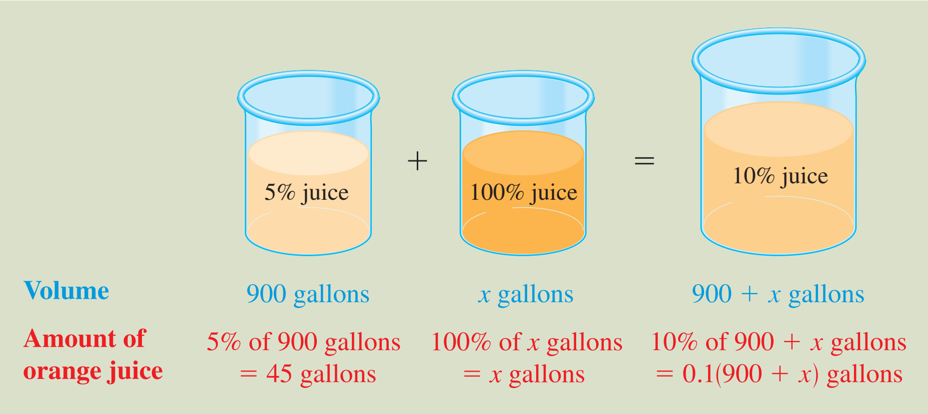


FIGURE 4

We now translate the information in the figure into the language of algebra:

In Words	In Algebra
Amount of orange juice to be added	x
Amount of the mixture	$900 + x$
Amount of orange juice in the first vat	$0.05(900) = 45$
Amount of orange juice in the second vat	$1 \cdot x = x$
Amount of orange juice in the mixture	$0.10(900 + x)$

Set up the model. To set up the model, we use the fact that the total amount of orange juice in the mixture is equal to the orange juice in the first two vats:

amount of orange juice in first vat	+	amount of orange juice in second vat	=	amount of orange juice in mixture

$$45 + x = 0.1(900 + x) \qquad \text{From Figure 4}$$

Solve. Now we solve for x:

$$45 + x = 90 + 0.1x \qquad \text{Distributive Property}$$

$$0.9x = 45 \qquad \text{Subtract } 0.1x \text{ and } 45$$

$$x = \frac{45}{0.9} = 50 \qquad \text{Divide by } 0.9$$

The manufacturer should add 50 gal of pure orange juice to the soda.

CHECK YOUR ANSWER

$$\text{amount of juice before mixing} = 5\% \text{ of } 900 \text{ gal} + 50 \text{ gal pure juice}$$

$$= 45 \text{ gal} + 50 \text{ gal} = 95 \text{ gal}$$

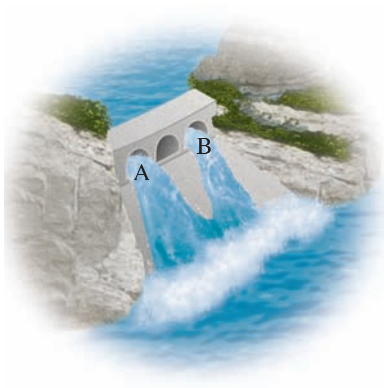
$$\text{amount of juice after mixing} = 10\% \text{ of } 950 \text{ gal} = 95 \text{ gal}$$

Amounts are equal. ✓

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **47** ■

▼ Problems About the Time Needed to Do a Job

When solving a problem that involves determining how long it takes several workers to complete a job, we use the fact that if a person or machine takes H time units to complete the task, then in one time unit the fraction of the task that has been completed is $1/H$. For example, if a worker takes 5 hours to mow a lawn, then in 1 hour the worker will mow 1/5 of the lawn.



EXAMPLE 6 | Time Needed to Do a Job

Because of an anticipated heavy rainstorm, the water level in a reservoir must be lowered by 1 ft. Opening spillway A lowers the level by this amount in 4 hours, whereas opening the smaller spillway B does the job in 6 hours. How long will it take to lower the water level by 1 ft if both spillways are opened?

SOLUTION Identify the variable. We are asked to find the time needed to lower the level by 1 ft if both spillways are open. So let

$x =$ the time (in hours) it takes to lower the water level by 1 ft if both spillways are open

Translate from words to algebra. Finding an equation relating x to the other quantities in this problem is not easy. Certainly x is not simply $4 + 6$, because that would mean that together the two spillways require longer to lower the water level than either spillway alone. Instead, *we look at the fraction of the job that can be done in 1 hour by each spillway*.

In Words	In Algebra
Time it takes to lower level 1 ft with A and B together	x h
Distance A lowers level in 1 h	$\frac{1}{4}$ ft
Distance B lowers level in 1 h	$\frac{1}{6}$ ft
Distance A and B together lower levels in 1 h	$\frac{1}{x}$ ft

Set up the model. Now we set up the model:

$$\text{fraction done by A} + \text{fraction done by B} = \text{fraction done by both}$$

$$\frac{1}{4} + \frac{1}{6} = \frac{1}{x}$$

Solve. Now we solve for x:

$$3x + 2x = 12 \qquad \text{Multiply by the LCD, } 12x$$

$$5x = 12 \qquad \text{Add}$$

$$x = \frac{12}{5} \qquad \text{Divide by 5}$$

It will take $2\frac{2}{5}$ hours, or 2 h 24 min, to lower the water level by 1 ft if both spillways are open.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **55** ■

▼ Problems About Distance, Rate, and Time

The next example deals with distance, rate (speed), and time. The formula to keep in mind here is

$$\boxed{\text{distance} = \text{rate} \times \text{time}}$$

where the rate is either the constant speed or average speed of a moving object. For example, driving at 60 mi/h for 4 hours takes you a distance of $60 \cdot 4 = 240$ mi.

EXAMPLE 7 | Distance, Speed, and Time

Bill left his house at 2:00 P.M. and rode his bicycle down Main Street at a speed of 12 mi/h. When his friend Mary arrived at his house at 2:10 P.M., Bill's mother told her the direction in which Bill had gone, and Mary cycled after him at a speed of 16 mi/h. At what time did Mary catch up with Bill?

SOLUTION Identify the variable. We are asked to find the time that it took Mary to catch up with Bill. Let

$$t = \text{the time (in hours) it took Mary to catch up with Bill}$$

Translate from words to algebra. In problems involving motion, it is often helpful to organize the information in a table, using the formula distance = rate × time. First we fill in the "Speed" column in the table, since we are told the speeds at which Mary and Bill cycled. Then we fill in the "Time" column. (Because Bill had a 10-minute, or $\frac{1}{6}$-hour head start, he cycled for $t + \frac{1}{6}$ hours.) Finally, we multiply these columns to calculate the entries in the "Distance" column.

	Distance (mi)	Speed (mi/h)	Time (h)
Mary	$16t$	16	t
Bill	$12\left(t + \frac{1}{6}\right)$	12	$t + \frac{1}{6}$

Set up the model. At the instant when Mary caught up with Bill, they had both cycled the same distance. We use this fact to set up the model for this problem:

$$\text{distance traveled by Mary} = \text{distance traveled by Bill}$$

$$16t = 12\left(t + \tfrac{1}{6}\right) \qquad \text{From table}$$

Solve. Now we solve for t:

$$16t = 12t + 2 \qquad \text{Distributive Property}$$

$$4t = 2 \qquad \text{Subtract } 12t$$

$$t = \tfrac{1}{2} \qquad \text{Divide by 4}$$

Mary caught up with Bill after cycling for half an hour, that is, at 2:40 P.M.

CHECK YOUR ANSWER

Bill traveled for $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ h, so

$$\text{distance Bill traveled} = 12 \text{ mi/h} \times \tfrac{2}{3} \text{ h} = 8 \text{ mi}$$

$$\text{distance Mary traveled} = 16 \text{ mi/h} \times \tfrac{1}{2} \text{ h} = 8 \text{ mi}$$

Distances are equal. ✓

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **59** ■

1.5 EXERCISES

CONCEPTS

1. Explain in your own words what it means for an equation to model a real-world situation, and give an example.

2. In the formula $I = Prt$ for simple interest, P stands for ________, r for ________, and t for ________.

3. Give a formula for the area of the geometric figure.

(a) A square of side x: $A =$ ________.

(b) A rectangle of length l and width w: $A =$ ________.

(c) A circle of radius r: $A =$ ________.

4. Balsamic vinegar contains 5% acetic acid, so a 32-oz bottle of balsamic vinegar contains ________ ounces of acetic acid.

5. A painter paints a wall in x hours, so the fraction of the wall that she paints in 1 hour is ________.

6. The formula $d = rt$ models the distance d traveled by an object moving at the constant rate r in time t. Find formulas for the following quantities.

$r =$ ________ $t =$ ________

SKILLS

7–20 ■ Express the given quantity in terms of the indicated variable.

7. The sum of three consecutive integers; $n =$ first integer of the three

8. The sum of three consecutive integers; $n =$ middle integer of the three

9. The sum of three consecutive even integers; $n =$ first integer of the three

10. The sum of the squares of two consecutive integers; $n =$ first integer of the two

11. The average of three test scores if the first two scores are 78 and 82; $s =$ third test score

12. The average of four quiz scores if each of the first three scores is 8; $q =$ fourth quiz score

13. The interest obtained after one year on an investment at $2\frac{1}{2}\%$ simple interest per year; $x =$ number of dollars invested

14. The total rent paid for an apartment if the rent is \$795 a month; $n =$ number of months

15. The area (in ft^2) of a rectangle that is five times as long as it is wide; $w =$ width of the rectangle (in ft)

16. The perimeter (in cm) of a rectangle that is 4 cm longer than it is wide; $w =$ width of the rectangle (in cm)

17. The time (in hours) it takes to travel a given distance at 55 mi/h; $d =$ given distance (in mi)

18. The distance (in mi) that a car travels in 45 min; $s =$ speed of the car (in mi/h)

19. The concentration (in oz/gal) of salt in a mixture of 3 gal of brine containing 25 oz of salt to which some pure water has been added; $x =$ volume of pure water added (in gal)

20. The value (in cents) of the change in a purse that contains twice as many nickels as pennies, four more dimes than nickels, and as many quarters as dimes and nickels combined; $p =$ number of pennies

APPLICATIONS

21. Renting a Truck A rental company charges \$65 a day and 20 cents a mile for renting a truck. Michael rents a truck for 3 days, and his bill comes to \$275. How many miles did he drive?

22. Cell Phone Costs A cell phone company charges a monthly fee of \$10 for the first 1000 text messages and 10 cents for each additional text message. Miriam's bill for text messages for the month of June is \$38.50. How many text messages did she send that month?

23. Average Linh has obtained scores of 82, 75, and 71 on her midterm algebra exams. If the final exam counts twice as much as a midterm, what score must she make on her final exam to get an average score of 80? (Assume that the maximum possible score on each test is 100.)

24. Average In a class of 25 students, the average score is 84. Six students in the class each received a maximum score of 100, and three students each received a score of 60. What is the average score of the remaining students?

25. Investments Phyllis invested \$12,000, a portion earning a simple interest rate of $4\frac{1}{2}\%$ per year and the rest earning a rate of 4% per year. After 1 year the total interest earned on these investments was \$525. How much money did she invest at each rate?

26. Investments If Ben invests \$4000 at 4% interest per year, how much additional money must he invest at $5\frac{1}{2}\%$ annual interest to ensure that the interest he receives each year is $4\frac{1}{2}\%$ of the total amount invested?

27. Investments What annual rate of interest would you have to earn on an investment of \$3500 to ensure receiving \$262.50 interest after 1 year?

28. Investments Jack invests \$1000 at a certain annual interest rate, and he invests another \$2000 at an annual rate that is one-half percent higher. If he receives a total of \$190 interest in 1 year, at what rate is the \$1000 invested?

29. Salaries An executive in an engineering firm earns a monthly salary plus a Christmas bonus of \$8500. If she earns a total of \$97,300 per year, what is her monthly salary?

30. Salaries A woman earns 15% more than her husband. Together they make \$69,875 per year. What is the husband's annual salary?

31. Overtime Pay Helen earns \$7.50 an hour at her job, but if she works more than 35 hours in a week, she is paid $1\frac{1}{2}$ times her regular salary for the overtime hours worked. One week her gross pay was \$352.50. How many overtime hours did she work that week?

32. Labor Costs A plumber and his assistant work together to replace the pipes in an old house. The plumber charges \$45 an hour for his own labor and \$25 an hour for his assistant's labor. The plumber works twice as long as his assistant on this job, and the labor charge on the final bill is \$4025. How long did the plumber and his assistant work on this job?

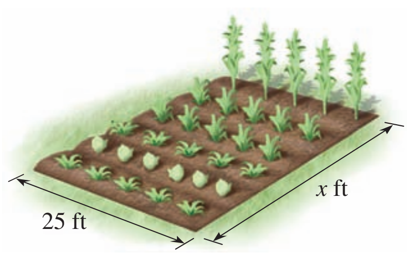
33. A Riddle A movie star, unwilling to give his age, posed the following riddle to a gossip columnist: "Seven years ago, I was eleven times as old as my daughter. Now I am four times as old as she is." How old is the movie star?

34. Career Home Runs During his major league career, Hank Aaron hit 41 more home runs than Babe Ruth hit during his career. Together they hit 1469 home runs. How many home runs did Babe Ruth hit?

35. Value of Coins A change purse contains an equal number of pennies, nickels, and dimes. The total value of the coins is \$1.44. How many coins of each type does the purse contain?

36. Value of Coins Mary has \$3.00 in nickels, dimes, and quarters. If she has twice as many dimes as quarters and five more nickels than dimes, how many coins of each type does she have?

37. Length of a Garden A rectangular garden is 25 ft wide. If its area is 1125 ft^2, what is the length of the garden?

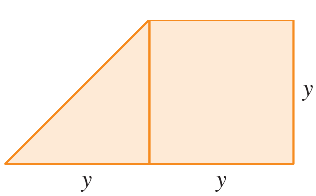


38. Width of a Pasture A pasture is twice as long as it is wide. Its area is 115,200 ft^2. How wide is the pasture?

39. Dimensions of a Lot A square plot of land has a building 60 ft long and 40 ft wide at one corner. The rest of the land outside the building forms a parking lot. If the parking lot has area 12,000 ft^2, what are the dimensions of the entire plot of land?

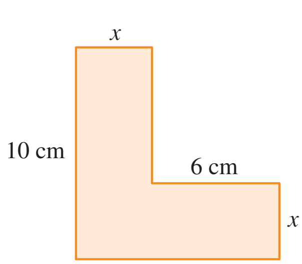
40. Dimensions of a Lot A half-acre building lot is five times as long as it is wide. What are its dimensions? [*Note:* 1 acre = 43,560 ft^2.]

41. Geometry Find the length y in the figure if the shaded area is 120 in^2.



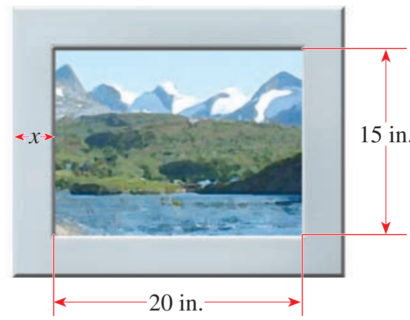
area = 120 in^2

42. Geometry Find the length x in the figure if the shaded area is 144 cm^2.

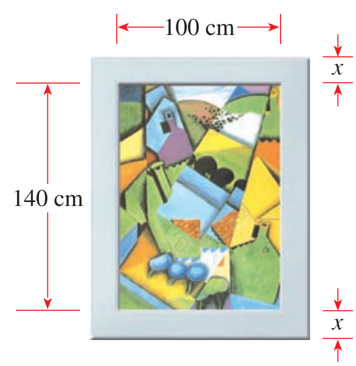


area = 144 cm^2

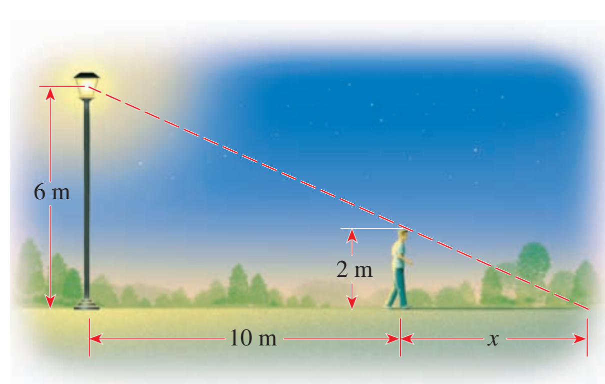
43. Framing a Painting Ali paints with watercolors on a sheet of paper 20 in. wide by 15 in. high. He then places this sheet on a mat so that a uniformly wide strip of the mat shows all around the picture. The perimeter of the mat is 102 in. How wide is the strip of the mat showing around the picture?



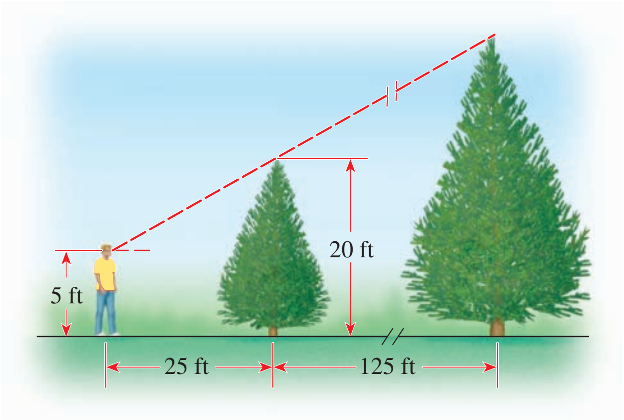
44. Dimensions of a Poster A poster has a rectangular printed area 100 cm by 140 cm and a blank strip of uniform width around the edges. The perimeter of the poster is $1\frac{1}{2}$ times the perimeter of the printed area. What is the width of the blank strip?



45. Length of a Shadow A man is walking away from a lamppost with a light source 6 m above the ground. The man is 2 m tall. How long is the man's shadow when he is 10 m from the lamppost? [*Hint:* Use similar triangles.]



46. Height of a Tree A woodcutter determines the height of a tall tree by first measuring a smaller one 125 ft away, then moving so that his eyes are in the line of sight along the tops of the trees and measuring how far he is standing from the small tree (see the figure). Suppose the small tree is 20 ft tall, the man is 25 ft from the small tree, and his eye level is 5 ft above the ground. How tall is the taller tree?



47. **Mixture Problem** What quantity of a 60% acid solution must be mixed with a 30% solution to produce 300 mL of a 50% solution?

48. **Mixture Problem** What quantity of pure acid must be added to 300 mL of a 50% acid solution to produce a 60% acid solution?

49. **Mixture Problem** A jeweler has five rings, each weighing 18 g, made of an alloy of 10% silver and 90% gold. She decides to melt down the rings and add enough silver to reduce the gold content to 75%. How much silver should she add?

50. **Mixture Problem** A pot contains 6 L of brine at a concentration of 120 g/L. How much of the water should be boiled off to increase the concentration to 200 g/L?

51. **Mixture Problem** The radiator in a car is filled with a solution of 60% antifreeze and 40% water. The manufacturer of the antifreeze suggests that for summer driving, optimal cooling of the engine is obtained with only 50% antifreeze. If the capacity of the radiator is 3.6 L, how much coolant should be drained and replaced with water to reduce the antifreeze concentration to the recommended level?

52. **Mixture Problem** A health clinic uses a solution of bleach to sterilize petri dishes in which cultures are grown. The sterilization tank contains 100 gal of a solution of 2% ordinary household bleach mixed with pure distilled water. New research indicates that the concentration of bleach should be 5% for complete sterilization. How much of the solution should be drained and replaced with bleach to increase the bleach content to the recommended level?

53. **Mixture Problem** A bottle contains 750 mL of fruit punch with a concentration of 50% pure fruit juice. Jill drinks 100 mL of the punch and then refills the bottle with an equal amount of a cheaper brand of punch. If the concentration of juice in the bottle is now reduced to 48%, what was the concentration in the punch that Jill added?

54. **Mixture Problem** A merchant blends tea that sells for \$3.00 a pound with tea that sells for \$2.75 a pound to produce 80 lb of a mixture that sells for \$2.90 a pound. How many pounds of each type of tea does the merchant use in the blend?

55. **Sharing a Job** Candy and Tim share a paper route. It takes Candy 70 min to deliver all the papers, and it takes Tim 80 min. How long does it take the two when they work together?

56. **Sharing a Job** Stan and Hilda can mow the lawn in 40 min if they work together. If Hilda works twice as fast as Stan, how long does it take Stan to mow the lawn alone?

57. **Sharing a Job** Betty and Karen have been hired to paint the houses in a new development. Working together, the women can paint a house in two-thirds the time that it takes Karen working alone. Betty takes 6 h to paint a house alone. How long does it take Karen to paint a house working alone?

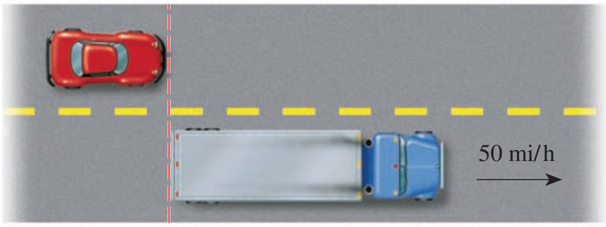
58. **Sharing a Job** Next-door neighbors Bob and Jim use hoses from both houses to fill Bob's swimming pool. They know that it takes 18 h using both hoses. They also know that Bob's hose, used alone, takes 20% less time than Jim's hose alone. How much time is required to fill the pool by each hose alone?

59. **Distance, Speed, and Time** Wendy took a trip from Davenport to Omaha, a distance of 300 mi. She traveled part of the way by bus, which arrived at the train station just in time for Wendy to complete her journey by train. The bus averaged 40 mi/h, and the train averaged 60 mi/h. The entire trip took $5\frac{1}{2}$ h. How long did Wendy spend on the train?

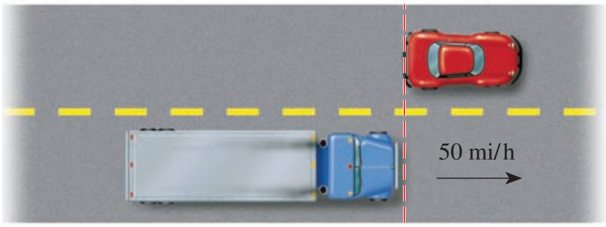
60. **Distance, Speed, and Time** Two cyclists, 90 mi apart, start riding toward each other at the same time. One cycles twice as fast as the other. If they meet 2 h later, at what average speed is each cyclist traveling?

61. **Distance, Speed, and Time** A pilot flew a jet from Montreal to Los Angeles, a distance of 2500 mi. On the return trip, the average speed was 20% faster than the outbound speed. The round-trip took 9 h 10 min. What was the speed from Montreal to Los Angeles?

62. **Distance, Speed, and Time** A woman driving a car 14 ft long is passing a truck 30 ft long. The truck is traveling at 50 mi/h. How fast must the woman drive her car so that she can pass the truck completely in 6 s, from the position shown in figure (a) to the position shown in figure (b)? [*Hint:* Use feet and seconds instead of miles and hours.]



(a)



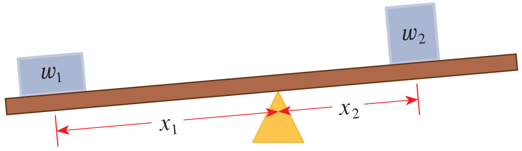
(b)

63. Law of the Lever The figure shows a lever system, similar to a seesaw that you might find in a children's playground. For the system to balance, the product of the weight and its distance from the fulcrum must be the same on each side; that is,

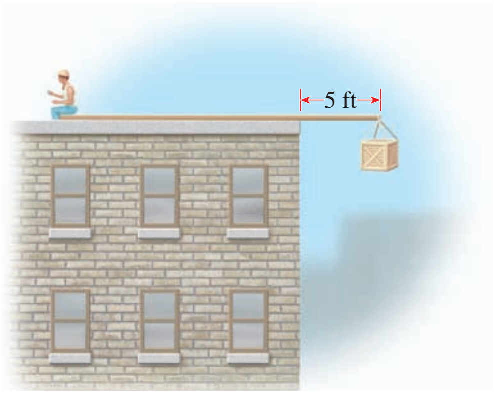
$$w_1x_1 = w_2x_2$$

This equation is called the **law of the lever** and was first discovered by Archimedes (see page 529).

A woman and her son are playing on a seesaw. The boy is at one end, 8 ft from the fulcrum. If the son weighs 100 lb and the mother weighs 125 lb, where should the woman sit so that the seesaw is balanced?

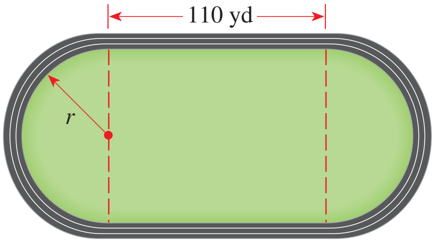


64. Law of the Lever A plank 30 ft long rests on top of a flat-roofed building, with 5 ft of the plank projecting over the edge, as shown in the figure. A worker weighing 240 lb sits on one end of the plank. What is the largest weight that can be hung on the projecting end of the plank if it is to remain in balance? (Use the law of the lever stated in Exercise 63.)

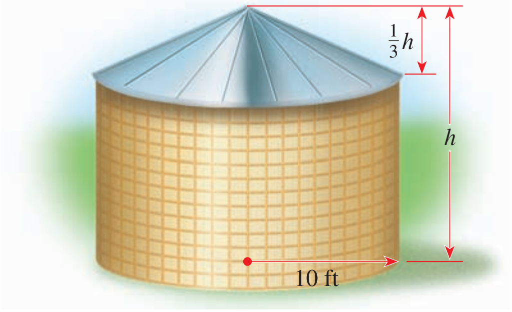


65. Dimensions of a Lot A rectangular parcel of land is 50 ft wide. The length of a diagonal between opposite corners is 10 ft more than the length of the parcel. What is the length of the parcel?

66. Dimensions of a Track A running track has the shape shown in the figure, with straight sides and semicircular ends. If the length of the track is 440 yd and the two straight parts are each 110 yd long, what is the radius of the semicircular parts (to the nearest yard)?



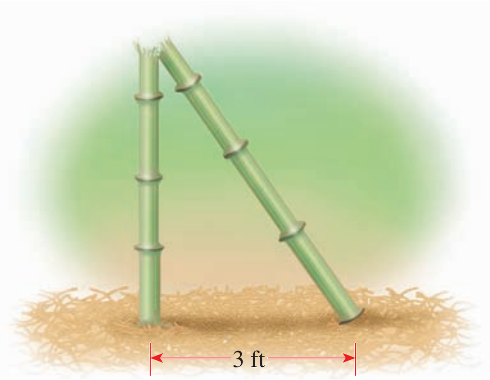
67. Dimensions of a Structure A storage bin for corn consists of a cylindrical section made of wire mesh, surmounted by a conical tin roof, as shown in the figure. The height of the roof is one-third the height of the entire structure. If the total volume of the structure is 1400π ft^3 and its radius is 10 ft, what is its height? [*Hint:* Use the volume formulas listed on the inside back cover of this book.]



68. An Ancient Chinese Problem This problem is taken from a Chinese mathematics textbook called *Chui-chang suan-shu*, or *Nine Chapters on the Mathematical Art*, which was written about 250 B.C.

A 10-ft-long stem of bamboo is broken in such a way that its tip touches the ground 3 ft from the base of the stem, as shown in the figure. What is the height of the break?

[*Hint:* Use the Pythagorean Theorem.]



DISCOVERY ■ DISCUSSION ■ WRITING

69. Historical Research Read the biographical notes on Euclid (page 57), Pythagoras (page 253), and Archimedes (page 529). Choose one of these mathematicians, and find out more about him from the library or on the Internet. Write a short essay on your findings. Include both biographical information and a description of the mathematics for which he is famous.

DISCOVERY PROJECT **Equations Through the Ages**

In this project we investigate equations that were created and solved by the ancient peoples of Egypt, Babylon, India, and China. You can find the project at the book companion website: **www.stewartmath.com**

1.6 Solving Quadratic Equations

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Solve quadratic equations by factoring ▶ Solve quadratic equations by completing the square ▶ Solve quadratic equations by using the Quadratic Formula ▶ Model with quadratic equations

Linear Equations

$4x = -7$

$6x - 8 = 21$

$2 + 3x = \frac{1}{2} - \frac{3}{4}x$

Quadratic Equations

$x^2 - 2x - 8 = 0$

$3x + 10 = 4x^2$

$\frac{1}{2}x^2 + \frac{1}{3}x - \frac{1}{6} = 0$

In Section P.8 we learned how to solve linear equations, which are first-degree equations such as $2x + 1 = 5$ or $4 - 3x = 2$. In this section we learn how to solve quadratic equations, which are second-degree equations such as $x^2 + 2x - 3 = 0$ or $2x^2 + 3 = 5x$. We will also see that many real-life problems can be modeled by using quadratic equations.

QUADRATIC EQUATIONS

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$.

▼ Solving Quadratic Equations by Factoring

Some quadratic equations can be solved by factoring and using the following basic property of real numbers.

ZERO-PRODUCT PROPERTY

$$AB = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0$$

This means that if we can factor the left-hand side of a quadratic (or other) equation, then we can solve it by setting each factor equal to 0 in turn. This method works only when the right-hand side of the equation is 0.

EXAMPLE 1 | Solving a Quadratic Equation by Factoring

Solve the equation $x^2 + 5x = 24$.

SOLUTION We must first rewrite the equation so that the right-hand side is 0:

$$\begin{aligned} x^2 + 5x &= 24 && \text{Given equation} \\ x^2 + 5x - 24 &= 0 && \text{Subtract 24} \\ (x - 3)(x + 8) &= 0 && \text{Factor} \\ x - 3 = 0 \quad \text{or} \quad x + 8 &= 0 && \text{Zero-Product Property} \\ x = 3 \qquad\qquad x &= -8 && \text{Solve} \end{aligned}$$

The solutions are $x = 3$ and $x = -8$.

CHECK YOUR ANSWERS

$x = 3$:

$(3)^2 + 5(3) = 9 + 15 = 24$ ✓

$x = -8$:

$(-8)^2 + 5(-8) = 64 - 40 = 24$ ✓

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5** ■

Do you see why one side of the equation must be 0 in Example 1? Factoring the equation as $x(x + 5) = 24$ does not help us find the solutions, since 24 can be factored in infinitely many ways, such as $6 \cdot 4$, $\frac{1}{2} \cdot 48$, $\left(-\frac{2}{5}\right) \cdot (-60)$, and so on.

Solving Quadratic Equations by Completing the Square

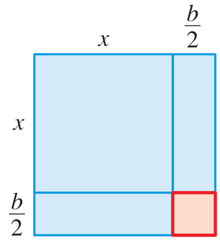
As we saw in Section P.8, Example 5(b), if a quadratic equation is of the form $(x \pm a)^2 = c$, then we can solve it by taking the square root of each side. In an equation of this form, the left-hand side is a *perfect square*: the square of a linear expression in x. So if a quadratic equation does not factor readily, then we can solve it by **completing the square**.

Completing the Square

The area of the blue region is

$$x^2 + 2\left(\frac{b}{2}\right)x = x^2 + bx$$

Add a small square of area $(b/2)^2$ to "complete" the square.



COMPLETING THE SQUARE

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$, the square of half the coefficient of x. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

To complete the square, we add a constant to a quadratic expression to make it a perfect square. For example, to make

$$x^2 + 6x$$

a perfect square, we must add $\left(\frac{6}{2}\right)^2 = 9$. Then

$$x^2 + 6x + 9 = (x + 3)^2$$

is a perfect square. The table gives some more examples of completing the square.

Expression	Add	Complete the square
$x^2 + 8x$	$\left(\frac{8}{2}\right)^2 = 16$	$x^2 + 8x + 16 = (x + 4)^2$
$x^2 - 12x$	$\left(-\frac{12}{2}\right)^2 = 36$	$x^2 - 12x + 36 = (x - 6)^2$
$x^2 + 3x$	$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$	$x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$
$x^2 - \sqrt{3}x$	$\left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$	$x^2 - \sqrt{3}x + \frac{3}{4} = \left(x - \frac{\sqrt{3}}{2}\right)^2$

EXAMPLE 2 | Solving Quadratic Equations by Completing the Square

Solve each equation.

(a) $x^2 - 8x + 13 = 0$

(b) $3x^2 - 12x + 6 = 0$

SOLUTION

(a)

$x^2 - 8x + 13 = 0$	Given equation
$x^2 - 8x = -13$	Subtract 13
$x^2 - 8x + 16 = -13 + 16$	Complete the square: add $\left(\frac{-8}{2}\right)^2 = 16$
$(x - 4)^2 = 3$	Perfect square
$x - 4 = \pm\sqrt{3}$	Take square root
$x = 4 \pm \sqrt{3}$	Add 4

When completing the square, make sure the coefficient of x^2 is 1. If it isn't, you must factor this coefficient from both terms that contain x:

$$ax^2 + bx = a\left(x^2 + \frac{b}{a}x\right)$$

Then complete the square inside the parentheses. Remember that the term added inside the parentheses is multiplied by a.

(b) After subtracting 6 from each side of the equation, we must factor the coefficient of x^2 (the 3) from the left side to put the equation in the correct form for completing the square:

$$\begin{aligned} 3x^2 - 12x + 6 &= 0 && \text{Given equation} \\ 3x^2 - 12x &= -6 && \text{Subtract 6} \\ 3(x^2 - 4x) &= -6 && \text{Factor 3 from LHS} \end{aligned}$$

Now we complete the square by adding $(-2)^2 = 4$ *inside* the parentheses. Since everything inside the parentheses is multiplied by 3, this means that we are actually adding $3 \cdot 4 = 12$ to the left side of the equation. Thus we must add 12 to the right side as well:

$$\begin{aligned} 3(x^2 - 4x + 4) &= -6 + 3 \cdot 4 && \text{Complete the square: add 4} \\ 3(x - 2)^2 &= 6 && \text{Perfect square} \\ (x - 2)^2 &= 2 && \text{Divide by 3} \\ x - 2 &= \pm\sqrt{2} && \text{Take square root} \\ x &= 2 \pm \sqrt{2} && \text{Add 2} \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **19** AND **27**

The Quadratic Formula

We can use the technique of completing the square to derive a formula for the roots of the general quadratic equation $ax^2 + bx + c = 0$.

THE QUADRATIC FORMULA

The roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

PROOF First, we divide each side of the equation by a and move the constant to the right side, giving

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Divide by } a$$

We now complete the square by adding $(b/2a)^2$ to each side of the equation:

$$\begin{aligned} x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 && \text{Complete the square: Add } \left(\frac{b}{2a}\right)^2 \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{-4ac + b^2}{4a^2} && \text{Perfect square} \\ x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} && \text{Take square root} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Subtract } \frac{b}{2a} \end{aligned}$$

The Quadratic Formula could be used to solve the equations in Examples 1 and 2. You should carry out the details of these calculations.

EXAMPLE 3 | Using the Quadratic Formula

Find all real solutions of each equation.

(a) $3x^2 - 5x - 1 = 0$ **(b)** $4x^2 + 12x + 9 = 0$ **(c)** $x^2 + 2x + 2 = 0$

SOLUTION

(a) In this quadratic equation $a = 3$, $b = -5$, and $c = -1$:

$$b = -5$$

$$3x^2 - 5x - 1 = 0$$

$$a = 3 \qquad c = -1$$

By the Quadratic Formula,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} = \frac{5 \pm \sqrt{37}}{6}$$

If approximations are desired, we can use a calculator to obtain

$$x = \frac{5 + \sqrt{37}}{6} \approx 1.8471 \qquad \text{and} \qquad x = \frac{5 - \sqrt{37}}{6} \approx -0.1805$$

Another Method

$$4x^2 + 12x + 9 = 0$$
$$(2x + 3)^2 = 0$$
$$2x + 3 = 0$$
$$x = -\tfrac{3}{2}$$

(b) Using the Quadratic Formula with $a = 4$, $b = 12$, and $c = 9$ gives

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4} = \frac{-12 \pm 0}{8} = -\frac{3}{2}$$

This equation has only one solution, $x = -\frac{3}{2}$.

(c) Using the Quadratic Formula with $a = 1$, $b = 2$, and $c = 2$ gives

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm \sqrt{-1}$$

Since the square of any real number is nonnegative, $\sqrt{-1}$ is undefined in the real number system. The equation has no real solution.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **33, 39,** AND **45**

In Section 3.5 we study the complex number system, in which the square roots of negative numbers do exist. The equation in Example 3(c) does have solutions in the complex number system.

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FRANÇOIS VIÈTE (1540–1603) had a successful political career before taking up mathematics late in life. He became one of the most famous French mathematicians of the 16th century. Viète introduced a new level of abstraction in algebra by using letters to stand for *known* quantities in an equation. Before Viète's time, each equation had to be solved on its own. For instance, the quadratic equations

$$3x^2 + 2x + 8 = 0$$

$$5x^2 - 6x + 4 = 0$$

had to be solved separately by completing the square. Viète's idea was to consider all quadratic equations at once by writing

$$ax^2 + bx + c = 0$$

where a, b, and c are known quantities. Thus he made it possible to write a *formula* (in this case, the Quadratic Formula) involving a, b, and c that can be used to solve all such equations in one fell swoop.

Viète's mathematical genius proved quite valuable during a war between France and Spain. To communicate with their troops, the Spaniards used a complicated code that Viète managed to decipher. Unaware of Viète's accomplishment, the Spanish king, Philip II, protested to the Pope, claiming that the French were using witchcraft to read his messages.

The Discriminant

The quantity $b^2 - 4ac$ that appears under the square root sign in the Quadratic Formula is called the *discriminant* of the equation $ax^2 + bx + c = 0$ and is given the symbol D. If $D < 0$, then $\sqrt{b^2 - 4ac}$ is undefined, and the quadratic equation has no real solution, as in Example 3(c). If $D = 0$, then the equation has only one real solution, as in Example 3(b). Finally, if $D > 0$, then the equation has two distinct real solutions, as in Example 3(a). The following box summarizes these observations.

THE DISCRIMINANT

The **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ is $D = b^2 - 4ac$.

1. If $D > 0$, then the equation has two distinct real solutions.
2. If $D = 0$, then the equation has exactly one real solution.
3. If $D < 0$, then the equation has no real solution.

EXAMPLE 4 | Using the Discriminant

Use the discriminant to determine how many real solutions each equation has.

(a) $x^2 + 4x - 1 = 0$ **(b)** $4x^2 - 12x + 9 = 0$ **(c)** $\frac{1}{3}x^2 - 2x + 4 = 0$

SOLUTION

(a) The discriminant is $D = 4^2 - 4(1)(-1) = 20 > 0$, so the equation has two distinct real solutions.

(b) The discriminant is $D = (-12)^2 - 4 \cdot 4 \cdot 9 = 0$, so the equation has exactly one real solution.

(c) The discriminant is $D = (-2)^2 - 4\left(\frac{1}{3}\right)4 = -\frac{4}{3} < 0$, so the equation has no real solution.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **63, 65,** AND **67**

In the next example we solve quadratic equations graphically. The graphs visually demonstrate why such equations can have one real solution, two real solutions, or no real solution.

EXAMPLE 5 | Solving a Quadratic Equation Algebraically and Graphically

Solve the quadratic equations algebraically and graphically.

(a) $x^2 - 4x + 2 = 0$ **(b)** $x^2 - 4x + 4 = 0$ **(c)** $x^2 - 4x + 6 = 0$

SOLUTION 1: Algebraic

The Quadratic Formula is discussed on page 123.

We use the Quadratic Formula to solve each equation.

(a) $$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 2}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

$D = 8 > 0$. Two real solutions.

There are two solutions, $x = 2 + \sqrt{2}$ and $x = 2 - \sqrt{2}$.

(b) $$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 4}}{2} = \frac{4 \pm \sqrt{0}}{2} = 2$$

$D = 0$. One real solution.

There is just one solution, $x = 2$.

(c) $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 6}}{2} = \dfrac{4 \pm \sqrt{-8}}{2}$

$D = -8$. No real solution.

There is no real solution.

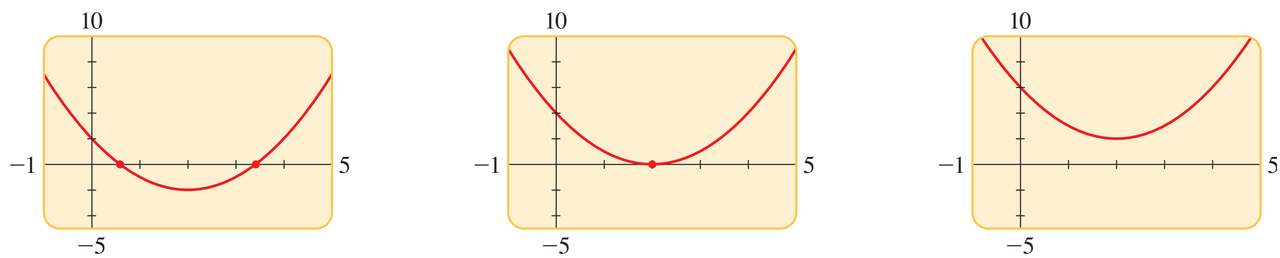
SOLUTION 2: Graphical

We graph the equations $y = x^2 - 4x + 2$, $y = x^2 - 4x + 4$, and $y = x^2 - 4x + 6$ in Figure 1. By determining the x-intercepts of the graphs, we find the following solutions.

(a) $x \approx 0.6$ and $x \approx 3.4$

(b) $x = 2$

(c) There is no x-intercept, so the equation has no solution.



(a) $y = x^2 - 4x + 2$ (b) $y = x^2 - 4x + 4$ (c) $y = x^2 - 4x + 6$

FIGURE 1

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **71, 73,** AND **75**

Modeling with Quadratic Equations

Let's look at some real-life problems that can be modeled by quadratic equations. The principles discussed in Section 1.5 for setting up equations as models are useful here as well.

EXAMPLE 6 | Dimensions of a Building Lot

A rectangular building lot is 8 ft longer than it is wide and has an area of 2900 ft^2. Find the dimensions of the lot.

SOLUTION Identify the variable. We are asked to find the width and length of the lot. So let

$$w = \text{width of lot}$$

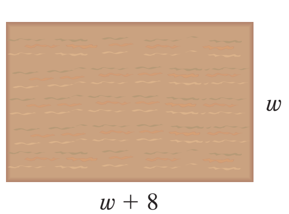


FIGURE 2

Translate from words to algebra. Then we translate the information given in the problem into the language of algebra (see Figure 2):

In Words	In Algebra
Width of lot	w
Length of lot	$w + 8$

Set up the model. Now we set up the model:

$$\text{width of lot} \times \text{length of lot} = \text{area of lot}$$

$$w(w + 8) = 2900$$

Solve. Now we solve for w.

$$w^2 + 8w = 2900 \qquad \text{Expand}$$

$$w^2 + 8w - 2900 = 0 \qquad \text{Subtract 2900}$$

$$(w - 50)(w + 58) = 0 \qquad \text{Factor}$$

$$w = 50 \quad \text{or} \quad w = -58 \qquad \text{Zero-Product Property}$$

Since the width of the lot must be a positive number, we conclude that $w = 50$ ft. The length of the lot is $w + 8 = 50 + 8 = 58$ ft.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **81**

EXAMPLE 7 | A Distance-Speed-Time Problem

A jet flew from New York to Los Angeles, a distance of 4200 km. The speed for the return trip was 100 km/h faster than the outbound speed. If the total trip took 13 hours, what was the jet's speed from New York to Los Angeles?

SOLUTION Identify the variable. We are asked for the speed of the jet from New York to Los Angeles. So let

$$s = \text{speed from New York to Los Angeles}$$

Then $$s + 100 = \text{speed from Los Angeles to New York}$$

Translate from words to algebra. Now we organize the information in a table. We fill in the "Distance" column first, since we know that the cities are 4200 km apart. Then we fill in the "Speed" column, since we have expressed both speeds (rates) in terms of the variable s. Finally, we calculate the entries for the "Time" column, using

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

	Distance (km)	Speed (km/h)	Time (h)
N.Y. to L.A.	4200	s	$\frac{4200}{s}$
L.A. to N.Y.	4200	$s + 100$	$\frac{4200}{s + 100}$

Set up the model. The total trip took 13 hours, so we have the model

$$\boxed{\text{time from N.Y. to L.A.}} + \boxed{\text{time from L.A. to N.Y.}} = \boxed{\text{total time}}$$

$$\frac{4200}{s} + \frac{4200}{s + 100} = 13$$

Solve. Multiplying by the common denominator, $s(s + 100)$, we get

$$4200(s + 100) + 4200s = 13s(s + 100)$$

$$8400s + 420{,}000 = 13s^2 + 1300s$$

$$0 = 13s^2 - 7100s - 420{,}000$$

Although this equation does factor, with numbers this large it is probably quicker to use the Quadratic Formula and a calculator:

$$s = \frac{7100 \pm \sqrt{(-7100)^2 - 4(13)(-420{,}000)}}{2(13)}$$

$$= \frac{7100 \pm 8500}{26}$$

$$s = 600 \qquad \text{or} \qquad s = \frac{-1400}{26} \approx -53.8$$

Since s represents speed, we reject the negative answer and conclude that the jet's speed from New York to Los Angeles was 600 km/h.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **91** ■

EXAMPLE 8 | The Path of a Projectile

This formula depends on the fact that acceleration due to gravity is constant near the earth's surface. Here we neglect the effect of air resistance.

An object thrown or fired straight upward at an initial speed of v_0 ft/s will reach a height of h feet after t seconds, where h and t are related by the formula

$$h = -16t^2 + v_0 t$$

Suppose that a bullet is shot straight upward with an initial speed of 800 ft/s. Its path is shown in Figure 3.

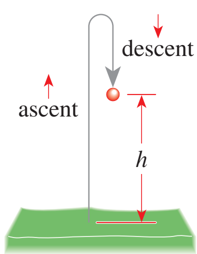


FIGURE 3

(a) When does the bullet fall back to ground level?

(b) When does it reach a height of 6400 ft?

(c) When does it reach a height of 2 mi?

(d) How high is the highest point the bullet reaches?

SOLUTION Since the initial speed in this case is $v_0 = 800$ ft/s, the formula is

$$h = -16t^2 + 800t$$

(a) Ground level corresponds to $h = 0$, so we must solve the equation

$$0 = -16t^2 + 800t \qquad \text{Set } h = 0$$

$$0 = -16t(t - 50) \qquad \text{Factor}$$

Thus $t = 0$ or $t = 50$. This means the bullet starts ($t = 0$) at ground level and returns to ground level after 50 s.

(b) Setting $h = 6400$ gives the equation

$$6400 = -16t^2 + 800t \qquad \text{Set } h = 6400$$

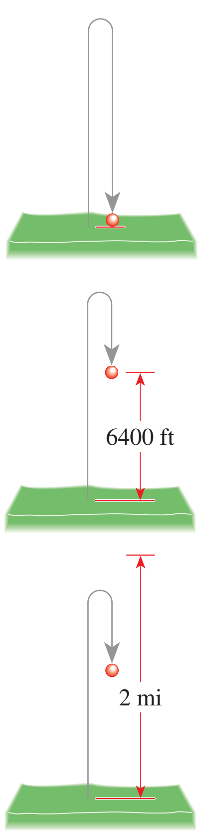
$$16t^2 - 800t + 6400 = 0 \qquad \text{All terms to LHS}$$

$$t^2 - 50t + 400 = 0 \qquad \text{Divide by 16}$$

$$(t - 10)(t - 40) = 0 \qquad \text{Factor}$$

$$t = 10 \qquad \text{or} \qquad t = 40 \qquad \text{Solve}$$

The bullet reaches 6400 ft after 10 s (on its ascent) and again after 40 s (on its descent to earth).



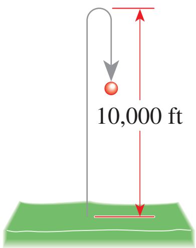
(c) Two miles is $2 \times 5280 = 10{,}560$ ft:

$$10{,}560 = -16t^2 + 800t \qquad \text{Set } h = 10{,}560$$

$$16t^2 - 800t + 10{,}560 = 0 \qquad \text{All terms to LHS}$$

$$t^2 - 50t + 660 = 0 \qquad \text{Divide by 16}$$

The discriminant of this equation is $D = (-50)^2 - 4(660) = -140$, which is negative. Thus the equation has no real solution. The bullet never reaches a height of 2 mi.



(d) Each height that the bullet reaches is attained twice: once on its ascent and once on its descent. The only exception is the highest point of its path, which is reached only once. This means that for the highest value of h, the following equation has only one solution for t:

$$h = -16t^2 + 800t$$

$$16t^2 - 800t + h = 0 \qquad \text{All terms to LHS}$$

This in turn means that the discriminant D of the equation is 0, so

$$D = (-800)^2 - 4(16)h = 0$$

$$640{,}000 - 64h = 0$$

$$h = 10{,}000$$

The maximum height reached is 10,000 ft.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **97**

1.6 EXERCISES

CONCEPTS

1. The Quadratic Formula gives us the solutions of the equation $ax^2 + bx + c = 0$.

(a) State the Quadratic Formula: $x =$ ________.

(b) In the equation $\frac{1}{2}x^2 - x - 4 = 0$, $a =$ ________, $b =$ ________, and $c =$ ________. So the solution of the equation is $x =$ ________.

2. Explain how you would use each method to solve the equation $x^2 - 4x - 5 = 0$.

(a) By factoring: ____________

(b) By completing the square: ____________

(c) By using the Quadratic Formula: ____________

3. For the quadratic equation $ax^2 + bx + c = 0$ the discriminant is $D =$ ________. The discriminant tells us how many real solutions a quadratic equation has.

If $D > 0$, the equation has ________ real solution(s).

If $D = 0$, the equation has ________ real solution(s).

If $D < 0$, the equation has ________ real solution(s).

4. Make up quadratic equations that have the following number of solutions:

Two solutions: ____________

One solution: ____________

No solution: ____________

SKILLS

5–18 ■ Find all real solutions of the equation by factoring.

5. $x^2 - x = 12$ **6.** $x^2 - 5x = -4$

7. $x^2 - 7x + 12 = 0$ **8.** $x^2 + 8x + 12 = 0$

9. $3x^2 - 5x - 2 = 0$ **10.** $4x^2 - 4x - 15 = 0$

11. $2s^2 = 5s + 3$ **12.** $4y^2 - 9y = 28$

13. $12z^2 - 44z = 45$ **14.** $4w^2 = 4w + 3$

15. $6x^2 + 5x = 4$ **16.** $3x^2 + 1 = 4x$

17. $x^2 = 5(x + 100)$ **18.** $6x(x - 1) = 21 - x$

19–30 ■ Find all real solutions of the equation by completing the square.

19. $x^2 + 2x - 5 = 0$ **20.** $x^2 - 4x + 2 = 0$

21. $x^2 - 6x - 11 = 0$ **22.** $x^2 + 3x - \frac{7}{4} = 0$

23. $x^2 + x - \frac{3}{4} = 0$ **24.** $x^2 - 5x + 1 = 0$

25. $x^2 + 22x + 21 = 0$ **26.** $x^2 - 18x = 19$

27. $2x^2 + 8x + 1 = 0$ **28.** $3x^2 - 6x - 1 = 0$

29. $2x^2 + 7x + 4 = 0$ **30.** $4x^2 + 5x - 8 = 0$

31–52 ■ Find all real solutions of the equation.

31. $x^2 - 2x - 15 = 0$ **32.** $x^2 + 5x - 6 = 0$

33. $x^2 - 7x + 10 = 0$ **34.** $x^2 + 30x + 200 = 0$

35. $2x^2 + x - 3 = 0$ **36.** $3x^2 + 7x + 4 = 0$

37. $x^2 + 12x - 27 = 0$ **38.** $8x^2 - 6x - 9 = 0$

39. $3x^2 + 6x - 5 = 0$ **40.** $x^2 - 6x + 1 = 0$

41. $z^2 - \frac{3}{2}z + \frac{9}{16} = 0$ **42.** $2y^2 - y - \frac{1}{2} = 0$

43. $4x^2 + 16x - 9 = 0$ **44.** $0 = x^2 - 4x + 1$

45. $w^2 = 3(w - 1)$ **46.** $3 + 5z + z^2 = 0$

47. $x^2 - \sqrt{5}x + 1 = 0$ **48.** $\sqrt{6}x^2 + 2x - \sqrt{\frac{3}{2}} = 0$

49. $10y^2 - 16y + 5 = 0$ **50.** $25x^2 + 70x + 49 = 0$

51. $3x^2 + 2x + 2 = 0$ **52.** $5x^2 - 7x + 5 = 0$

53–56 ■ Use the quadratic formula and a calculator to find all real solutions, rounded to three decimals.

53. $x^2 - 0.011x - 0.064 = 0$ **54.** $x^2 - 2.450x + 1.500 = 0$

55. $x^2 - 2.450x + 1.501 = 0$ **56.** $x^2 - 1.800x + 0.810 = 0$

57–62 ■ Solve the equation for the indicated variable.

57. $h = \frac{1}{2}gt^2 + v_0t$; for t

58. $S = \dfrac{n(n+1)}{2}$; for n

59. $A = 2x^2 + 4xh$; for x

60. $A = 2\pi r^2 + 2\pi rh$; for r

61. $\dfrac{1}{s+a} + \dfrac{1}{s+b} = \dfrac{1}{c}$; for s

62. $\dfrac{1}{r} + \dfrac{2}{1-r} = \dfrac{4}{r^2}$; for r

63–68 ■ Use the discriminant to determine the number of real solutions of the equation. Do not solve the equation.

63. $x^2 - 6x + 1 = 0$

64. $x^2 = 6x - 9$

65. $x^2 + 2.20x + 1.21 = 0$

66. $x^2 + 2.21x + 1.21 = 0$

67. $4x^2 + 5x + \frac{13}{8} = 0$

68. $x^2 + rx - s = 0 \quad (s > 0)$

69–70 ■ Solve the equation for x.

69. $a^2x^2 + 2ax + 1 = 0 \quad (a \neq 0)$

70. $ax^2 - (2a+1)x + (a+1) = 0 \quad (a \neq 0)$

71–76 ■ Find all real solutions of the equation both algebraically and graphically.

71. $x^2 - x - 6 = 0$

72. $x^2 + 9 = 0$

73. $x^2 + 6x + 9 = 0$

74. $16x^2 + 8x + 1 = 0$

75. $x^2 - 6x + 14 = 0$

76. $x^2 + 2x - 10 = 0$

77–78 ■ Find all values of k that ensure that the given equation has exactly one solution.

77. $4x^2 + kx + 25 = 0$

78. $kx^2 + 36x + k = 0$

APPLICATIONS

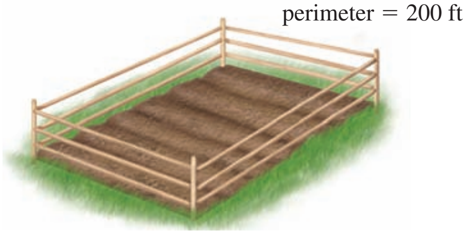
79. Number Problem Find two numbers whose sum is 55 and whose product is 684.

80. Number Problem The sum of the squares of two consecutive even integers is 1252. Find the integers.

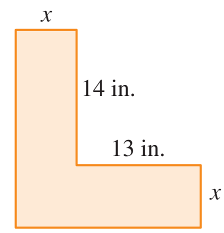
81. Dimensions of a Garden A rectangular garden is 10 ft longer than it is wide. Its area is 875 ft^2. What are its dimensions?

82. Dimensions of a Room A rectangular bedroom is 7 ft longer than it is wide. Its area is 228 ft^2. What is the width of the room?

83. Dimensions of a Garden A farmer has a rectangular garden plot surrounded by 200 ft of fence. Find the length and width of the garden if its area is 2400 ft^2.

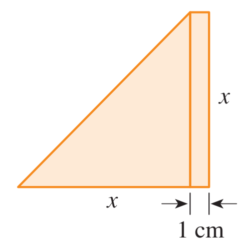


84. Geometry Find the length x if the shaded area is 160 in^2.



area = 160 in^2

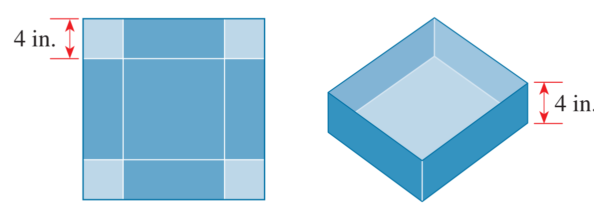
85. Geometry Find the length x if the shaded area is 1200 cm^2.



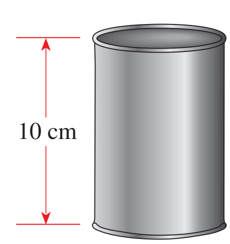
area = 1200 cm^2

86. Profit A small-appliance manufacturer finds that the profit P (in dollars) generated by producing x microwave ovens per week is given by the formula $P = \frac{1}{10}x(300 - x)$ provided that $0 \le x \le 200$. How many ovens must be manufactured in a given week to generate a profit of \$1250?

87. Dimensions of a Box A box with a square base and no top is to be made from a square piece of cardboard by cutting 4-in. squares from each corner and folding up the sides, as shown in the figure. The box is to hold 100 in^3. How big a piece of cardboard is needed?



88. Dimensions of a Can A cylindrical can has a volume of 40π cm^3 and is 10 cm tall. What is its diameter? [*Hint:* Use the volume formula listed on the inside back cover of this book.]



89. **Dimensions of a Lot** A parcel of land is 6 ft longer than it is wide. Each diagonal from one corner to the opposite corner is 174 ft long. What are the dimensions of the parcel?

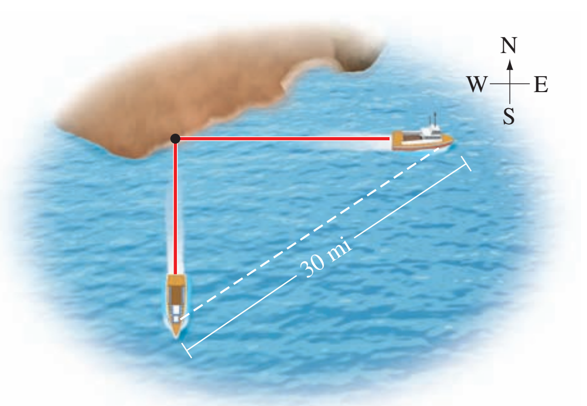
90. **Height of a Flagpole** A flagpole is secured on opposite sides by two guy wires, each of which is 5 ft longer than the pole. The distance between the points where the wires are fixed to the ground is equal to the length of one guy wire. How tall is the flagpole (to the nearest inch)?

91. **Distance, Speed, and Time** A salesman drives from Ajax to Barrington, a distance of 120 mi, at a steady speed. He then increases his speed by 10 mi/h to drive the 150 mi from Barrington to Collins. If the second leg of his trip took 6 min more time than the first leg, how fast was he driving between Ajax and Barrington?

92. **Distance, Speed, and Time** Kiran drove from Tortula to Cactus, a distance of 250 mi. She increased her speed by 10 mi/h for the 360-mi trip from Cactus to Dry Junction. If the total trip took 11 h, what was her speed from Tortula to Cactus?

93. **Distance, Speed, and Time** It took a crew 2 h 40 min to row 6 km upstream and back again. If the rate of flow of the stream was 3 km/h, what was the rowing speed of the crew in still water?

94. **Speed of a Boat** Two fishing boats depart a harbor at the same time, one traveling east, the other south. The eastbound boat travels at a speed 3 mi/h faster than the southbound boat. After two hours the boats are 30 mi apart. Find the speed of the southbound boat.



95–96 ■ **Falling-Body Problems** Suppose an object is dropped from a height h_0 above the ground. Then its height after t seconds is given by $h = -16t^2 + h_0$, where h is measured in feet. Use this information to solve the problem.

95. If a ball is dropped from 288 ft above the ground, how long does it take to reach ground level?

96. A ball is dropped from the top of a building 96 ft tall.
(a) How long will it take to fall half the distance to ground level?
(b) How long will it take to fall to ground level?

97–98 ■ **Falling-Body Problems** Use the formula $h = -16t^2 + v_0 t$ discussed in Example 8.

97. A ball is thrown straight upward at an initial speed of $v_0 = 40$ ft/s.
(a) When does the ball reach a height of 24 ft?
(b) When does it reach a height of 48 ft?
(c) What is the greatest height reached by the ball?
(d) When does the ball reach the highest point of its path?
(e) When does the ball hit the ground?

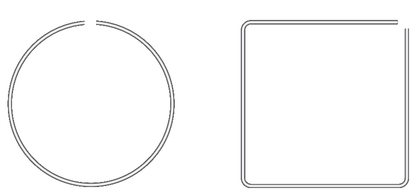
98. How fast would a ball have to be thrown upward to reach a maximum height of 100 ft? [*Hint:* Use the discriminant of the equation $16t^2 - v_0 t + h = 0$.]

99. **Fish Population** The fish population in a certain lake rises and falls according to the formula

$$F = 1000(30 + 17t - t^2)$$

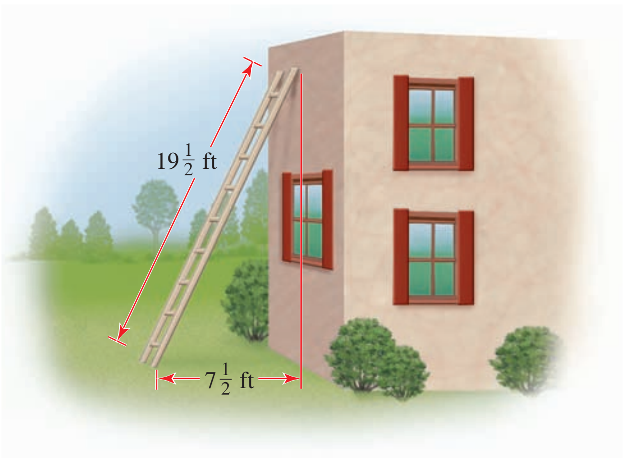
Here F is the number of fish at time t, where t is measured in years since January 1, 2002, when the fish population was first estimated.
(a) On what date will the fish population again be the same as it was on January 1, 2002?
(b) By what date will all the fish in the lake have died?

100. **Comparing Areas** A wire 360 in. long is cut into two pieces. One piece is formed into a square, and the other is formed into a circle. If the two figures have the same area, what are the lengths of the two pieces of wire (to the nearest tenth of an inch)?



101. **Width of a Lawn** A factory is to be built on a lot measuring 180 ft by 240 ft. A local building code specifies that a lawn of uniform width and equal in area to the factory must surround the factory. What must the width of this lawn be, and what are the dimensions of the factory?

102. Reach of a Ladder A $19\frac{1}{2}$-foot ladder leans against a building. The base of the ladder is $7\frac{1}{2}$ ft from the building. How high up the building does the ladder reach?



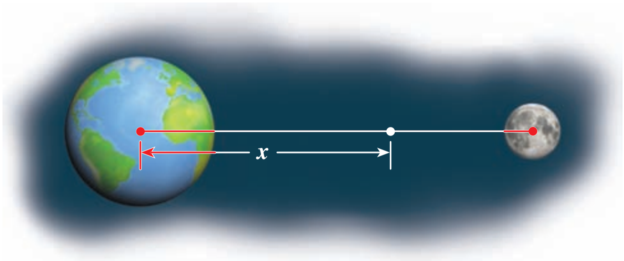
103. Sharing a Job Henry and Irene working together can wash all the windows of their house in 1 h 48 min. Working alone, it takes Henry $1\frac{1}{2}$ h more than Irene to do the job. How long does it take each person working alone to wash all the windows?

104. Sharing a Job Jack, Kay, and Lynn deliver advertising flyers in a small town. If each person works alone, it takes Jack 4 h to deliver all the flyers, and it takes Lynn 1 h longer than it takes Kay. Working together, they can deliver all the flyers in 40% of the time it takes Kay working alone. How long does it take Kay to deliver all the flyers alone?

105. Gravity If an imaginary line segment is drawn between the centers of the earth and the moon, then the net gravitational force F acting on an object situated on this line segment is

$$F = \frac{-K}{x^2} + \frac{0.012K}{(239 - x)^2}$$

where $K > 0$ is a constant and x is the distance of the object from the center of the earth, measured in thousands of miles. How far from the center of the earth is the "dead spot" where no net gravitational force acts upon the object? (Express your answer to the nearest thousand miles.)



DISCOVERY ▪ DISCUSSION ▪ WRITING

106. Relationship Between Roots and Coefficients The Quadratic Formula gives us the roots of a quadratic equation from its coefficients. We can also obtain the coefficients from the roots. For example, find the roots of the equation $x^2 - 9x + 20 = 0$ and show that the product of the roots is the constant term 20 and the sum of the roots is 9, the negative of the coefficient of x. Show that the same relationship between roots and coefficients holds for the following equations:

$$x^2 - 2x - 8 = 0$$
$$x^2 + 4x + 2 = 0$$

Use the Quadratic Formula to prove that in general, if the equation $x^2 + bx + c = 0$ has roots r_1 and r_2, then $c = r_1 r_2$ and $b = -(r_1 + r_2)$.

1.7 Solving Other Types of Equations

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Solve basic polynomial equations ▶ Solve equations involving radicals ▶ Solve equations of quadratic type ▶ Model with equations

So far, we have learned how to solve linear and quadratic equations. In this section we study other types of equations, including those that involve higher powers, fractional expressions, and radicals.

▼ Polynomial Equations

Some equations can be solved by factoring and using the Zero-Product Property, which says that if a product equals 0, then at least one of the factors must equal 0.

EXAMPLE 1 | Solving an Equation by Factoring

Solve the equation $x^5 = 9x^3$.

SOLUTION We bring all terms to one side and then factor:

$$x^5 - 9x^3 = 0 \qquad \text{Subtract } 9x^3$$

$$x^3(x^2 - 9) = 0 \qquad \text{Factor } x^3$$

$$x^3(x-3)(x+3) = 0 \qquad \text{Difference of squares}$$

$$x^3 = 0 \quad \text{or} \quad x - 3 = 0 \quad \text{or} \quad x + 3 = 0 \qquad \text{Zero-Product Property}$$

$$x = 0 \qquad x = 3 \qquad x = -3 \qquad \text{Solve}$$

The solutions are $x = 0$, $x = 3$, and $x = -3$. You should check that each of these satisfies the original equation.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5**

To divide each side of the equation in Example 1 by the common factor x^3 would be wrong, because in doing so, we would lose the solution $x = 0$. Never divide both sides of an equation by an expression that contains the variable unless you know that the expression cannot equal 0.

EXAMPLE 2 | Factoring by Grouping

Solve the equation $x^3 + 3x^2 - 4x - 12 = 0$.

SOLUTION The left-hand side of the equation can be factored by grouping the terms in pairs:

$$(x^3 + 3x^2) - (4x + 12) = 0 \qquad \text{Group terms}$$

$$x^2(x+3) - 4(x+3) = 0 \qquad \text{Factor } x^2 \text{ and } 4$$

$$(x^2 - 4)(x+3) = 0 \qquad \text{Factor } x + 3$$

$$(x-2)(x+2)(x+3) = 0 \qquad \text{Difference of squares}$$

$$x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x + 3 = 0 \qquad \text{Zero-Product Property}$$

$$x = 2 \qquad x = -2 \qquad x = -3 \qquad \text{Solve}$$

The solutions are $x = 2$, -2, and -3.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **19**

EXAMPLE 3 | An Equation Involving Fractional Expressions

Solve the equation $\dfrac{3}{x} + \dfrac{5}{x+2} = 2$.

SOLUTION To simplify the equation, we multiply each side by the common denominator:

$$\left(\frac{3}{x} + \frac{5}{x+2}\right)x(x+2) = 2x(x+2) \qquad \text{Multiply by LCD } x(x+2)$$

$$3(x+2) + 5x = 2x^2 + 4x \qquad \text{Expand}$$

$$8x + 6 = 2x^2 + 4x \qquad \text{Expand LHS}$$

$$0 = 2x^2 - 4x - 6 \qquad \text{Subtract } 8x + 6$$

$$0 = x^2 - 2x - 3 \qquad \text{Divide both sides by 2}$$

$$0 = (x-3)(x+1) \qquad \text{Factor}$$

$$x - 3 = 0 \quad \text{or} \quad x + 1 = 0 \qquad \text{Zero-Product Property}$$

$$x = 3 \qquad x = -1 \qquad \text{Solve}$$

CHECK YOUR ANSWERS

$x = 3$:

$$\text{LHS} = \frac{3}{3} + \frac{5}{3+2} = 2$$

$$\text{RHS} = 2$$

$$\text{LHS} = \text{RHS} \quad ✓$$

$x = -1$:

$$\text{LHS} = \frac{3}{-1} + \frac{5}{-1+2} = 2$$

$$\text{RHS} = 2$$

$$\text{LHS} = \text{RHS} \quad ✓$$

We must check our answers because multiplying by an expression that contains the variable can introduce extraneous solutions (see the *Warning* on page 133). From *Check Your Answers* we see that the solutions are $x = 3$ and -1.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **25**

Equations Involving Radicals

When you solve an equation that involves radicals, you must be especially careful to check your final answers. The next example demonstrates why.

EXAMPLE 4 | An Equation Involving a Radical

Solve the equation $2x = 1 - \sqrt{2 - x}$.

SOLUTION To eliminate the square root, we first isolate it on one side of the equal sign, then square:

$$\begin{aligned}
2x - 1 &= -\sqrt{2-x} && \text{Subtract 1}\\
(2x-1)^2 &= 2 - x && \text{Square each side}\\
4x^2 - 4x + 1 &= 2 - x && \text{Expand LHS}\\
4x^2 - 3x - 1 &= 0 && \text{Add } -2 + x\\
(4x+1)(x-1) &= 0 && \text{Factor}\\
4x + 1 = 0 \quad \text{or} &\quad x - 1 = 0 && \text{Zero-Product Property}\\
x = -\tfrac{1}{4} \qquad &\quad x = 1 && \text{Solve}
\end{aligned}$$

CHECK YOUR ANSWERS

$x = -\frac{1}{4}$:

$$\text{LHS} = 2\left(-\tfrac{1}{4}\right) = -\tfrac{1}{2}$$

$$\begin{aligned}
\text{RHS} &= 1 - \sqrt{2 - \left(-\tfrac{1}{4}\right)}\\
&= 1 - \sqrt{\tfrac{9}{4}}\\
&= 1 - \tfrac{3}{2} = -\tfrac{1}{2}
\end{aligned}$$

$$\text{LHS} = \text{RHS} \quad ✓$$

$x = 1$:

$$\text{LHS} = 2(1) = 2$$

$$\begin{aligned}
\text{RHS} &= 1 - \sqrt{2 - 1}\\
&= 1 - 1 = 0
\end{aligned}$$

$$\text{LHS} \neq \text{RHS} \quad ✗$$

The values $x = -\frac{1}{4}$ and $x = 1$ are only potential solutions. We must check them to see whether they satisfy the original equation. From *Check Your Answers* we see that $x = -\frac{1}{4}$ is a solution but $x = 1$ is not. The only solution is $x = -\frac{1}{4}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41**

When we solve an equation, we may end up with one or more **extraneous solutions**, that is, potential solutions that do not satisfy the original equation. In Example 4 the value $x = 1$ is an extraneous solution. Extraneous solutions may be introduced when we square each side of an equation because the operation of squaring can turn a false equation into a true one. For example, $-1 \neq 1$, but $(-1)^2 = 1^2$. Thus the squared equation may be true for more values of the variable than the original equation. That is why you must always check your answers to make sure that each satisfies the original equation.

Equations of Quadratic Type

An equation of the form $aW^2 + bW + c = 0$, where W is an algebraic expression, is an equation of **quadratic type**. We solve equations of quadratic type by substituting for the algebraic expression, as we see in the next two examples.

EXAMPLE 5 | An Equation of Quadratic Type

Solve the equation $\left(1 + \dfrac{1}{x}\right)^2 - 6\left(1 + \dfrac{1}{x}\right) + 8 = 0$.

SOLUTION We could solve this equation by multiplying it out first. But it's easier to think of the expression $1 + \frac{1}{x}$ as the unknown in this equation and give it a new name, W. This turns the equation into a quadratic equation in the new variable W:

$$\left(1+\frac{1}{x}\right)^2 - 6\left(1+\frac{1}{x}\right) + 8 = 0 \qquad \text{Given equation}$$

$$W^2 - 6W + 8 = 0 \qquad \text{Let } W = 1 + \frac{1}{x}$$

$$(W-4)(W-2) = 0 \qquad \text{Factor}$$

$$W - 4 = 0 \quad \text{or} \quad W - 2 = 0 \qquad \text{Zero-Product Property}$$

$$W = 4 \qquad W = 2 \qquad \text{Solve}$$

Now we change these values of W back into the corresponding values of x:

$$1 + \frac{1}{x} = 4 \qquad 1 + \frac{1}{x} = 2 \qquad W = 1 + \frac{1}{x}$$

$$\frac{1}{x} = 3 \qquad \frac{1}{x} = 1 \qquad \text{Subtract 1}$$

$$x = \frac{1}{3} \qquad x = 1 \qquad \text{Take reciprocals}$$

The solutions are $x = \frac{1}{3}$ and $x = 1$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **49**

EXAMPLE 6 | A Fourth-Degree Equation of Quadratic Type

Find all solutions of the equation $x^4 - 8x^2 + 8 = 0$.

SOLUTION If we set $W = x^2$, then we get a quadratic equation in the new variable W:

$$(x^2)^2 - 8x^2 + 8 = 0 \qquad \text{Write } x^4 \text{ as } (x^2)^2$$

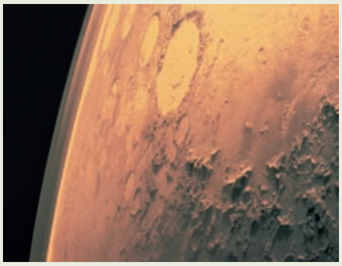
$$W^2 - 8W + 8 = 0 \qquad \text{Let } W = x^2$$

$$W = \frac{-(-8) \pm \sqrt{(-8)^2 - 4\cdot 8}}{2} = 4 \pm 2\sqrt{2} \qquad \text{Quadratic Formula}$$

$$x^2 = 4 \pm 2\sqrt{2} \qquad W = x^2$$

$$x = \pm\sqrt{4 \pm 2\sqrt{2}} \qquad \text{Take square roots}$$

MATHEMATICS IN THE MODERN WORLD



Courtesy of NASA

Error-Correcting Codes

The pictures sent back by the *Pathfinder* spacecraft from the surface of Mars on July 4, 1997, were astoundingly clear. But few watching these pictures were aware of the complex mathematics used to accomplish that feat. The distance to Mars is enormous, and the background noise (or static) is many times stronger than the original signal emitted by the spacecraft. So when scientists receive the signal, it is full of errors. To get a clear picture, the errors must be found and corrected. This same problem of errors is routinely encountered in transmitting bank records when you use an ATM or voice when you are talking on the telephone.

To understand how errors are found and corrected, we must first understand that to transmit pictures, sound, or text, we transform them into bits (the digits 0 or 1; see page 97). To help the receiver recognize errors, the message is "coded" by inserting additional bits. For example, suppose you want to transmit the message "10100." A very simple-minded code is as follows: Send each digit a million times. The person receiving the message reads it in blocks of a million digits. If the first block is mostly 1's, the person concludes that you are probably trying to transmit a 1, and so on. To say that this code is not efficient is a bit of an understatement; it requires sending a million times more data than the original message. Another method inserts "check digits." For example, for each block of eight digits insert a ninth digit; the inserted digit is 0 if there is an even number of 1's in the block and 1 if there is an odd number. So if a single digit is wrong (a 0 changed to a 1 or vice versa), the check digits allow us to recognize that an error has occurred. This method does not tell us where the error is, so we can't correct it. Modern error-correcting codes use interesting mathematical algorithms that require inserting relatively few digits but that allow the receiver to not only recognize, but also correct, errors. The first error-correcting code was developed in the 1940s by Richard Hamming at MIT. It is interesting to note that the English language has a built-in error correcting mechanism; to test it, try reading this error-laden sentence: Gve mo libty ox giv ne deth.

So there are four solutions:

$$\sqrt{4+2\sqrt{2}} \qquad \sqrt{4-2\sqrt{2}} \qquad -\sqrt{4+2\sqrt{2}} \qquad -\sqrt{4-2\sqrt{2}}$$

Using a calculator, we obtain the approximations $x \approx 2.61, 1.08, -2.61, -1.08$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **51**

EXAMPLE 7 | An Equation Involving Fractional Powers

Find all solutions of the equation $x^{1/3} + x^{1/6} - 2 = 0$.

SOLUTION This equation is of quadratic type because if we let $W = x^{1/6}$, then $W^2 = (x^{1/6})^2 = x^{1/3}$:

$$\begin{aligned} x^{1/3} + x^{1/6} - 2 &= 0 && \text{Given equation} \\ W^2 + W - 2 &= 0 && \text{Let } W = x^{1/6} \\ (W-1)(W+2) &= 0 && \text{Factor} \end{aligned}$$

$$\begin{array}{lclll} W - 1 = 0 & \text{or} & W + 2 = 0 & & \text{Zero-Product Property} \\ W = 1 & & W = -2 & & \text{Solve} \\ x^{1/6} = 1 & & x^{1/6} = -2 & & W = x^{1/6} \\ x = 1^6 = 1 & & x = (-2)^6 = 64 & & \text{Take the 6th power} \end{array}$$

From *Check Your Answers* we see that $x = 1$ is a solution but $x = 64$ is not. The only solution is $x = 1$.

CHECK YOUR ANSWERS

$x = 1$:

$\text{LHS} = 1^{1/3} + 1^{1/6} - 2 = 0$

$\text{RHS} = 0$

$\text{LHS} = \text{RHS}$ ✓

$x = 64$:

$\text{LHS} = 64^{1/3} + 64^{1/6} - 2$

$= 4 + 2 - 2 = 4$

$\text{RHS} = 0$

$\text{LHS} \neq \text{RHS}$ ✗

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **57**

▼ Applications

Many real-life problems can be modeled with the types of equations that we have studied in this section.

EXAMPLE 8 | Dividing a Lottery Jackpot

A group of people come forward to claim a \$1,000,000 lottery jackpot, which the winners are to share equally. Before the jackpot is divided, three more winning ticket holders show up. As a result, the share of each of the original winners is reduced by \$75,000. How many winners were in the original group?

SOLUTION Identify the variable. We are asked for the number of people in the original group. So let

$$x = \text{number of winners in the original group}$$

Translate from words to algebra. We translate the information in the problem as follows:

In Words	In Algebra
Number of winners in original group	x
Number of winners in final group	$x + 3$
Winnings per person, originally	$\dfrac{1{,}000{,}000}{x}$
Winnings per person, finally	$\dfrac{1{,}000{,}000}{x+3}$

Set up the model. Now we set up the model:

$$\boxed{\text{winnings per person, originally}} - \boxed{\$75{,}000} = \boxed{\text{winnings per person, finally}}$$

$$\frac{1{,}000{,}000}{x} - 75{,}000 = \frac{1{,}000{,}000}{x+3}$$

Solve. We now solve for x:

$$1{,}000{,}000(x+3) - 75{,}000x(x+3) = 1{,}000{,}000x \qquad \text{Multiply by LCD } x(x+3)$$

$$40(x+3) - 3x(x+3) = 40x \qquad \text{Divide by 25,000}$$

$$x^2 + 3x - 40 = 0 \qquad \text{Expand, simplify, and divide by 3}$$

$$(x+8)(x-5) = 0 \qquad \text{Factor}$$

$$x + 8 = 0 \quad \text{or} \quad x - 5 = 0 \qquad \text{Zero-Product Property}$$

$$x = -8 \qquad\qquad x = 5 \qquad \text{Solve}$$

Since we can't have a negative number of people, we conclude that there were five winners in the original group.

CHECK YOUR ANSWER

$$\text{winnings per person, originally} = \frac{\$1{,}000{,}000}{5} = \$200{,}000$$

$$\text{winnings per person, finally} = \frac{\$1{,}000{,}000}{8} = \$125{,}000$$

$$\$200{,}000 - \$75{,}000 = \$125{,}000 \quad ✓$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **81**

EXAMPLE 9 | Energy Expended in Bird Flight

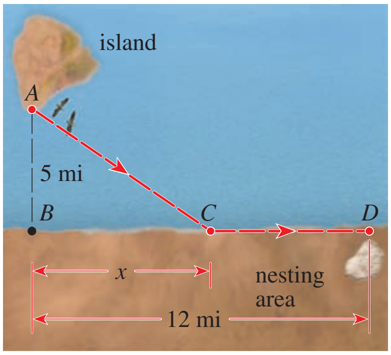


FIGURE 1

Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours, because air generally rises over land and falls over water in the daytime, so flying over water requires more energy. A bird is released from point A on an island, 5 mi from point B, the nearest point on a straight shoreline. The bird flies to a point C on the shoreline and then flies along the shoreline to its nesting area at point D, as shown in Figure 1. Suppose the bird has 170 kcal of energy reserves. It uses 10 kcal/mi flying over land and 14 kcal/mi flying over water.

(a) Where should the point C be located so that the bird uses exactly 170 kcal of energy during its flight?

(b) Does the bird have enough energy reserves to fly directly from A to D?

SOLUTION

(a) Identify the variable. We are asked to find the location of C. So let

$$x = \text{distance from } B \text{ to } C$$

Translate from words to algebra. From the figure, and from the fact that

$$\text{energy used} = \text{energy per mile} \times \text{miles flown}$$

we determine the following:

In Words	In Algebra	
Distance from B to C	x	
Distance flown over water (from A to C)	$\sqrt{x^2 + 25}$	Pythagorean Theorem
Distance flown over land (from C to D)	$12 - x$	
Energy used over water	$14\sqrt{x^2 + 25}$	
Energy used over land	$10(12 - x)$	

Set up the model. Now we set up the model:

$$\boxed{\text{total energy used}} = \boxed{\text{energy used over water}} + \boxed{\text{energy used over land}}$$

$$170 = 14\sqrt{x^2 + 25} + 10(12 - x)$$

Solve. To solve this equation, we eliminate the square root by first bringing all other terms to the left of the equal sign and then squaring each side:

$$170 - 10(12 - x) = 14\sqrt{x^2 + 25} \qquad \text{Isolate square root term on RHS}$$

$$50 + 10x = 14\sqrt{x^2 + 25} \qquad \text{Simplify LHS}$$

$$(50 + 10x)^2 = (14)^2(x^2 + 25) \qquad \text{Square each side}$$

$$2500 + 1000x + 100x^2 = 196x^2 + 4900 \qquad \text{Expand}$$

$$0 = 96x^2 - 1000x + 2400 \qquad \text{All terms to RHS}$$

This equation could be factored, but because the numbers are so large, it is easier to use the Quadratic Formula and a calculator:

$$x = \frac{1000 \pm \sqrt{(-1000)^2 - 4(96)(2400)}}{2(96)} = \frac{1000 \pm 280}{192}$$

$$x = 6\tfrac{2}{3} \qquad \text{or} \qquad x = 3\tfrac{3}{4}$$

Point C should be either $6\frac{2}{3}$ mi or $3\frac{3}{4}$ mi from point B so that the bird uses exactly 170 kcal of energy during its flight.

(b) By the Pythagorean Theorem (see page 253), the length of the route directly from A to D is $\sqrt{5^2 + 12^2} = 13$ mi, so the energy the bird requires for that route is $14 \times 13 = 182$ kcal. This is more energy than the bird has available, so it can't use this route.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **89**

We were able to solve all the equations in this section algebraically; however, not all equations can be solved this way. If our model leads to an equation that cannot be solved algebraically, we can solve it graphically as described in Section 1.4.

1.7 EXERCISES

CONCEPTS

1. **(a)** To solve the equation $x^3 - 4x^2 = 0$, we ________ the left-hand side.

   (b) The solutions of the equation $x^2(x - 4) = 0$ are ________.

2. Solve the equation $\sqrt{2x} + x = 0$ by doing the following steps.

   (a) Isolate the radical: ________________.

   (b) Square both sides: ________________.

   (c) The solutions of the resulting quadratic equation are ________________.

   (d) The solution(s) that satisfy the original equation are ________________.

3. The equation $(x + 1)^2 - 5(x + 1) + 6 = 0$ is of ________ type. To solve the equation, we set $W =$ ________. The resulting quadratic equation is ________________.

4. The equation $x^6 + 7x^3 - 8 = 0$ is of ________ type. To solve the equation, we set $W =$ ________. The resulting quadratic equation is ________________.

SKILLS

5–22 ■ Find all real solutions of the equation.

5. $x^3 = 64x$
6. $x^7 = 25x^5$
7. $x^6 - 81x^2 = 0$
8. $x^5 - 16x = 0$
9. $4z^5 - 10z^2 = 0$
10. $125t^{10} - 2t^7 = 0$
11. $x^5 + 8x^2 = 0$
12. $x^4 + 64x = 0$
13. $x^3 - 5x^2 + 6x = 0$
14. $x^4 - x^3 - 6x^2 = 0$
15. $x^4 + 4x^3 + 2x^2 = 0$
16. $y^5 - 8y^4 + 4y^3 = 0$
17. $(2r + 1)^6 - 16(2r + 1)^4 = 0$
18. $(x - 2)^5 - 9(x - 2)^3 = 0$
19. $x^3 - 5x^2 - 2x + 10 = 0$
20. $2x^3 + x^2 - 18x - 9 = 0$
21. $x^3 - x^2 + x - 1 = x^2 + 1$
22. $7x^3 - x + 1 = x^3 + 3x^2 + x$

23–34 ■ Find all real solutions of the equation.

23. $z + \dfrac{4}{z + 1} = 3$
24. $\dfrac{10}{m + 5} + 15 = 3m$
25. $\dfrac{1}{x - 1} + \dfrac{1}{x + 2} = \dfrac{5}{4}$
26. $\dfrac{10}{x} - \dfrac{12}{x - 3} + 4 = 0$
27. $\dfrac{x^2}{x + 100} = 50$
28. $1 + \dfrac{2x}{(x + 3)(x + 4)} = \dfrac{2}{x + 3} + \dfrac{4}{x + 4}$
29. $\dfrac{x + 5}{x - 2} = \dfrac{5}{x + 2} + \dfrac{28}{x^2 - 4}$
30. $\dfrac{y}{3y + 1} + \dfrac{2}{3y - 1} = \dfrac{4}{9y^2 - 1}$
31. $\dfrac{x}{2x + 7} - \dfrac{x + 1}{x + 3} = 1$
32. $\dfrac{1}{x - 1} - \dfrac{2}{x^2} = 0$
33. $\dfrac{x + \dfrac{2}{x}}{3 + \dfrac{4}{x}} = 5x$
34. $\dfrac{3 + \dfrac{1}{x}}{2 - \dfrac{4}{x}} = x$

35–46 ■ Find all real solutions of the equation.

35. $5 = \sqrt{4x - 3}$
36. $\sqrt{8x - 1} = 3$
37. $\sqrt{2x - 1} = \sqrt{3x - 5}$
38. $\sqrt{3 + x} = \sqrt{x^2 + 1}$
39. $\sqrt{x + 2} = x$
40. $\sqrt{4 - 6x} = 2x$
41. $\sqrt{2x + 1} + 1 = x$
42. $x - \sqrt{9 - 3x} = 0$
43. $x - \sqrt{x - 1} = 3$
44. $\sqrt{3 - x} + 2 = 1 - x$
45. $x - \sqrt{x + 3} = \dfrac{x}{2}$
46. $x + 2\sqrt{x - 7} = 10$

47–56 ■ Find all real solutions of the equation.

47. $(x + 5)^2 - 3(x + 5) - 10 = 0$
48. $\left(\dfrac{x + 1}{x}\right)^2 + 4\left(\dfrac{x + 1}{x}\right) + 3 = 0$
49. $\left(\dfrac{1}{x + 1}\right)^2 - 2\left(\dfrac{1}{x + 1}\right) - 8 = 0$
50. $\left(\dfrac{x}{x + 2}\right)^2 = \dfrac{4x}{x + 2} - 4$
51. $x^4 - 13x^2 + 40 = 0$
52. $x^4 - 5x^2 + 4 = 0$
53. $2x^4 + 4x^2 + 1 = 0$
54. $x^6 - 2x^3 - 3 = 0$
55. $x^6 - 26x^3 - 27 = 0$
56. $x^8 + 15x^4 = 16$

57–64 ■ Find all real solutions of the equation.

57. $x^{4/3} - 5x^{2/3} + 6 = 0$
58. $\sqrt{x} - 3\sqrt[4]{x} - 4 = 0$
59. $4(x + 1)^{1/2} - 5(x + 1)^{3/2} + (x + 1)^{5/2} = 0$
60. $2(x - 4)^{7/3} - (x - 4)^{4/3} - (x - 4)^{1/3} = 0$
61. $x^{3/2} + 8x^{1/2} + 16x^{-1/2} = 0$
62. $x^{1/2} + 3x^{-1/2} = 10x^{-3/2}$
63. $x^{1/2} - 3x^{1/3} = 3x^{1/6} - 9$
64. $x - 5\sqrt{x} + 6 = 0$

65–72 ■ Find all real solutions of the equation.

65. $\dfrac{1}{x^3} + \dfrac{4}{x^2} + \dfrac{4}{x} = 0$

66. $4x^{-4} - 16x^{-2} + 4 = 0$

67. $\sqrt{\sqrt{x+5} + x} = 5$

68. $\sqrt[3]{4x^2 - 4x} = x$

69. $x^2\sqrt{x+3} = (x+3)^{3/2}$

70. $\sqrt{11 - x^2} - \dfrac{2}{\sqrt{11 - x^2}} = 1$

71. $\sqrt{x + \sqrt{x+2}} = 2$

72. $\sqrt{1 + \sqrt{x + \sqrt{2x+1}}} = \sqrt{5 + \sqrt{x}}$

73–76 ■ Solve the equation graphically. Compare your answer to the one obtained in the indicated exercise.

73. $x^3 - 5x^2 - 2x + 10 = 0$ (Exercise 19)

74. $\sqrt{2x+1} + 1 = x$ (Exercise 41)

75. $x^{4/3} - 5x^{2/3} + 6 = 0$ (Exercise 57)

76. $x^2\sqrt{x+3} = (x+3)^{3/2}$ (Exercise 69)

77–80 ■ Solve the equation for the variable x. The constants a and b represent positive real numbers.

77. $x^4 - 5ax^2 + 4a^2 = 0$

78. $a^3x^3 + b^3 = 0$

79. $\sqrt{x+a} + \sqrt{x-a} = \sqrt{2}\sqrt{x+6}$

80. $\sqrt{x} - a\sqrt[3]{x} + b\sqrt[6]{x} - ab = 0$

APPLICATIONS

81. Chartering a Bus A social club charters a bus at a cost of \$900 to take a group of members on an excursion to Atlantic City. At the last minute, five people in the group decide not to go. This raises the transportation cost per person by \$2. How many people originally intended to take the trip?

82. Buying a Cottage A group of friends decides to buy a vacation home for \$120,000, sharing the cost equally. If they can find one more person to join them, each person's contribution will drop by \$6000. How many people are in the group?

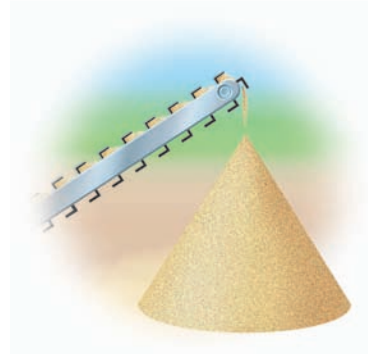
83. Fish Population A large pond is stocked with fish. The fish population P is modeled by the formula $P = 3t + 10\sqrt{t} + 140$, where t is the number of days since the fish were first introduced into the pond. How many days will it take for the fish population to reach 500?

84. The Lens Equation If F is the focal length of a convex lens and an object is placed at a distance x from the lens, then its image will be at a distance y from the lens, where F, x, and y are related by the *lens equation*

$$\frac{1}{F} = \frac{1}{x} + \frac{1}{y}$$

Suppose that a lens has a focal length of 4.8 cm and that the image of an object is 4 cm closer to the lens than the object itself. How far from the lens is the object?

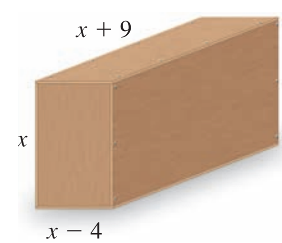
85. Volume of Grain Grain is falling from a chute onto the ground, forming a conical pile whose diameter is always three times its height. How high is the pile (to the nearest hundredth of a foot) when it contains 1000 ft^3 of grain?



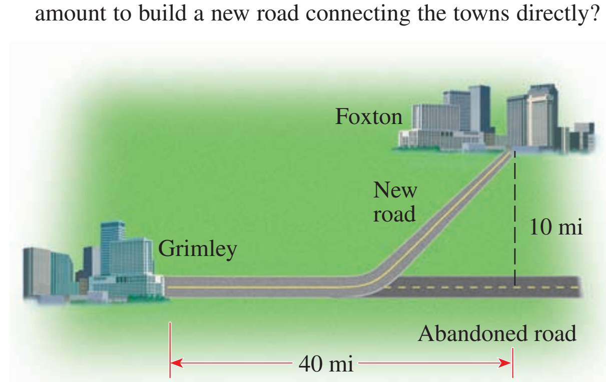
86. Radius of a Tank A spherical tank has a capacity of 750 gallons. Using the fact that 1 gallon is about 0.1337 ft^3, find the radius of the tank (to the nearest hundredth of a foot).

87. Radius of a Sphere A jeweler has three small solid spheres made of gold, of radius 2 mm, 3 mm, and 4 mm. He decides to melt these down and make just one sphere out of them. What will the radius of this larger sphere be?

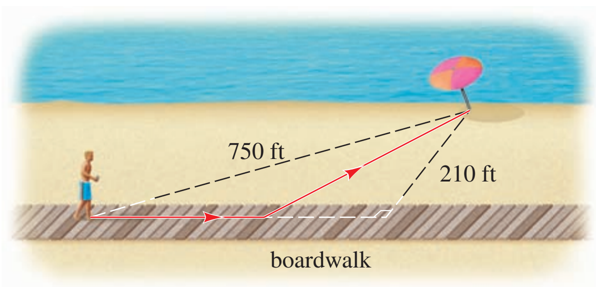
88. Dimensions of a Box A large plywood box has a volume of 180 ft^3. Its length is 9 ft greater than its height, and its width is 4 ft less than its height. What are the dimensions of the box?



89. Construction Costs The town of Foxton lies 10 mi north of an abandoned east-west road that runs through Grimley, as shown in the figure. The point on the abandoned road closest to Foxton is 40 mi from Grimley. County officials are about to build a new road connecting the two towns. They have determined that restoring the old road would cost \$100,000 per mile, while building a new road would cost \$200,000 per mile. How much of the abandoned road should be used (as indicated in the figure) if the officials intend to spend exactly \$6.8 million? Would it cost less than this amount to build a new road connecting the towns directly?



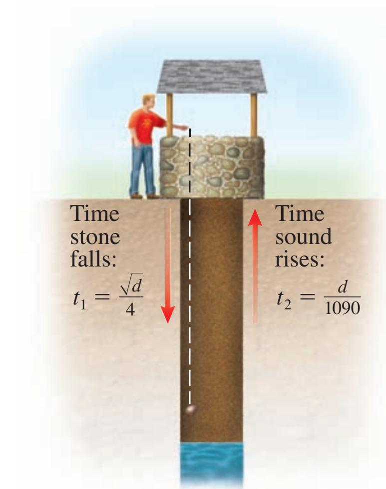
90. Distance, Speed, and Time A boardwalk is parallel to and 210 ft inland from a straight shoreline. A sandy beach lies between the boardwalk and the shoreline. A man is standing on the boardwalk, exactly 750 ft across the sand from his beach umbrella, which is right at the shoreline. The man walks 4 ft/s on the boardwalk and 2 ft/s on the sand. How far should he walk on the boardwalk before veering off onto the sand if he wishes to reach his umbrella in exactly 4 min 45 s?



91. Dimensions of a Lot A city lot has the shape of a right triangle whose hypotenuse is 7 ft longer than one of the other sides. The perimeter of the lot is 392 ft. How long is each side of the lot?

92. TV Monitors Two television monitors sitting beside each other on a shelf in an appliance store have the same screen height. One has a conventional screen, which is 5 in. wider than it is high. The other has a wider, high-definition screen, which is 1.8 times as wide as it is high. The diagonal measure of the wider screen is 14 in. more than the diagonal measure of the smaller. What is the height of the screens, rounded to the nearest 0.1 in.?

93. Depth of a Well One method for determining the depth of a well is to drop a stone into it and then measure the time it takes until the splash is heard. If d is the depth of the well (in feet) and t_1 the time (in seconds) it takes for the stone to fall, then $d = 16t_1^2$, so $t_1 = \sqrt{d}/4$. Now if t_2 is the time it takes for the sound to travel back up, then $d = 1090t_2$ because the speed of sound is 1090 ft/s. So $t_2 = d/1090$. Thus the total time elapsed between dropping the stone and hearing the splash is $t_1 + t_2 = \sqrt{d}/4 + d/1090$. How deep is the well if this total time is 3 s? (See the following figure.)



DISCOVERY ■ DISCUSSION ■ WRITING

94. Solving an Equation in Different Ways We have learned several different ways to solve an equation in this section. Some equations can be tackled by more than one method. For example, the equation $x - \sqrt{x} - 2 = 0$ is of quadratic type: We can solve it by letting $\sqrt{x} = u$ and $x = u^2$ and factoring. Or we could solve for $\sqrt{x}$, square each side, and then solve the resulting quadratic equation. Solve the following equations using both methods indicated, and show that you get the same final answers.

(a) $x - \sqrt{x} - 2 = 0$ Quadratic type; solve for the radical, and square

(b) $\dfrac{12}{(x-3)^2} + \dfrac{10}{x-3} + 1 = 0$ Quadratic type; multiply by LCD

1.8 Solving Inequalities

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Solve Linear Inequalities ▶ Solve Nonlinear Inequalities ▶ Model with Inequalities

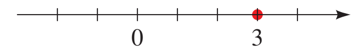
x	$4x + 7 \le 19$
1	$11 \le 19$ ✓
2	$15 \le 19$ ✓
3	$19 \le 19$ ✓
4	$23 \le 19$ ✗
5	$27 \le 19$ ✗

Some problems in algebra lead to **inequalities** instead of equations. An inequality looks just like an equation, except that in the place of the equal sign is one of the symbols, $<$, $>$, $\le$, or $\ge$. Here is an example of an inequality in the one variable x:

$$4x + 7 \le 19$$

The table in the margin shows that some numbers satisfy the inequality and some numbers do not.

To **solve** an inequality that contains a variable means to find all values of the variable that make the inequality true. Unlike an equation, an inequality generally has infinitely many solutions, which form an interval or a union of intervals on the real line. The following illustration shows how an inequality differs from its corresponding equation:

		Solution	**Graph**
Equation:	$4x + 7 = 19$	$x = 3$	
Inequality:	$4x + 7 \leq 19$	$x \leq 3$	

To solve inequalities, we use the following rules to isolate the variable on one side of the inequality sign. These rules tell us when two inequalities are *equivalent* (the symbol $\Leftrightarrow$ means "is equivalent to"). In these rules, the symbols A, B, and C stand for real numbers or algebraic expressions. Here we state the rules for inequalities involving the symbol $\leq$, but they apply to all four inequality symbols.

RULES FOR INEQUALITIES

Rule	Description
1. $A \leq B \quad \Leftrightarrow \quad A + C \leq B + C$	**Adding** the same quantity to each side of an inequality gives an equivalent inequality.
2. $A \leq B \quad \Leftrightarrow \quad A - C \leq B - C$	**Subtracting** the same quantity from each side of an inequality gives an equivalent inequality.
3. If $C > 0$, then $A \leq B \quad \Leftrightarrow \quad CA \leq CB$	**Multiplying** each side of an inequality by the same *positive* quantity gives an equivalent inequality.
4. If $C < 0$, then $A \leq B \quad \Leftrightarrow \quad CA \geq CB$	**Multiplying** each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality.
5. If $A > 0$ and $B > 0$, then $A \leq B \quad \Leftrightarrow \quad \dfrac{1}{A} \geq \dfrac{1}{B}$	**Taking reciprocals** of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality.
6. If $A \leq B$ and $C \leq D$, then $A + C \leq B + D$	Inequalities can be added.

Pay special attention to Rules 3 and 4. Rule 3 says that we can multiply (or divide) each side of an inequality by a *positive* number, but Rule 4 says that if we multiply each side of an inequality by a *negative* number, then we reverse the direction of the inequality. For example, if we start with the inequality

$$3 < 5$$

and multiply by 2, we get

$$6 < 10$$

but if we multiply by -2, we get

$$-6 > -10$$

Solving Linear Inequalities

An inequality is **linear** if each term is constant or a multiple of the variable. To solve a linear inequality, we isolate the variable on one side of the inequality sign.

EXAMPLE 1 | Solving a Linear Inequality

Solve the inequality $3x < 9x + 4$, and sketch the solution set.

SOLUTION

$$\begin{aligned} 3x &< 9x + 4 && \text{Given inequality} \\ 3x - 9x &< 9x + 4 - 9x && \text{Subtract } 9x \\ -6x &< 4 && \text{Simplify} \\ \left(-\tfrac{1}{6}\right)(-6x) &> \left(-\tfrac{1}{6}\right)(4) && \text{Multiply by } -\tfrac{1}{6} \text{ (or divide by } -6) \\ x &> -\tfrac{2}{3} && \text{Simplify} \end{aligned}$$

Multiplying by the negative number $-\frac{1}{6}$ *reverses* the direction of the inequality.

The solution set consists of all numbers greater than $-\frac{2}{3}$. In other words, the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. It is graphed in Figure 1.

FIGURE 1

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **23**

EXAMPLE 2 | Solving a Pair of Simultaneous Inequalities

Solve the inequalities $4 \le 3x - 2 < 13$.

SOLUTION The solution set consists of all values of x that satisfy both of the inequalities $4 \le 3x - 2$ and $3x - 2 < 13$. Using Rules 1 and 3, we see that the following inequalities are equivalent:

$$\begin{aligned} 4 \le 3x - 2 &< 13 && \text{Given inequality} \\ 6 \le 3x &< 15 && \text{Add 2} \\ 2 \le x &< 5 && \text{Divide by 3} \end{aligned}$$

Therefore the solution set is $[2, 5)$, as shown in Figure 2.

FIGURE 2

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **33**

Solving Nonlinear Inequalities

To solve inequalities involving squares and other powers of the variable, we use factoring, together with the following principle.

> **THE SIGN OF A PRODUCT OR QUOTIENT**
>
> If a product or a quotient has an *even* number of *negative* factors, then its value is *positive*.
>
> If a product or a quotient has an *odd* number of *negative* factors, then its value is *negative*.

For example, to solve the inequality $x^2 - 5x \le -6$, we first move all terms to the left-hand side and factor to get

$$(x - 2)(x - 3) \le 0$$

This form of the inequality says that the product $(x - 2)(x - 3)$ must be negative or zero, so to solve the inequality, we must determine where each factor is negative or positive (because the sign of a product depends on the sign of the factors). The details are explained in Example 3, in which we use the following guidelines.

GUIDELINES FOR SOLVING NONLINEAR INEQUALITIES

1. **Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the Intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals that are determined by these numbers.
4. **Make a Table or Diagram.** Use test values to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
5. **Solve.** Determine the solution of the inequality from the last row of the sign table. Be sure to check whether the inequality is satisfied by some or all of the endpoints of the intervals. (This may happen if the inequality involves $\le$ or $\ge$.)

The factoring technique that is described in these guidelines works only if all nonzero terms appear on one side of the inequality symbol. If the inequality is not written in this form, first rewrite it, as indicated in Step 1.

EXAMPLE 3 | Solving a Quadratic Inequality

Solve the inequality $x^2 \le 5x - 6$.

SOLUTION We will follow the guidelines above.

Move all terms to one side. We move all the terms to the left-hand side:

$$\begin{aligned} x^2 &\le 5x - 6 && \text{Given inequality} \\ x^2 - 5x + 6 &\le 0 && \text{Subtract } 5x \text{ and add } 6 \end{aligned}$$

Factor. Factoring the left-hand side of the inequality, we get

$$(x - 2)(x - 3) \le 0 \qquad \text{Factor}$$

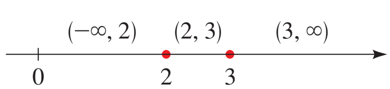


FIGURE 3

Find the intervals. The factors of the left-hand side are $x - 2$ and $x - 3$. These factors are zero when x is 2 and 3, respectively. As shown in Figure 3, the numbers 2 and 3 divide the real line into the three intervals

$$(-\infty, 2), (2, 3), (3, \infty)$$

The factors $x - 2$ and $x - 3$ change sign only at 2 and 3, respectively. So these factors maintain their sign on each of these three intervals.

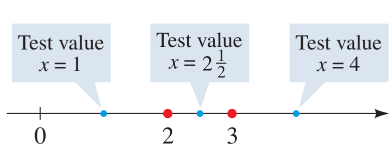


FIGURE 4

Make a table or diagram. To determine the sign of each factor on each of the intervals that we found, we use **test values**. We choose a number inside each interval and check the sign of the factors $x - 2$ and $x - 3$ at the number we chose. For the interval $(-\infty, 2)$, let's choose the test value 1 (see Figure 4). Substituting 1 for x in the factors $x - 2$ and $x - 3$, we get

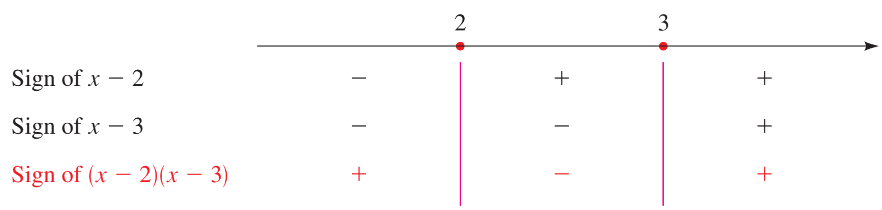
$$\begin{aligned} x - 2 &= 1 - 2 = -1 < 0 \\ x - 3 &= 1 - 3 = -2 < 0 \end{aligned}$$

So both factors are negative on this interval. Notice that we need to check only one test value for each interval because the factors $x - 2$ and $x - 3$ do not change sign on any of the three intervals we found.

Using the test values $x = 2\frac{1}{2}$ and $x = 4$ for the intervals $(2, 3)$ and $(3, \infty)$ (see Figure 4), respectively, we construct the following sign table. The final row of the table is obtained from the fact that the expression in the last row is the product of the two factors.

Interval	$(-\infty, 2)$	$(2, 3)$	$(3, \infty)$
Sign of $x - 2$ Sign of $x - 3$	$-$ $-$	$+$ $-$	$+$ $+$
Sign of $(x - 2)(x - 3)$	$+$	$-$	$+$

If you prefer, you can represent this information on a real line, as in the following sign diagram. The vertical lines indicate the points at which the real line is divided into intervals:



Solve. We read from the table or the diagram that $(x - 2)(x - 3)$ is negative on the interval $(2, 3)$. Thus the solution of the inequality $(x - 2)(x - 3) \le 0$ is

$$\{x \mid 2 \le x \le 3\} = [2, 3]$$

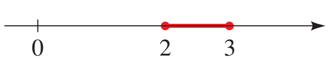


FIGURE 5

We have included the endpoints 2 and 3 because we seek values of x such that the product is either less than *or equal to* zero. The solution is illustrated in Figure 5.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **43**

EXAMPLE 4 | Solving an Inequality

Solve the inequality $2x^2 - x > 1$.

SOLUTION We will follow the guidelines on page 144.

Move all terms to one side. We move all the terms to the left-hand side:

$$2x^2 - x > 1 \qquad \text{Given inequality}$$

$$2x^2 - x - 1 > 0 \qquad \text{Subtract 1}$$

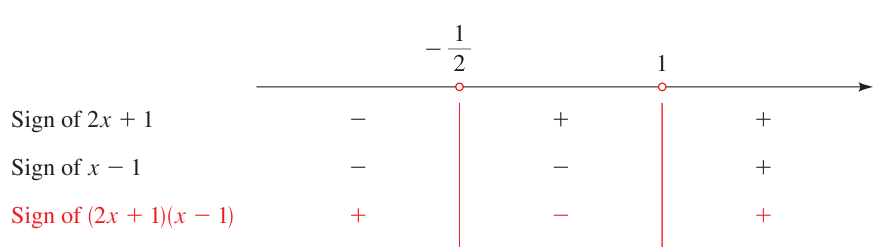
Factor. Factoring the left-hand side of the inequality, we get

$$(2x + 1)(x - 1) > 0 \qquad \text{Factor}$$

Find the intervals. The factors of the left-hand side are $2x + 1$ and $x - 1$. These factors are zero when x is $-\frac{1}{2}$ and 1. These numbers divide the real line into the intervals

$$\left(-\infty, -\tfrac{1}{2}\right), \left(-\tfrac{1}{2}, 1\right), (1, \infty)$$

Make a diagram. We make the following diagram, using test points to determine the sign of each factor in each interval:



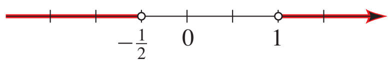


FIGURE 6

Solve. From the diagram we see that $(2x+1)(x-1) > 0$ for x in the interval $\left(-\infty, -\frac{1}{2}\right)$ or for x in $(1, \infty)$. So the solution set is the union of these two intervals:

$$\left(-\infty, -\tfrac{1}{2}\right) \cup (1, \infty)$$

The solution set is graphed in Figure 6.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45**

EXAMPLE 5 | Solving an Inequality with Repeated Factors

Solve the inequality $x(x-1)^2(x-3) < 0$.

SOLUTION All nonzero terms are already on one side of the inequality, and the nonzero side of the inequality is already factored. So we begin by finding the intervals for this inequality.

Find the intervals. The factors of the left-hand side are x, $(x-1)^2$, and $x-3$. These are zero when $x = 0, 1, 3$. These numbers divide the real line into the intervals

$$(-\infty, 0), (0, 1), (1, 3), (3, \infty)$$

Make a diagram. We make the following diagram, using test points to determine the sign of each factor in each interval:

		0		1		3	
Sign of x	−		+		+		+
Sign of $(x-1)^2$	+		+		+		+
Sign of $(x-3)$	−		−		−		+
Sign of $x(x-1)^2(x-3)$	+		−		−		+

Solve. From the diagram we see that $x(x-1)^2(x-3) < 0$ for x in the interval $(0, 1)$ or for x in $(1, 3)$. So the solution set is the union of these two intervals:

$$(0, 1) \cup (1, 3)$$

The solution set is graphed in Figure 7.

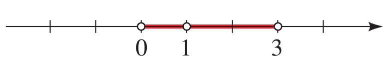


FIGURE 7

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **57**

EXAMPLE 6 | Solving an Inequality Involving a Quotient

Solve the inequality $\dfrac{1+x}{1-x} \geq 1$.

SOLUTION Move all terms to one side. We move the terms to the left-hand side and simplify using a common denominator:

$$\frac{1+x}{1-x} \geq 1 \qquad \text{Given inequality}$$

$$\frac{1+x}{1-x} - 1 \geq 0 \qquad \text{Subtract 1}$$

$$\frac{1+x}{1-x} - \frac{1-x}{1-x} \geq 0 \qquad \text{Common denominator } 1-x$$

$$\frac{1+x-1+x}{1-x} \geq 0 \qquad \text{Combine the fractions}$$

$$\frac{2x}{1-x} \geq 0 \qquad \text{Simplify}$$

It is tempting to multiply both sides of the inequality by $1-x$ (as you would if this were an *equation*). But this doesn't work because we don't know whether $1-x$ is positive or negative, so we can't tell whether the inequality needs to be reversed. (See Exercise 109.)

Find the intervals. The factors of the left-hand side are $2x$ and $1 - x$. These are zero when x is 0 and 1. These numbers divide the real line into the intervals

$$(-\infty, 0), (0, 1), (1, \infty)$$

Make a diagram. We make the following diagram using test points to determine the sign of each factor in each interval:

		0		1	
Sign of $2x$	$-$		$+$		$+$
Sign of $1 - x$	$+$		$+$		$-$
Sign of $\dfrac{2x}{1-x}$	$-$		$+$		$-$

Solve. From the diagram we see that $\dfrac{2x}{1-x} \geq 0$ for x in the interval $[0, 1)$. We include the endpoint 0 because the original inequality requires that the quotient be greater than *or equal to* 1. However, we do not include the other endpoint 1 because the quotient in the inequality is not defined at 1. So the solution set is the interval

$$[0, 1)$$

FIGURE 8

The solution set is graphed in Figure 8.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **65**

Example 6 shows that we should always check the endpoints of the solution set to see whether they satisfy the original inequality.

Solving Inequalities Graphically

Inequalities can be solved graphically. To describe the method, we solve the inequality of Example 3 graphically:

$$x^2 - 5x + 6 \leq 0$$

We first use a graphing calculator to draw the graph of the equation

$$y = x^2 - 5x + 6$$

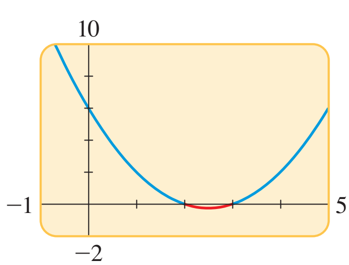


FIGURE 9 $x^2 - 5x + 6 \leq 0$

Our goal is to find those values of x for which $y \leq 0$. These are simply the x-values for which the graph lies below the x-axis. From Figure 9 we see that the solution of the inequality is the interval $[2, 3]$.

EXAMPLE 7 | Solving an Inequality Graphically

Solve the inequality $3.7x^2 + 1.3x - 1.9 \leq 2.0 - 1.4x$.

SOLUTION We graph the equations

$$y_1 = 3.7x^2 + 1.3x - 1.9 \qquad \text{and} \qquad y_2 = 2.0 - 1.4x$$

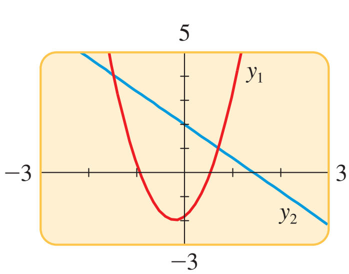


FIGURE 10
$y_1 = 3.7x^2 + 1.3x - 1.9$
$y_2 = 2.0 - 1.4x$

in the same viewing rectangle in Figure 10. We are interested in those values of x for which $y_1 \leq y_2$; these are points for which the graph of y_2 lies on or above the graph of y_1. To determine the appropriate interval, we look for the x-coordinates of points where the graphs intersect. We conclude that the solution is (approximately) the interval $[-1.45, 0.72]$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **79**

EXAMPLE 8 | Solving an Inequality Graphically

Solve the inequality $x^3 - 5x^2 \geq -8$.

SOLUTION We write the inequality as

$$x^3 - 5x^2 + 8 \geq 0$$

and then graph the equation

$$y = x^3 - 5x^2 + 8$$

in the viewing rectangle $[-6, 6]$ by $[-15, 15]$, as shown in Figure 11. The solution of the inequality consists of those intervals on which the graph lies on or above the x-axis. By moving the cursor to the x-intercepts, we find that, rounded to one decimal place, the solution is $[-1.1, 1.5] \cup [4.6, \infty)$.

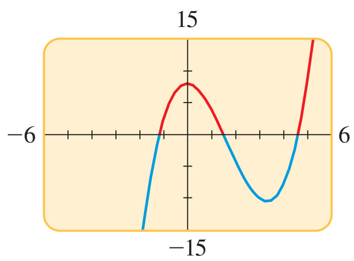


FIGURE 11 $x^3 - 5x^2 + 8 \geq 0$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **81**

Modeling with Inequalities

Modeling real-life problems frequently leads to inequalities because we are often interested in determining when one quantity is more (or less) than another.

EXAMPLE 9 | Carnival Tickets

A carnival has two plans for tickets:

Plan A: $5 entrance fee and 25¢ each ride

Plan B: $2 entrance fee and 50¢ each ride

How many rides would you have to take for Plan A to be less expensive than Plan B?

SOLUTION Identify the variable. We are asked for the number of rides for which Plan A is less expensive than Plan B. So let

$$x = \text{number of rides}$$

Translate from words to algebra. The information in the problem may be organized as follows:

In Words	In Algebra
Number of rides	x
Cost with Plan A	$5 + 0.25x$
Cost with Plan B	$2 + 0.50x$

Set up the model. Now we set up the model:

$$\boxed{\text{cost with Plan A}} < \boxed{\text{cost with Plan B}}$$

$$5 + 0.25x < 2 + 0.50x$$

Solve. We solve for x.

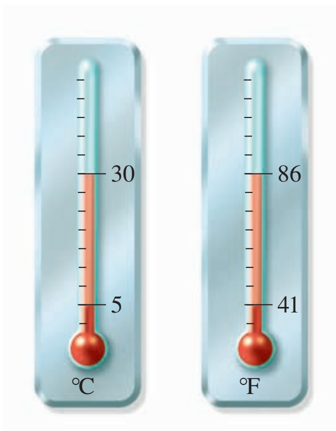
$$3 + 0.25x < 0.50x \qquad \text{Subtract 2}$$

$$3 < 0.25x \qquad \text{Subtract } 0.25x$$

$$12 < x \qquad \text{Divide by 0.25}$$

So if you plan to take *more than* 12 rides, Plan A is less expensive.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **95**



EXAMPLE 10 | Relationship Between Fahrenheit and Celsius Scales

The instructions on a bottle of medicine indicate that the bottle should be stored at a temperature between 5 °C and 30 °C. What range of temperatures does this correspond to on the Fahrenheit scale?

SOLUTION The relationship between degrees Celsius (C) and degrees Fahrenheit (F) is given by the equation $C = \frac{5}{9}(F - 32)$. Expressing the statement on the bottle in terms of inequalities, we have

$$5 < C < 30$$

So the corresponding Fahrenheit temperatures satisfy the inequalities

$5 < \frac{5}{9}(F - 32) < 30$	Substitute $C = \frac{5}{9}(F - 32)$
$\frac{9}{5} \cdot 5 < F - 32 < \frac{9}{5} \cdot 30$	Multiply by $\frac{9}{5}$
$9 < F - 32 < 54$	Simplify
$9 + 32 < F < 54 + 32$	Add 32
$41 < F < 86$	Simplify

The medicine should be stored at a temperature between 41°F and 86 °F.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **93**

1.8 EXERCISES

CONCEPTS

1. Fill in the blank with an appropriate inequality sign.

(a) If $x < 5$, then $x - 3$ ________ 2.

(b) If $x \le 5$, then $3x$ ________ 15.

(c) If $x \ge 2$, then $-3x$ ________ -6.

(d) If $x < -2$, then $-x$ ________ 2.

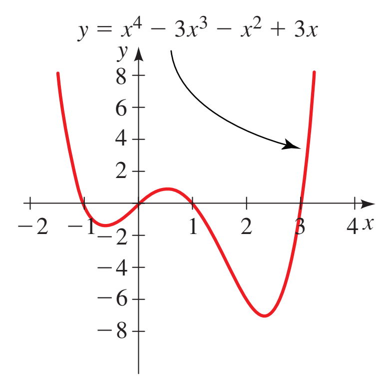
2. To solve the nonlinear inequality $x^2 \le -5x + 14$, we first move all terms to one side to get the inequality ______________, and then factor to get the inequality ______________. The numbers _____ and _____ divide the real line into the intervals ________________________. Complete the table.

Interval			
Sign of Sign of			
Sign of			

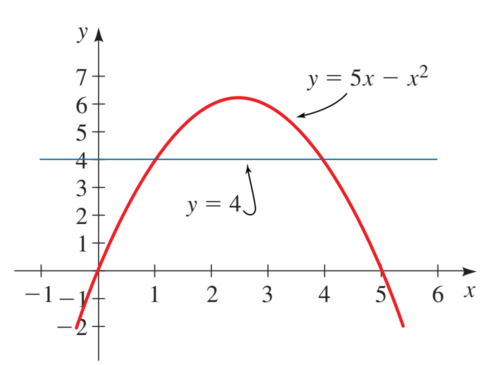
The solution of the inequality is

____

3. The figure shows a graph of $y = x^4 - 3x^3 - x^2 + 3x$. Use the graph to find the solutions of the inequality $x^4 - 3x^3 - x^2 + 3x \le 0$.



4. The figure shows the graphs of $y = 5x - x^2$ and $y = 4$. Use the graphs to find the solutions of the inequality $5x - x^2 > 4$.



SKILLS

5–14 ■ Let $S = \{-2, -1, 0, \frac{1}{2}, 1, \sqrt{2}, 2, 4\}$. Determine which elements of S satisfy the inequality.

5. $x - 3 > 0$ **6.** $x + 1 < 2$

7. $3 - 2x \le \frac{1}{2}$ **8.** $2x - 1 \ge x$

9. $1 < 2x - 4 \le 7$ **10.** $-2 \le 3 - x < 2$

11. $\dfrac{1}{x} \le \dfrac{1}{2}$ **12.** $\dfrac{3}{x} \ge 6$

13. $1 - x^2 \le -1$ **14.** $x^2 + 2 < 4$

15–38 ■ Solve the linear inequality. Express the solution using interval notation and graph the solution set.

15. $2x \le 7$ **16.** $-4x \ge 10$

17. $2x - 5 > 3$ **18.** $3x + 11 < 5$

19. $7 - x \ge 5$ **20.** $5 - 3x \le -16$

21. $2x + 1 < 0$ **22.** $0 < 5 - 2x$

23. $3x + 11 \le 6x + 8$ **24.** $6 - x \ge 2x + 9$

25. $\frac{1}{2}x - \frac{2}{3} > 2$ **26.** $\frac{2}{5}x + 1 < \frac{1}{5} - 2x$

27. $\frac{1}{3}x + 2 < \frac{1}{6}x - 1$ **28.** $\frac{2}{3} - \frac{1}{2}x \ge \frac{1}{6} + x$

29. $4 - 3x \le -(1 + 8x)$ **30.** $2(7x - 3) \le 12x + 16$

31. $2 \le x + 5 < 4$ **32.** $5 \le 3x - 4 \le 14$

33. $-1 < 2x - 5 < 7$ **34.** $1 < 3x + 4 \le 16$

35. $-2 < 8 - 2x \le -1$ **36.** $-3 \le 3x + 7 \le \frac{1}{2}$

37. $\dfrac{2}{3} \ge \dfrac{2x - 3}{12} > \dfrac{1}{6}$ **38.** $-\dfrac{1}{2} \le \dfrac{4 - 3x}{5} \le \dfrac{1}{4}$

39–60 ■ Solve the nonlinear inequality. Express the solution using interval notation and graph the solution set.

39. $(x + 2)(x - 3) < 0$ **40.** $(x - 5)(x + 4) \ge 0$

41. $x(2x + 7) \ge 0$ **42.** $x(2 - 3x) \le 0$

43. $x^2 - 3x - 18 \le 0$ **44.** $x^2 + 5x + 6 > 0$

45. $2x^2 + x \ge 1$ **46.** $x^2 < x + 2$

47. $3x^2 - 3x < 2x^2 + 4$ **48.** $5x^2 + 3x \ge 3x^2 + 2$

49. $x^2 > 3(x + 6)$ **50.** $x^2 + 2x > 3$

51. $x^2 < 4$ **52.** $x^2 \ge 9$

53. $(x + 2)(x - 1)(x - 3) \le 0$

54. $(x - 5)(x - 2)(x + 1) > 0$

55. $(x - 4)(x + 2)^2 < 0$ **56.** $(x + 3)^2(x + 1) > 0$

57. $(x - 2)^2(x - 3)(x + 1) \le 0$

58. $x^2(x^2 - 1) \ge 0$

59. $x^3 - 4x > 0$ **60.** $16x \le x^3$

61–78 ■ Solve the linear inequality. Express the solution using interval notation and graph the solution set.

61. $\dfrac{x - 3}{x + 1} \ge 0$ **62.** $\dfrac{2x + 6}{x - 2} < 0$

63. $\dfrac{4x}{2x + 3} > 2$ **64.** $-2 < \dfrac{x + 1}{x - 3}$

65. $\dfrac{2x + 1}{x - 5} \le 3$ **66.** $\dfrac{3 + x}{3 - x} \ge 1$

67. $\dfrac{4}{x} < x$ **68.** $\dfrac{x}{x + 1} > 3x$

69. $1 + \dfrac{2}{x + 1} \le \dfrac{2}{x}$ **70.** $\dfrac{3}{x - 1} - \dfrac{4}{x} \ge 1$

71. $\dfrac{6}{x - 1} - \dfrac{6}{x} \ge 1$ **72.** $\dfrac{x}{2} \ge \dfrac{5}{x + 1} + 4$

73. $\dfrac{x + 2}{x + 3} < \dfrac{x - 1}{x - 2}$ **74.** $\dfrac{1}{x + 1} + \dfrac{1}{x + 2} \le 0$

75. $\dfrac{(x - 1)(x + 2)}{(x - 2)^2} \ge 0$ **76.** $\dfrac{(2x - 1)(x - 3)^2}{x - 4} < 0$

77. $x^4 > x^2$ **78.** $x^5 > x^2$

79–86 ■ Solve the inequality graphically. Express the solution using interval notation.

79. $x^2 \le 3x + 10$ **80.** $0.5x^2 + 0.875x \le 0.25$

81. $x^3 + 11x \le 6x^2 + 6$ **82.** $16x^3 + 24x^2 > -9x - 1$

83. $x^{1/3} < x$ **84.** $\sqrt{0.5x^2 + 1} \le 2|x|$

85. $(x + 1)^2 < (x - 1)^2$ **86.** $(x + 1)^2 \le x^3$

87–90 ■ Determine the values of the variable for which the expression is defined as a real number.

87. $\sqrt{16 - 9x^2}$ **88.** $\sqrt{3x^2 - 5x + 2}$

89. $\left(\dfrac{1}{x^2 - 5x - 14}\right)^{1/2}$ **90.** $\sqrt[4]{\dfrac{1 - x}{2 + x}}$

91. Solve the inequality for x, assuming that a, b, and c are positive constants.
(a) $a(bx - c) \ge bc$ **(b)** $a \le bx + c < 2a$

92. Suppose that a, b, c, and d are positive numbers such that

$$\frac{a}{b} < \frac{c}{d}$$

Show that $\dfrac{a}{b} < \dfrac{a + c}{b + d} < \dfrac{c}{d}$.

APPLICATIONS

93. Temperature Scales Use the relationship between C and F given in Example 8 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \le C \le 30$.

94. Temperature Scales What interval on the Celsius scale corresponds to the temperature range $50 \le F \le 95$?

95. Car Rental Cost A car rental company offers two plans for renting a car:

Plan A: \$30 per day and 10¢ per mile

Plan B: \$50 per day with free unlimited mileage

For what range of miles will Plan B save you money?

96. Long-Distance Cost A telephone company offers two long-distance plans:

Plan A: $25 per month and 5¢ per minute

Plan B: $5 per month and 12¢ per minute

For how many minutes of long-distance calls would Plan B be financially advantageous?

97. Driving Cost It is estimated that the annual cost of driving a certain new car is given by the formula

$$C = 0.35m + 2200$$

where m represents the number of miles driven per year and C is the cost in dollars. Jane has purchased such a car and decides to budget between $6400 and $7100 for next year's driving costs. What is the corresponding range of miles that she can drive her new car?

98. Air Temperature As dry air moves upward, it expands and, in so doing, cools at a rate of about 1°C for each 100-meter rise, up to about 12 km.

(a) If the ground temperature is 20°C, write a formula for the temperature at height h.

(b) What range of temperatures can be expected if an airplane takes off and reaches a maximum height of 5 km?

99. Airline Ticket Price A charter airline finds that on its Saturday flights from Philadelphia to London all 120 seats will be sold if the ticket price is $200. However, for each $3 increase in ticket price, the number of seats sold decreases by one.

(a) Find a formula for the number of seats sold if the ticket price is P dollars.

(b) Over a certain period the number of seats sold for this flight ranged between 90 and 115. What was the corresponding range of ticket prices?

100. Accuracy of a Scale A coffee merchant sells a customer 3 lb of Hawaiian Kona at $6.50 per pound. The merchant's scale is accurate to within ± 0.03 lb. By how much could the customer have been overcharged or undercharged because of possible inaccuracy in the scale?

101. Gravity The gravitational force F exerted by the earth on an object having a mass of 100 kg is given by the equation

$$F = \frac{4,000,000}{d^2}$$

where d is the distance (in km) of the object from the center of the earth, and the force F is measured in newtons (N). For what distances will the gravitational force exerted by the earth on this object be between 0.0004 N and 0.01 N?

102. Bonfire Temperature In the vicinity of a bonfire the temperature T in °C at a distance of x meters from the center of the fire was given by

$$T = \frac{600,000}{x^2 + 300}$$

At what range of distances from the fire's center was the temperature less than 500°C?

103. Falling Ball Using calculus, it can be shown that if a ball is thrown upward with an initial velocity of 16 ft/s from the top of a building 128 ft high, then its height h above the ground t seconds later will be

$$h = 128 + 16t - 16t^2$$

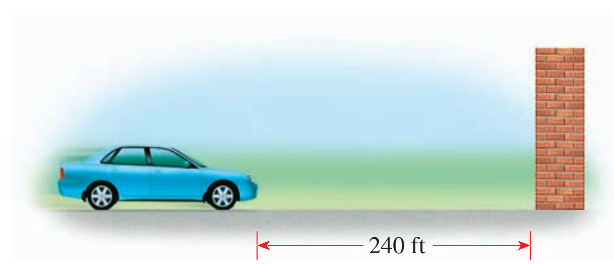
During what time interval will the ball be at least 32 ft above the ground?

104. Gas Mileage The gas mileage g (measured in mi/gal) for a particular vehicle, driven at v mi/h, is given by the formula $g = 10 + 0.9v - 0.01v^2$, as long as v is between 10 mi/h and 75 mi/h. For what range of speeds is the vehicle's mileage 30 mi/gal or better?

105. Stopping Distance For a certain model of car the distance d required to stop the vehicle if it is traveling at v mi/h is given by the formula

$$d = v + \frac{v^2}{20}$$

where d is measured in feet. Kerry wants her stopping distance not to exceed 240 ft. At what range of speeds can she travel?



106. Manufacturer's Profit If a manufacturer sells x units of a certain product, revenue R and cost C (in dollars) are given by

$$R = 20x$$
$$C = 2000 + 8x + 0.0025x^2$$

Use the fact that

$$\text{profit} = \text{revenue} - \text{cost}$$

to determine how many units the manufacturer should sell to enjoy a profit of at least $2400.

107. Fencing a Garden A determined gardener has 120 ft of deer-resistant fence. She wants to enclose a rectangular vegetable garden in her backyard, and she wants the area that is enclosed to be at least 800 ft^2. What range of values is possible for the length of her garden?

DISCOVERY ■ DISCUSSION ■ WRITING

108. Do Powers Preserve Order? If $a < b$, is $a^2 < b^2$? (Check both positive and negative values for a and b.) If $a < b$, is $a^3 < b^3$? On the basis of your observations, state a general rule about the relationship between a^n and b^n when $a < b$ and n is a positive integer.

109. What's Wrong Here? It is tempting to try to solve an inequality as if it were an equation. For instance, we might try to solve $1 < 3/x$ by multiplying both sides by x, to get $x < 3$, so the solution would be $(-\infty, 3)$. But that's wrong; for example, $x = -1$ lies in this interval but does not satisfy the original inequality. Explain why this method doesn't work (think about the *sign* of x). Then solve the inequality correctly.

1.9 Solving Absolute Value Equations and Inequalities

LEARNING OBJECTIVES **After completing this section, you will be able to:**

| Solve Absolute Value Equations ► Solve Absolute Value Inequalities

Recall from Section P.2 that the absolute value of a number a is given by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

and that it represents the distance from a to the origin on the real number line (see Figure 1). More generally, $|x - a|$ is the distance between x and a on the real number line. Figure 2 illustrates the fact that the distance between 2 and 5 is 3.

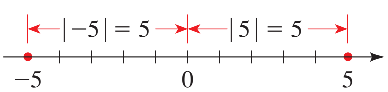


FIGURE 1

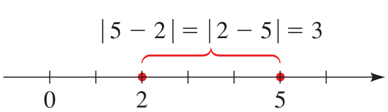


FIGURE 2

▼ Absolute Value Equations

We use the following property to solve equations that involve absolute value.

$$\boxed{|x| = C \qquad \text{is equivalent to} \qquad x = \pm C}$$

This property says that to solve an absolute value equation, we must solve *two* separate equations. For example, the equation $|x| = 5$ is equivalent to the two equations $x = 5$ and $x = -5$.

EXAMPLE 1 | Solving an Absolute Value Equation

Solve the equation $|2x - 5| = 3$.

SOLUTION The equation $|2x - 5| = 3$ is equivalent to two equations:

$2x - 5 = 3$	or	$2x - 5 = -3$	
$2x = 8$		$2x = 2$	Add 5
$x = 4$		$x = 1$	Divide by 2

The solutions are 1 and 4.

CHECK YOUR ANSWERS

$x = 1$:

$$\text{LHS} = |2 \cdot 1 - 5| = |-3| = 3 = \text{RHS} \quad ✓$$

$x = 4$:

$$\text{LHS} = |2 \cdot 4 - 5| = |3| = 3 = \text{RHS} \quad ✓$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **13** ■

EXAMPLE 2 | Solving an Absolute Value Equation

Solve the equation $3|x - 7| + 5 = 14$.

SOLUTION We first isolate the absolute value on one side of the equal sign.

$$3|x - 7| + 5 = 14 \quad \text{Given equation}$$
$$3|x - 7| = 9 \quad \text{Subtract 5}$$
$$|x - 7| = 3 \quad \text{Divide by 3}$$
$$x - 7 = 3 \quad \text{or} \quad x - 7 = -3 \quad \text{Take cases}$$
$$x = 10 \qquad\qquad x = 4 \quad \text{Add 7}$$

The solutions are 4 and 10.

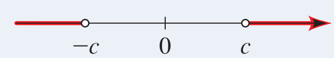
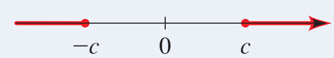
PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17**

Absolute Value Inequalities

We use the following properties to solve inequalities that involve absolute value.

These properties hold when x is replaced by any algebraic expression.

PROPERTIES OF ABSOLUTE VALUE INEQUALITIES

Inequality	Equivalent form	Graph
1. $\lvert x \rvert < c$	$-c < x < c$	
2. $\lvert x \rvert \le c$	$-c \le x \le c$	
3. $\lvert x \rvert > c$	$x < -c$ or $c < x$	
4. $\lvert x \rvert \ge c$	$x \le -c$ or $c \le x$	

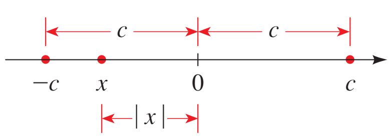


FIGURE 3

These properties can be proved by using the definition of absolute value. To prove Property 1, for example, note that the inequality $|x| < c$ says that the distance from x to 0 is less than c, and from Figure 3 you can see that this is true if and only if x is between c and $-c$.

EXAMPLE 3 | Solving an Absolute Value Inequality

Solve the inequality $|x - 5| < 2$.

SOLUTION 1 The inequality $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2 \quad \text{Property 1}$$
$$3 < x < 7 \quad \text{Add 5}$$

The solution set is the open interval $(3, 7)$.

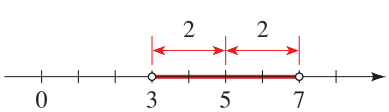


FIGURE 4

SOLUTION 2 Geometrically, the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 4 we see that this is the interval $(3, 7)$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **27**

EXAMPLE 4 | Solving an Absolute Value Inequality

Solve the inequality $|3x + 2| \ge 4$.

SOLUTION By Property 4 the inequality $|3x + 2| \ge 4$ is equivalent to

$$3x + 2 \ge 4 \quad \text{or} \quad 3x + 2 \le -4$$
$$3x \ge 2 \qquad\qquad 3x \le -6 \quad \text{Subtract 2}$$
$$x \ge \tfrac{2}{3} \qquad\qquad x \le -2 \quad \text{Divide by 3}$$

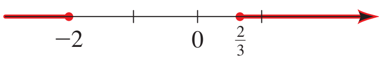


FIGURE 5

So the solution set is

$$\{x \mid x \leq -2 \quad \text{or} \quad x \geq \tfrac{2}{3}\} = (-\infty, -2] \cup [\tfrac{2}{3}, \infty)$$

The solution set is graphed in Figure 5.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **31** ■

EXAMPLE 5 | Piston Tolerances

The specifications for a car engine indicate that the pistons have diameter 3.8745 in. with a tolerance of 0.0015 in. This means that the diameters can vary from the indicated specification by as much as 0.0015 in. and still be acceptable.

(a) Find an inequality involving absolute values that describes the range of possible diameters for the pistons.

(b) Solve the inequality.

SOLUTION

(a) Let d represent the actual diameter of a piston. Since the difference between the actual diameter (d) and the specified diameter (3.8745) is less than 0.0015, we have

$$|d - 3.8745| \leq 0.0015$$

(b) The inequality is equivalent to

$$-0.0015 \leq d - 3.8745 \leq 0.0015 \qquad \text{Property 1}$$

$$3.8730 \leq d \leq 3.8760 \qquad \text{Add 3.8745}$$

Acceptable piston diameters may vary between 3.8730 in. and 3.8760 in.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **57** ■

1.9 EXERCISES

CONCEPTS

1. The equation $|x| = 3$ has the two solutions ________ and ________.

2. The solution of the inequality $|x| \leq 3$ is the interval ________.

3. The solution of the inequality $|x| \geq 3$ is a union of two intervals ________ $\cup$ ________.

4. (a) The set of all points on the real line whose distance from zero is less than 3 can be described by the absolute value inequality $|x|$________.

(b) The set of all points on the real line whose distance from zero is greater than 3 can be described by the absolute value inequality $|x|$ ________.

SKILLS

5–22 ■ Solve the equation.

5. $|4x| = 24$ **6.** $|6x| = 15$

7. $5|x| + 3 = 28$ **8.** $\frac{1}{2}|x| - 7 = 2$

9. $|x - 3| = 2$ **10.** $|2x - 3| = 7$

11. $|x + 4| = 0.5$ **12.** $|x - 4| = -3$

13. $|4x + 7| = 9$ **14.** $|\frac{1}{2}x - 2| = 1$

15. $4 - |3x + 6| = 1$ **16.** $|5 - 2x| + 6 = 14$

17. $3|x + 5| + 6 = 15$ **18.** $20 + |2x - 4| = 15$

19. $8 + 5|\frac{1}{3}x - \frac{5}{6}| = 33$ **20.** $|\frac{3}{5}x + 2| - \frac{1}{2} = 4$

21. $|x - 1| = |3x + 2|$ **22.** $|x + 3| = |2x + 1|$

23–48 ■ Solve the inequality. Express the answer using interval notation.

23. $|x| \leq 4$ **24.** $|3x| < 15$

25. $|2x| > 7$ **26.** $\frac{1}{2}|x| \geq 1$

27. $|x - 5| \leq 3$ **28.** $|x - 9| > 9$

29. $|x + 1| \geq 1$ **30.** $|x + 4| \leq 0$

31. $|x + 5| \geq 2$ **32.** $|x + 1| \geq 3$

33. $|2x - 3| \leq 0.4$ **34.** $|5x - 2| < 6$

35. $\left|\dfrac{x - 2}{3}\right| < 2$ **36.** $\left|\dfrac{x + 1}{2}\right| \geq 4$

37. $|x + 6| < 0.001$ **38.** $|x - a| < d$

39. $4|x+2|-3<13$

40. $3-|2x+4|\le 1$

41. $8-|2x-1|\ge 6$

42. $7|x+2|+5>4$

43. $\frac{1}{2}|4x+\frac{1}{3}|>\frac{5}{6}$

44. $2|\frac{1}{2}x+3|+3\le 51$

45. $1\le |x|\le 4$

46. $0<|x-5|\le\frac{1}{2}$

47. $\dfrac{1}{|x+7|}>2$

48. $\dfrac{1}{|2x-3|}\le 5$

49–52 ■ A phrase that describes a set of real numbers is given. Express the phrase as an inequality involving an absolute value.

49. All real numbers x less than 3 units from 0

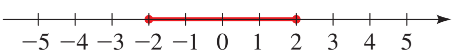
50. All real numbers x more than 2 units from 0

51. All real numbers x at least 5 units from 7

52. All real numbers x at most 4 units from 2

53–56 ■ A set of real numbers is graphed. Find an inequality involving an absolute value that describes the set.

53.



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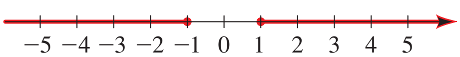

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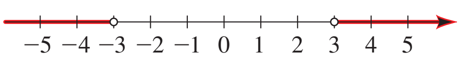

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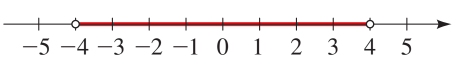
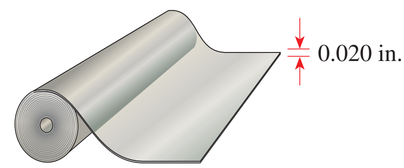


APPLICATIONS

57. Thickness of a Laminate A company manufactures industrial laminates (thin nylon-based sheets) of thickness 0.020 in., with a tolerance of 0.003 in.

(a) Find an inequality involving absolute values that describes the range of possible thickness for the laminate.

(b) Solve the inequality that you found in part (a).



58. Range of Height The average height of adult males is 68.2 in., and 95% of adult males have height h that satisfies the inequality

$$\left|\frac{h-68.2}{2.9}\right|\le 2$$

Solve the inequality to find the range of heights.

DISCOVERY ■ DISCUSSION ■ WRITING

59. Using Distances to Solve Absolute Value Inequalities Recall that $|a-b|$ is the distance between a and b on the number line. For any number x, what do $|x-1|$ and $|x-3|$ represent? Use this interpretation to solve the inequality $|x-1|<|x-3|$ geometrically. In general, if $a<b$, what is the solution of the inequality $|x-a|<|x-b|$?

CHAPTER 1 | REVIEW

■ PROPERTIES AND FORMULAS

The Distance Formula (p. 75)

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$d(A,B)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

The Midpoint Formula (p. 76)

The midpoint of the line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Intercepts (p. 82)

To find the ***x*-intercepts** of the graph of an equation, set $y=0$ and solve for x.

To find the ***y*-intercepts** of the graph of an equation, set $x=0$ and solve for y.

Circles (p. 84)

The circle with center (0, 0) and radius r has equation

$$x^2+y^2=r^2$$

The circle with center (h, k) and radius r has equation

$$(x-h)^2+(y-k)^2=r^2$$

Symmetry (p. 86)

The graph of an equation is **symmetric with respect to the *x*-axis** if the equation remains unchanged when you replace y by $-y$.

The graph of an equation is **symmetric with respect to the *y*-axis** if the equation remains unchanged when you replace x by $-x$.

The graph of an equation is **symmetric with respect to the origin** if the equation remains unchanged when you replace x by $-x$ and y by $-y$.

Slope of a Line (p. 91)

The slope of the nonvertical line that contains the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m=\frac{\text{rise}}{\text{run}}=\frac{y_2-y_1}{x_2-x_1}$$

Equations of Lines (pp. 92–95)

If a line has slope m, has y-intercept b, and contains the point (x_1, y_1), then:

the **point-slope form** of its equation is

$$y - y_1 = m(x - x_1)$$

the **slope-intercept form** of its equation is

$$y = mx + b$$

The equation of any line can be expressed in the **general form**

$$Ax + By + C = 0$$

(where A and B can't both be 0).

Vertical and Horizontal Lines (p. 94)

The **vertical** line containing the point (a, b) has the equation $x = a$.

The **horizontal** line containing the point (a, b) has the equation $y = b$.

Parallel and Perpendicular Lines (pp. 95–96)

Two lines with slopes m_1 and m_2 are

parallel if and only if $m_1 = m_2$

perpendicular if and only if $m_1 m_2 = -1$

Zero-Product Property (p. 121)

If $AB = 0$ then $A = 0$ or $B = 0$.

Completing the Square (p. 122)

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Quadratic Formula (pp. 123–125)

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

Its solutions are given by the **Quadratic Formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The **discriminant** is $D = b^2 - 4ac$.

If $D > 0$, the equation has two real solutions.

If $D = 0$, the equation has one solution.

If $D < 0$, the equation has two complex solutions.

Inequalities (p. 142)

Adding the same quantity to each side of an inequality gives an equivalent inequality:

$$A < B \quad \Leftrightarrow \quad A + C < B + C$$

Multiplying each side of an inequality by the same *positive* quantity gives an equivalent inequality. Multiplying each side by the same *negative* quantity reverses the direction of the inequality:

$$A < B \quad \Leftrightarrow \quad CA < CB \quad \text{if } C > 0$$
$$A < B \quad \Leftrightarrow \quad CA > CB \quad \text{if } C < 0$$

Absolute Value Equations (p. 152)

To solve an absolute value equation, we use

$$|x| = C \quad \Leftrightarrow \quad x = C \quad \text{or} \quad x = -C$$

Absolute Value Inequalities (p. 153)

To solve an absolute value inequality, we use

$$|x| < C \quad \Leftrightarrow \quad -C < x < C$$
$$|x| > C \quad \Leftrightarrow \quad x < -C \quad \text{or} \quad x > C$$

■ LEARNING OBJECTIVES SUMMARY

Section	After completing this chapter, you should be able to . . .	Review Exercises
1.1	▪ Graph points and regions in the coordinate plane	1–6
	▪ Use the Distance Formula	1–4, 7, 10
	▪ Use the Midpoint Formula	1–4, 10
1.2	▪ Graph equations	15–24, 33–36
	▪ Find intercepts	25–32, 33–36
	▪ Find equations of circles	1–4, 8–10, 11–14, 65–66
	▪ Graph circles in a coordinate plane	1–4, 11–14
	▪ Determine symmetry properties of an equation	25–32
1.3	▪ Find the slope of a line	1–4
	▪ Find the equation of a line given a point and the slope	1–4, 38, 39, 65–66
	▪ Find the equation of a line given the slope and y-intercept	1–4, 37–46
	▪ Find equations of horizontal and vertical lines	41–42
	▪ Graph equations of lines	1–4, 37–46
	▪ Find equations for parallel and perpendicular lines	43–46, 47–48
	▪ Make a linear model: interpret slope as rate of change	49–50
1.4	▪ Solve equations graphically	51–52, 57–60

1.5	▪ Make linear equations that model real-world situations	85–86, 89–90, 92
	▪ Use equations to solve problems about real-world situations	85–94
1.6	▪ Solve quadratic equations by factoring, completing the square, or using the Quadratic Formula	67–70, 73–74
	▪ Model with quadratic equations	88, 91, 93
1.7	▪ Solve basic polynomial equations	71–72, 75–76
	▪ Solve equations involving radicals	78–80
	▪ Solve equations of quadratic type	77–80
	▪ Model with equations	87, 94
1.8	▪ Solve linear inequalities	95–98
	▪ Solve nonlinear inequalities	53–56, 61–64, 99–104, 109
	▪ Model with inequalities	110
1.9	▪ Solve absolute value equations	81–84
	▪ Solve absolute value inequalities	105–108

EXERCISES

1–4 ▪ Two points P and Q are given.
(a) Plot P and Q on a coordinate plane.
(b) Find the distance from P to Q.
(c) Find the midpoint of the segment PQ.
(d) Find the slope of the line determined by P and Q, and find equations for the line in point-slope form and in slope-intercept form. Then sketch a graph of the line.
(e) Sketch the circle that passes through Q and has center P, and find the equation of this circle.

1. $P(0, 3)$, $Q(3, 7)$ **2.** $P(2, 0)$, $Q(-4, 8)$

3. $P(-6, 2)$, $Q(4, -14)$ **4.** $P(5, -2)$, $Q(-3, -6)$

5–6 ▪ Sketch the region given by the set.

5. $\{(x, y) \mid -4 < x < 4 \text{ and } -2 < y < 2\}$

6. $\{(x, y) \mid x \geq 4 \text{ or } y \geq 2\}$

7. Which of the points $A(4, 4)$ or $B(5, 3)$ is closer to the point $C(-1, -3)$?

8. Find an equation of the circle that has center $(2, -5)$ and radius $\sqrt{2}$.

9. Find an equation of the circle that has center $(-5, -1)$ and passes through the origin.

10. Find an equation of the circle that contains the points $P(2, 3)$ and $Q(-1, 8)$ and has the midpoint of the segment PQ as its center.

11–14 ▪ **(a)** Complete the square to determine whether the equation represents a circle or a point or has no graph. **(b)** If the equation is that of a circle, find its center and radius, and sketch its graph.

11. $x^2 + y^2 + 2x - 6y + 9 = 0$

12. $2x^2 + 2y^2 - 2x + 8y = \frac{1}{2}$

13. $x^2 + y^2 + 72 = 12x$

14. $x^2 + y^2 - 6x - 10y + 34 = 0$

15–24 ▪ Sketch the graph of the equation by making a table and plotting points.

15. $y = 2 - 3x$ **16.** $2x - y + 1 = 0$

17. $x + 3y = 21$ **18.** $x = 2y + 12$

19. $\frac{x}{2} - \frac{y}{7} = 1$ **20.** $\frac{x}{4} + \frac{y}{5} = 0$

21. $y = 16 - x^2$ **22.** $8x + y^2 = 0$

23. $x = \sqrt{y}$ **24.** $y = -\sqrt{1 - x^2}$

25–32 ▪ **(a)** Test the equation for symmetry with respect to the x-axis, the y-axis, and the origin. **(b)** Find the x- and y-intercepts of the graph of the equation.

25. $y = 9 - x^2$ **26.** $6x + y^2 = 36$

27. $x^2 + (y - 1)^2 = 1$ **28.** $x^4 = 16 + y$

29. $9x^2 - 16y^2 = 144$ **30.** $y = \frac{4}{x}$

31. $x^2 + 4xy + y^2 = 1$ **32.** $x^3 + xy^2 = 5$

33–36 ▪ **(a)** Use a graphing device to graph the equation in an appropriate viewing rectangle. **(b)** Use the graph to find the x- and y-intercepts.

33. $y = x^2 - 6x$ **34.** $y = \sqrt{5 - x}$

35. $y = x^3 - 4x^2 - 5x$ **36.** $\frac{x^2}{4} + y^2 = 1$

37–46 ▪ A description of a line is given. **(a)** Find an equation for the line in slope-intercept form. **(b)** Find an equation for the line in general form. **(c)** Graph the line.

37. The line that has slope 2 and y-intercept 6

38. The line that has slope $-\frac{1}{2}$ and passes through the point $(6, -3)$

39. The line that passes through the points $(-1, -6)$ and $(2, -4)$

40. The line that has x-intercept 4 and y-intercept 12

41. The vertical line that passes through the point $(3, -2)$

42. The horizontal line with y-intercept 5

43. The line that passes through the point $(1, 1)$ and is parallel to the line $2x - 5y = 10$

44. The line that passes through the origin and is parallel to the line containing $(2, 4)$ and $(4, -4)$

45. The line that passes through the origin and is perpendicular to the line $y = \frac{1}{2}x - 10$

46. The line that passes through the point $(1, 7)$ and is perpendicular to the line $x - 3y + 16 = 0$

47–48 ■ The equations of two lines are given. Determine whether the lines are parallel, perpendicular, or neither.

47. $y = -\frac{1}{3}x - 1; \quad 9y + 3x + 3 = 0$

48. $5x - 8y = 3; \quad 10y + 16x = 1$

49. Hooke's Law states that if a weight w is attached to a hanging spring, then the stretched length s of the spring is linearly related to w. For a particular spring we have

$$s = 0.3w + 2.5$$

where s is measured in inches and w in pounds.

(a) What do the slope and s-intercept in this equation represent?

(b) How long is the spring when a 5-lb weight is attached?

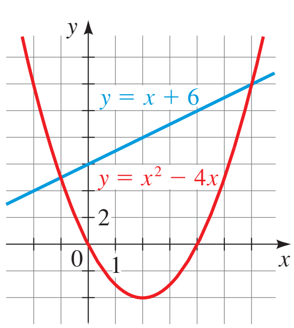
50. Margarita is hired by an accounting firm at a salary of \$60,000 per year. Three years later her annual salary has increased to \$70,500. Assume that her salary increases linearly.

(a) Find an equation that relates her annual salary S and the number of years t that she has worked for the firm.

(b) What do the slope and S-intercept of her salary equation represent?

(c) What will her salary be after 12 years with the firm?

51–56 ■ Graphs of the equations $y = x^2 - 4x$ and $y = x + 6$ are given. Use the graphs to solve the equation or inequality.



51. $x^2 - 4x = x + 6$

52. $x^2 - 4x = 0$

53. $x^2 - 4x \le x + 6$

54. $x^2 - 4x \ge x + 6$

55. $x^2 - 4x \ge 0$

56. $x^2 - 4x \le 0$

57–60 ■ Solve the equation graphically.

57. $x^2 - 4x = 2x + 7$

58. $\sqrt{x + 4} = x^2 - 5$

59. $x^4 - 9x^2 = x - 9$

60. $\left||x + 3| - 5\right| = 2$

61–64 ■ Solve the inequality graphically.

61. $4x - 3 \ge x^2$

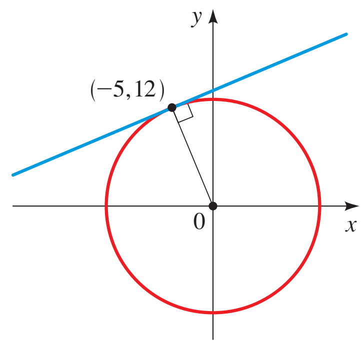
62. $x^3 - 4x^2 - 5x > 2$

63. $x^4 - 4x^2 < \frac{1}{2}x - 1$

64. $|x^2 - 16| - 10 \ge 0$

65–66 ■ Find equations for the circle and the line in the figure.

65.



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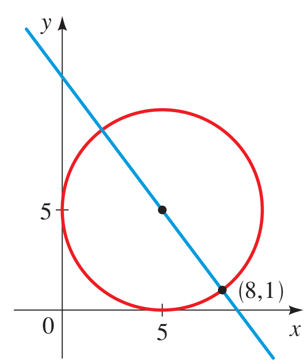


67–84 ■ Find all real solutions of the equation.

67. $x^2 - 9x + 14 = 0$

68. $x^2 + 24x + 144 = 0$

69. $2x^2 + x = 1$

70. $3x^2 + 5x - 2 = 0$

71. $4x^3 - 25x = 0$

72. $x^3 - 2x^2 - 5x + 10 = 0$

73. $3x^2 + 4x - 1 = 0$

74. $x^2 - 3x + 9 = 0$

75. $\dfrac{1}{x} + \dfrac{2}{x - 1} = 3$

76. $\dfrac{x}{x - 2} + \dfrac{1}{x + 2} = \dfrac{8}{x^2 - 4}$

77. $x^4 - 8x^2 - 9 = 0$

78. $x - 4\sqrt{x} = 32$

79. $x^{-1/2} - 2x^{1/2} + x^{3/2} = 0$

80. $(1 + \sqrt{x})^2 - 2(1 + \sqrt{x}) - 15 = 0$

81. $|x - 7| = 4$

82. $|3x| = 18$

83. $|2x - 5| = 9$

84. $4|3 - x| + 3 = 15$

85. A shopkeeper sells raisins for \$3.20 per pound and nuts for \$2.40 per pound. She decides to mix the raisins and nuts and sell 50 lb of the mixture for \$2.72 per pound. What quantities of raisins and nuts should she use?

86. Anthony leaves Kingstown at 2:00 P.M. and drives to Queensville, 160 mi distant, at 45 mi/h. At 2:15 P.M. Helen leaves Queensville and drives to Kingstown at 40 mi/h. At what time do they pass each other on the road?

87. A woman cycles 8 mi/h faster than she runs. Every morning she cycles 4 mi and runs $2\frac{1}{2}$ mi, for a total of 1 hour of exercise. How fast does she run?

88. The approximate distance d (in feet) that drivers travel after noticing that they must come to a sudden stop is given by the following formula, where x is the speed of the car (in mi/h):

$$d = x + \frac{x^2}{20}$$

If a car travels 75 ft before stopping, what was its speed before the brakes were applied?

89. Luc invests \$7000 in two bank accounts: One earns 1.5% simple interest per year, and the other earns 2.5% simple interest per year. After one year the total interest earned on these investments is \$120.25. How much money did he invest in each account?

90. Shania invests \$6000 at 3% simple interest per year. How much additional money must she invest at 1.25% simple interest per year to ensure that the interest she receives each year is \$300?

91. The hypotenuse of a right triangle has length 20 cm. The sum of the lengths of the other two sides is 28 cm. Find the lengths of the other two sides of the triangle.

92. Abbie paints twice as fast as Beth and three times as fast as Cathie. If it takes them 60 min to paint a living room with all three working together, how long would it take Abbie if she works alone?

93. A rectangular swimming pool is 8 ft deep everywhere and twice as long as it is wide. If the pool holds 8464 ft^3 of water, what are its dimensions?

94. A gardening enthusiast wishes to fence in three adjoining garden plots, one for each of his children, as shown in the figure. If each plot is to be 80 ft^2 in area and he has 88 ft of fencing material at hand, what dimensions should each plot have?

95–108 ■ Solve the inequality. Express the solution using interval notation, and graph the solution set on the real number line.

95. $3x - 2 > -11$

96. $12 - x \geq 7x$

97. $-1 < 2x + 5 \leq 3$

98. $3 - x \leq 2x - 7$

99. $x^2 + 4x - 12 > 0$

100. $x^2 \leq 1$

101. $\dfrac{2x + 5}{x + 1} \leq 1$

102. $2x^2 \geq x + 3$

103. $\dfrac{x - 4}{x^2 - 4} \leq 0$

104. $\dfrac{5}{x^3 - x^2 - 4x + 4} < 0$

105. $|x - 5| \leq 3$

106. $|x - 4| < 0.02$

107. $|2x + 1| \geq 1$

108. $|x - 1| < |x - 3|$
[*Hint:* Interpret the quantities as distances.]

109. For what values of x is the algebraic expression defined as a real number?

(a) $\sqrt{24 - x - 3x^2}$ **(b)** $\dfrac{1}{\sqrt[4]{x - x^4}}$

110. The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. Find the interval of values of the radius so that the volume is between 8 ft^3 and 12 ft^3, inclusive.

CHAPTER 1 TEST

1. Let $P(1, -3)$ and $Q(7, 5)$ be two points in the coordinate plane.
   (a) Plot P and Q in the coordinate plane.
   (b) Find the distance between P and Q.
   (c) Find the midpoint of the segment PQ.
   (d) Find the slope of the line that contains P and Q.
   (e) Find the perpendicular bisector of the line that contains P and Q.
   (f) Find an equation for the circle for which the segment PQ is a diameter.

2. Find the center and radius of each circle, and sketch its graph.
   (a) $x^2 + y^2 = \frac{25}{4}$ **(b)** $(x - 3)^2 + y^2 = 9$ **(c)** $x^2 + 6x + y^2 - 2y + 6 = 0$

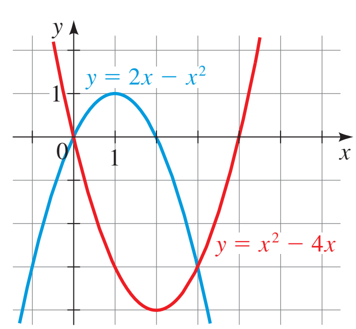
3. Test each equation for symmetry. Find the x- and y-intercepts, and sketch a graph of the equation.
   (a) $x = 4 - y^2$ **(b)** $y = |x - 2|$

4. A line has the general linear equation $3x - 5y = 15$.
   (a) Find the x- and y-intercepts of the graph of this line.
   (b) Graph the line. Use the intercepts that you found in part (a) to help you.
   (c) Write the equation of the line in slope-intercept form.
   (d) What is the slope of the line?
   (e) What is the slope of any line perpendicular to the given line?

5. Find an equation for the line with the given property.
   (a) It passes through the point $(3, -6)$ and is parallel to the line $3x + y - 10 = 0$.
   (b) It has x-intercept 6 and y-intercept 4.

6. A geologist uses a probe to measure the temperature T (in °C) of the soil at various depths below the surface, and finds that at a depth of x cm the temperature is given by the linear equation $T = 0.08x - 4$.
   (a) What is the temperature at a depth of one meter (100 cm)?
   (b) Sketch a graph of the linear equation.
   (c) What do the slope, the x-intercept, and the T-intercept of the graph of this equation represent?

7. Graphs of the equations $y = x^2 - 4x$ and $y = 2x - x^2$ are given. Use the graphs to solve the equation or inequality.
   (a) $x^2 - 4x = 2x - x^2$ **(b)** $x^2 - 4x > 2x - x^2$
   (c) $2x - x^2 = 0$ **(d)** $x^2 - 4x \le 0$



8. Solve the equation or inequality graphically, rounded to two decimals.
   (a) $x^3 - 9x - 1 = 0$ **(b)** $\frac{1}{2}x + 2 \ge \sqrt{x^2 + 1}$

9. Natasha drove from Bedingfield to Portsmouth at an average speed of 100 km/h to attend a job interview. On the way back she decided to slow down to enjoy the scenery, so she drove at just 75 km/h. Her trip involved a total of 3.5 hours of driving time. What is the distance between Bedingfield and Portsmouth?

10. Find all real solutions of each equation.

(a) $x^2 - x - 12 = 0$

(b) $2x^2 + 4x - 3 = 0$

(c) $3 - \sqrt{x - 3} = x$

(d) $x^{1/2} - 3x^{1/4} + 2 = 0$

(e) $x^4 - 16x^2 = 0$

(f) $3|x - 4| - 10 = 0$

11. A rectangular parcel of land is 70 ft longer than it is wide. Each diagonal between opposite corners is 130 ft. What are the dimensions of the parcel?

12. Solve each inequality. Sketch the solution on a real number line, and write the answer using interval notation.

(a) $-1 \leq 5 - 2x < 10$

(b) $x(x - 1)(x - 2) > 0$

(c) $|x - 3| < 2$

(d) $\dfrac{2x + 5}{x + 1} \leq 1$

13. A bottle of medicine must be stored at a temperature between 5°C and 10°C. What range does this correspond to on the Fahrenheit scale? [*Note:* The Fahrenheit (F) and Celsius (C) scales satisfy the relation $C = \frac{5}{9}(F - 32)$.]

14. For what values of x is the expression $\sqrt{4x - x^2}$ defined as a real number?

Fitting Lines to Data

A model is a representation of an object or process. For example, a toy Ferrari is a model of the actual car; a road map is a model of the streets in a city. A **mathematical model** is a mathematical representation (usually an equation) of an object or process. Once a mathematical model is made it can be used to obtain useful information or make predictions about the thing being modeled. In these *Focus on Modeling* sections we explore different ways in which mathematics is used to model real-world phenomena.

▼ The Line That Best Fits the Data

In Section 1.3 we used linear equations to model relationships between varying quantities. In practice, such relationships are discovered by collecting data. But real-world data seldom fall into a precise line. The **scatter plot** in Figure 1(a) shows the result of a study on childhood obesity. The graph plots the body mass index (BMI) versus the number of hours of television watched per day for 25 adolescent subjects. Of course, we would not expect the data to be exactly linear as in Figure 1(b). But there is a linear *trend* indicated by the blue line in Figure 1(a): The more hours a subject watches TV the higher the BMI. In this section we learn how to find the line that best fits the data.

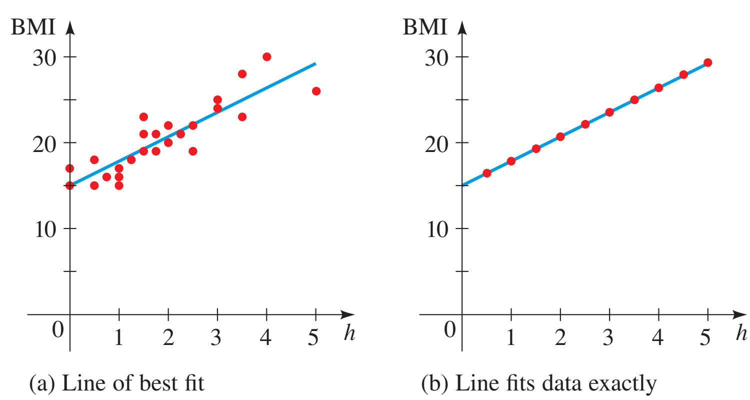


FIGURE 1

Table 1 gives the nationwide infant mortality rate for the period from 1950 to 2000. The *rate* is the number of infants who die before reaching their first birthday, out of every 1000 live births.

TABLE 1
U.S. Infant Mortality

Year	Rate
1950	29.2
1960	26.0
1970	20.0
1980	12.6
1990	9.2
2000	6.9

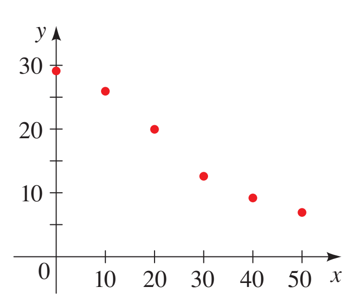


FIGURE 2 U.S. infant mortality rate

The scatter plot in Figure 2 shows that the data lie roughly on a straight line. We can try to fit a line visually to approximate the data points, but since the data aren't *exactly* linear, there are many lines that might seem to work. Figure 3 shows two attempts at "eyeballing" a line to fit the data.

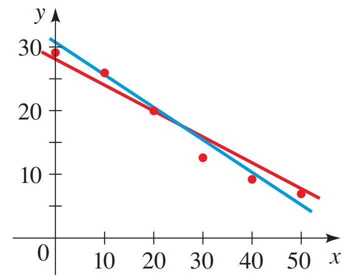


FIGURE 3 Visual attempts to fit line to data

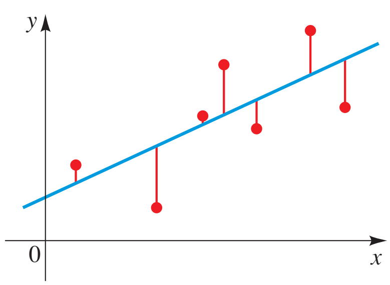


FIGURE 4 Distance from the data points to the line

Of all the lines that run through these data points, there is one that "best" fits the data, in the sense that it provides the most accurate linear model for the data. We now describe how to find this line.

It seems reasonable that the line of best fit is the line that is as close as possible to all the data points. This is the line for which the sum of the vertical distances from the data points to the line is as small as possible (see Figure 4). For technical reasons it is better to use the line where the sum of the squares of these distances is smallest. This is called the **regression line**. The formula for the regression line is found by using calculus, but fortunately, the formula is programmed into most graphing calculators. In Example 1 we see how to use a TI-83 calculator to find the regression line for the infant mortality data described above. (The process for other calculators is similar.)

EXAMPLE 1 | Regression Line for U.S. Infant Mortality Rates

(a) Find the regression line for the infant mortality data in Table 1.

(b) Graph the regression line on a scatter plot of the data.

(c) Use the regression line to estimate the infant mortality rates in 1995 and 2006.

SOLUTION

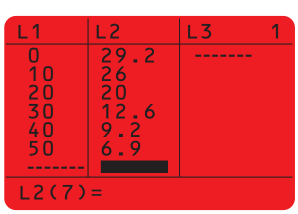


FIGURE 5 Entering the data

(a) To find the regression line using a TI-83 calculator, we must first enter the data into the lists L_1 and L_2, which are accessed by pressing the STAT key and selecting `Edit`. Figure 5 shows the calculator screen after the data have been entered. (Note that we are letting $x = 0$ correspond to the year 1950, so that $x = 50$ corresponds to 2000. This makes the equations easier to work with.) We then press the STAT key again and select `Calc`, then `4:LinReg(ax+b)`, which provides the output shown in Figure 6(a). This tells us that the regression line is

$$y = -0.48x + 29.4$$

Here x represents the number of years since 1950, and y represents the corresponding infant mortality rate.

(b) The scatter plot and the regression line have been plotted on a graphing calculator screen in Figure 6(b).

See Appendix C, *Using the TI-83/84 Graphing Calculator,* for specific graphing instructions.

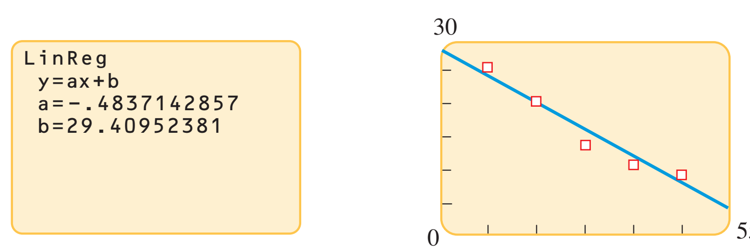


FIGURE 6 (a) Output of the `LinReg` command (b) Scatter plot and regression line

(c) The year 1995 is 45 years after 1950, so substituting 45 for x, we find that $y = -0.48(45) + 29.4 = 7.8$. So the infant mortality rate in 1995 was about 7.8. Similarly, substituting 56 for x, we find that the infant mortality rate predicted for 2006 was about $-0.48(56) + 29.4 \approx 2.5$. ■

An Internet search shows that the actual infant mortality rate was 7.6 in 1995 and 6.4 in 2006. So the regression line is fairly accurate for 1995 (the actual rate was slightly lower than the predicted rate), but it is considerably off for 2006 (the actual rate was more than twice the predicted rate). The reason is that infant mortality in the United States stopped declining and actually started rising in 2002, for the first time in more than a century. This shows that we have to be very careful about extrapolating linear models outside the domain over which the data are spread.

AP Photo/Michael Probst

Steven Hooker, 2008 Olympic gold medal winner, men's pole vault

▼ Examples of Regression Analysis

Since the modern Olympic Games began in 1896, achievements in track and field events have been improving steadily. One example in which the winning records have shown an upward linear trend is the pole vault. Pole vaulting began in the northern Netherlands as a practical activity: When traveling from village to village, people would vault across the many canals that crisscrossed the area to avoid having to go out of their way to find a bridge. Households maintained a supply of wooden poles of lengths appropriate for each member of the family. Pole vaulting for height rather than distance became a collegiate track and field event in the mid-1800s and was one of the events in the first modern Olympics. In the next example we find a linear model for the gold-medal-winning records in the men's Olympic pole vault.

EXAMPLE 2 | Regression Line for Olympic Pole Vault Records

Table 2 gives the men's Olympic pole vault records up to 2004.

(a) Find the regression line for the data.

(b) Make a scatter plot of the data, and graph the regression line. Does the regression line appear to be a suitable model for the data?

(c) What does the slope of the regression line represent?

(d) Use the model to predict the winning pole vault height for the 2008 Olympics.

TABLE 2
Men's Olympic Pole Vault Records

Year	x	Gold medalist	Height (m)	Year	x	Gold medalist	Height (m)
1896	-4	William Hoyt, USA	3.30	1956	56	Robert Richards, USA	4.56
1900	0	Irving Baxter, USA	3.30	1960	60	Don Bragg, USA	4.70
1904	4	Charles Dvorak, USA	3.50	1964	64	Fred Hansen, USA	5.10
1906	6	Fernand Gonder, France	3.50	1968	68	Bob Seagren, USA	5.40
1908	8	A. Gilbert, E. Cook, USA	3.71	1972	72	W. Nordwig, E. Germany	5.64
1912	12	Harry Babcock, USA	3.95	1976	76	Tadeusz Slusarski, Poland	5.64
1920	20	Frank Foss, USA	4.09	1980	80	W. Kozakiewicz, Poland	5.78
1924	24	Lee Barnes, USA	3.95	1984	84	Pierre Quinon, France	5.75
1928	28	Sabin Can, USA	4.20	1988	88	Sergei Bubka, USSR	5.90
1932	32	William Miller, USA	4.31	1992	92	M. Tarassob, Unified Team	5.87
1936	36	Earle Meadows, USA	4.35	1996	96	Jean Jaffione, France	5.92
1948	48	Guinn Smith, USA	4.30	2000	100	Nick Hysong, USA	5.90
1952	52	Robert Richards, USA	4.55	2004	104	Timothy Mack, USA	5.95

```
LinReg
 y=ax+b
 a=.0265652857
 b=3.400989881
```

Output of the `LinReg` function on the TI-83

SOLUTION

(a) Let $x = \text{year} - 1900$, so 1896 corresponds to $x = -4$, 1900 to $x = 0$, and so on. Using a calculator, we find the following regression line:

$$y = 0.0266x + 3.40$$

(b) The scatter plot and the regression line are shown in Figure 7. The regression line appears to be a good model for the data.

(c) The slope is the average rate of increase in the pole vault record per year. So on average, the pole vault record increased by 0.0266 m/yr.

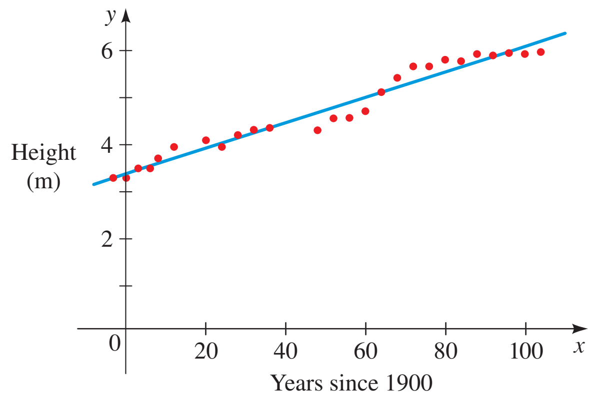


FIGURE 7 Scatter plot and regression line for pole vault data

(d) The year 2008 corresponds to $x = 108$ in our model. The model gives

$$y = 0.0266(108) + 3.40$$
$$\approx 6.27$$

So the model predicts that in 2008 the winning pole vault will be 6.27 m. ■

At the 2008 Olympics in Beijing, China, the men's Olympic gold medal in the pole vault was won by Steven Hooker of Australia, with a vault of 5.96 m. Although this height set an Olympic record, it was considerably lower than the 6.27 m predicted by the model of Example 2. In Problem 10 we find a regression line for the pole vault data from 1972 to 2004. Do the problem to see whether this restricted set of more recent data provides a better predictor for the 2008 record.

Is a linear model really appropriate for the data of Example 2? In subsequent *Focus on Modeling* sections, we study regression models that use other types of functions, and we learn how to choose the best model for a given set of data.

In the next example we see how linear regression is used in medical research to investigate potential causes of diseases such as cancer.

TABLE 3
Asbestos–Tumor Data

Asbestos exposure (fibers/mL)	Percent that develop lung tumors
50	2
400	6
500	5
900	10
1100	26
1600	42
1800	37
2000	28
3000	50

EXAMPLE 3 | Regression Line for Links Between Asbestos and Cancer

When laboratory rats are exposed to asbestos fibers, some of the rats develop lung tumors. Table 3 lists the results of several experiments by different scientists.

(a) Find the regression line for the data.

(b) Make a scatter plot and graph the regression line. Does the regression line appear to be a suitable model for the data?

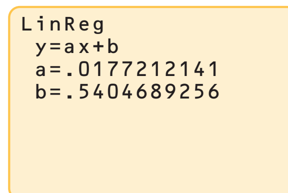
(c) What does the y-intercept of the regression line represent?

SOLUTION

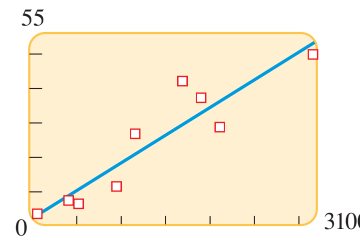
(a) Using a calculator, we find the following regression line (see Figure 8(a)):

$$y = 0.0177x + 0.5405$$

(b) The scatter plot and regression line are graphed in Figure 8(b). The regression line appears to be a reasonable model for the data.



(a) Output of the `LinReg` command



(b) Scatter plot and regression line

FIGURE 8 Linear regression for the asbestos–tumor data

(c) The y-intercept is the percentage of rats that develop tumors when no asbestos fibers are present. In other words, this is the percentage that normally develop lung tumors (for reasons other than asbestos).

How Good Is the Fit? The Correlation Coefficient

For any given set of two-variable data it is always possible to find a regression line, even if the data points do not tend to lie on a line and even if the variables don't seem to be related at all. Look at the three scatter plots in Figure 9. In the first scatter plot, the data points lie close to a line. In the second plot, there is still a linear trend but the points are more scattered. In the third plot there doesn't seem to be any trend at all, linear or otherwise.

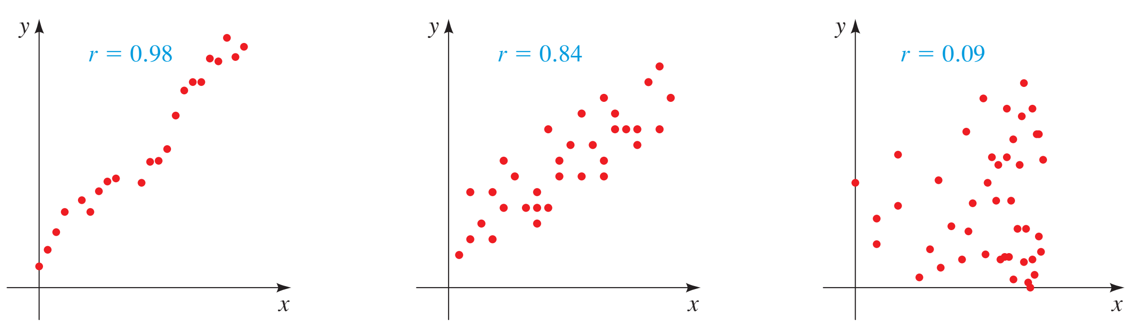


FIGURE 9

A graphing calculator can give us a regression line for each of these scatter plots. But how well do these lines represent or "fit" the data? To answer this question, statisticians have invented the **correlation coefficient**, usually denoted r. The correlation coefficient is a number between -1 and 1 that measures how closely the data follow the regression line—or, in other words, how strongly the variables are **correlated**. Many graphing calculators give the value of r when they compute a regression line. If r is close to -1 or 1, then the variables are strongly correlated—that is, the scatter plot follows the regression line closely. If r is close to 0, then the variables are weakly correlated or not correlated at all. (The sign of r depends on the slope of the regression line.) The correlation coefficients of the scatter plots in Figure 9 are indicated on the graphs. For the first plot, r is close to 1 because the data are very close to linear. The second plot also has a relatively large r, but it is not as large as the first, because the data, while fairly linear, are more diffuse. The third plot has an r close to 0, since there is virtually no linear trend in the data.

If two variables are correlated, it does not necessarily mean that a change in one variable *causes* a change in the other. For example, the mathematician John Allen Paulos points out that shoe size is strongly correlated to mathematics scores among schoolchildren. Does this mean that big feet cause high math scores? Certainly not—both shoe size and math skills increase independently as children get older. So it is important not to jump to conclusions: Correlation and causation are not the same thing. Correlation is a useful tool in bringing important cause-and-effect relationships to light; but to prove causation, we must explain the mechanism by which one variable affects the other. For example, the link between smoking and lung cancer was observed as a correlation long before science found the mechanism through which smoking causes lung cancer.

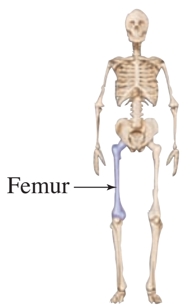
PROBLEMS

1. **Femur Length and Height** Anthropologists use a linear model that relates femur length to height. The model allows an anthropologist to determine the height of an individual when only a partial skeleton (including the femur) is found. In this problem we find the model by analyzing the data on femur length and height for the eight males given in the table.

   (a) Make a scatter plot of the data.

   (b) Find and graph a linear function that models the data.

   (c) An anthropologist finds a femur of length 58 cm. How tall was the person?



Femur length (cm)	Height (cm)
50.1	178.5
48.3	173.6
45.2	164.8
44.7	163.7
44.5	168.3
42.7	165.0
39.5	155.4
38.0	155.8

2. **Demand for Soft Drinks** A convenience store manager notices that sales of soft drinks are higher on hotter days, so he assembles the data in the table.

   (a) Make a scatter plot of the data.

   (b) Find and graph a linear function that models the data.

   (c) Use the model to predict soft drink sales if the temperature is 95°F.

High temperature (°F)	Number of cans sold
55	340
58	335
64	410
68	460
70	450
75	610
80	735
84	780

3. Tree Diameter and Age To estimate ages of trees, forest rangers use a linear model that relates tree diameter to age. The model is useful because tree diameter is much easier to measure than tree age (which requires special tools for extracting a representative cross section of the tree and counting the rings). To find the model, use the data in the table, which were collected for a certain variety of oaks.

(a) Make a scatter plot of the data.

(b) Find and graph a linear function that models the data.

(c) Use the model to estimate the age of an oak whose diameter is 18 in.

Diameter (in.)	Age (years)
2.5	15
4.0	24
6.0	32
8.0	56
9.0	49
9.5	76
12.5	90
15.5	89

4. Carbon Dioxide Levels The Mauna Loa Observatory, located on the island of Hawaii, has been monitoring carbon dioxide (CO_2) levels in the atmosphere since 1958. The table lists the average annual CO_2 levels measured in parts per million (ppm) from 1984 to 2010.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) Use the linear model in part (b) to estimate the CO_2 level in the atmosphere in 2005. Compare your answer with the actual CO_2 level of 379.7 that was measured in 2005.

Year	CO_2 level (ppm)
1984	344.3
1986	347.0
1988	351.3
1990	354.0
1992	356.3
1994	358.9
1996	362.7
1998	366.5
2000	369.4
2002	372.0
2004	377.5
2006	381.9
2008	385.6
2010	389.8

5. Temperature and Chirping Crickets Biologists have observed that the chirping rate of crickets of a certain species appears to be related to temperature. The table shows the chirping rates for various temperatures.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) Use the linear model in part (b) to estimate the chirping rate at 100°F.

Temperature (°F)	Chirping rate (chirps/min)
50	20
55	46
60	79
65	91
70	113
75	140
80	173
85	198
90	211

6. Extent of Arctic Sea Ice The National Snow and Ice Data Center monitors the amount of ice in the Arctic year round. The table gives approximate values for the sea ice extent in millions of square kilometers from 1980 to 2006, in two-year intervals.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) Use the linear model in part (b) to estimate the ice extent in the year 2010.

Year	Ice extent (million km^2)	Year	Ice extent (million km^2)
1980	7.9	1994	7.1
1982	7.4	1996	7.9
1984	7.2	1998	6.6
1986	7.6	2000	6.3
1988	7.5	2002	6.0
1990	6.2	2004	6.1
1992	7.6	2006	5.7

7. Mosquito Prevalence The table lists the relative abundance of mosquitoes (as measured by the mosquito positive rate) versus the flow rate (measured as a percentage of maximum flow) of canal networks in Saga City, Japan.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) Use the linear model in part (b) to estimate the mosquito positive rate if the canal flow is 70% of maximum.

Flow rate (%)	Mosquito positive rate (%)
0	22
10	16
40	12
60	11
90	6
100	2

8. Noise and Intelligibility Audiologists study the intelligibility of spoken sentences under different noise levels. Intelligibility, the MRT score, is measured as the percent of a spoken sentence that the listener can decipher at a certain noise level in decibels (dB). The table shows the results of one such test.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) Find the correlation coefficient. Is a linear model appropriate?

(d) Use the linear model in part (b) to estimate the intelligibility of a sentence at a 94-dB noise level.

Noise level (dB)	MRT score (%)
80	99
84	91
88	84
92	70
96	47
100	23
104	11

9. Life Expectancy The average life expectancy in the United States has been rising steadily over the past few decades, as shown in the table.

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) Use the linear model you found in part (b) to predict the life expectancy in the year 2006.

(d) Search the Internet or your campus library to find the actual 2006 average life expectancy. Compare to your answer in part (c).

Year	Life expectancy
1920	54.1
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	76.9

10. Olympic Pole Vault The graph in Figure 7 indicates that in recent years the winning Olympic men's pole vault height has fallen below the value predicted by the regression line in Example 2. This might have occurred because when the pole vault was a new event, there was much room for improvement in vaulters' performances, whereas now even the best training can produce only incremental advances. Let's see whether concentrating on more recent results gives a better predictor of future records.

(a) Use the data in Table 2 to complete the table of winning pole vault heights. (Note that we are using $x = 0$ to correspond to the year 1972, where this restricted data set begins.)

(b) Find the regression line for the data in part (a).

(c) Plot the data and the regression line on the same axes. Does the regression line seem to provide a good model for the data?

(d) What does the regression line predict as the winning pole vault height for the 2008 Olympics? Compare this predicted value to the actual 2008 winning height of 5.96 m, as described on page 165. Has this new regression line provided a better prediction than the line in Example 2?

Year	x	Height (m)
1972	0	5.64
1976	4	
1980	8	
1984		
1988		
1992		
1996		
2000		
2004		

11. Olympic Swimming Records The tables give the gold medal times in the men's and women's 100-m freestyle Olympic swimming event.

(a) Find the regression lines for the men's data and the women's data.

(b) Sketch both regression lines on the same graph. When do these lines predict that the women will overtake the men in the event? Does this conclusion seem reasonable?

MEN

Year	Gold medalist	Time (s)
1908	C. Daniels, USA	65.6
1912	D. Kahanamoku, USA	63.4
1920	D. Kahanamoku, USA	61.4
1924	J. Weissmuller, USA	59.0
1928	J. Weissmuller, USA	58.6
1932	Y. Miyazaki, Japan	58.2
1936	F. Csik, Hungary	57.6
1948	W. Ris, USA	57.3
1952	C. Scholes, USA	57.4
1956	J. Henricks, Australia	55.4
1960	J. Devitt, Australia	55.2
1964	D. Schollander, USA	53.4
1968	M. Wenden, Australia	52.2
1972	M. Spitz, USA	51.22
1976	J. Montgomery, USA	49.99
1980	J. Woithe, E. Germany	50.40
1984	R. Gaines, USA	49.80
1988	M. Biondi, USA	48.63
1992	A. Popov, Russia	49.02
1996	A. Popov, Russia	48.74
2000	P. van den Hoogenband, Netherlands	48.30
2004	P. van den Hoogenband, Netherlands	48.17
2008	A. Bernard, France	47.21

WOMEN

Year	Gold medalist	Time (s)
1912	F. Durack, Australia	82.2
1920	E. Bleibtrey, USA	73.6
1924	E. Lackie, USA	72.4
1928	A. Osipowich, USA	71.0
1932	H. Madison, USA	66.8
1936	H. Mastenbroek, Holland	65.9
1948	G. Andersen, Denmark	66.3
1952	K. Szoke, Hungary	66.8
1956	D. Fraser, Australia	62.0
1960	D. Fraser, Australia	61.2
1964	D. Fraser, Australia	59.5
1968	J. Henne, USA	60.0
1972	S. Nielson, USA	58.59
1976	K. Ender, E. Germany	55.65
1980	B. Krause, E. Germany	54.79
1984	(Tie) C. Steinseifer, USA	55.92
	N. Hogshead, USA	55.92
1988	K. Otto, E. Germany	54.93
1992	Z. Yong, China	54.64
1996	L. Jingyi, China	54.50
2000	I. DeBruijn, Netherlands	53.83
2004	J. Henry, Australia	53.84
2008	B. Steffen, Germany	53.12

12. Shoe Size and Height Do you think that shoe size and height are correlated? Find out by surveying the shoe sizes and heights of people in your class. (Of course, the data for men and women should be separate.) Find the correlation coefficient.

13. Demand for Candy Bars In this problem you will determine a linear demand equation that describes the demand for candy bars in your class. Survey your classmates to determine what price they would be willing to pay for a candy bar. Your survey form might look like the sample to the left.

(a) Make a table of the number of respondents who answered "yes" at each price level.

(b) Make a scatter plot of your data.

(c) Find and graph the regression line $y = mp + b$, which gives the number of responents y who would buy a candy bar if the price were p cents. This is the *demand equation*. Why is the slope m negative?

(d) What is the p-intercept of the demand equation? What does this intercept tell you about pricing candy bars?

Would you buy a candy bar from the vending machine in the hallway if the price is as indicated?

Price	Yes or No
30¢	
40¢	
50¢	
60¢	
70¢	
80¢	
90¢	
$1.00	
$1.10	
$1.20	

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CHAPTER 2

FUNCTIONS

Finding and Using Rules Many of our everyday activities are governed by precise rules. There is a rule that relates the amount of money you get paid to the number of hours you work, a rule that relates the grade you get in your algebra class to your exam scores, a rule that relates the amount of gas you use to the distance you drive. Rules like these are expressed in algebra by using *functions.* Rules are discovered in different ways: by collecting data and looking for patterns, by experimenting, or by reasoning about the properties of the process being studied. Once a rule has been found, it can be used to *predict* the outcome of a process. For example, the skydivers pictured here need to know the rule that relates the distance they fall to the amount of time they've been falling. Knowing this rule allows them to enjoy skydiving—safely!

In *Focus on Modeling* at the end of this chapter we explore different ways of using functions to model real-world situations.

2.1 FUNCTIONS

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Recognize functions in the real world ▶ Work with function notation ▶ Evaluate functions ▶ Find net change ▶ Find domains of functions ▶ Represent functions verbally, algebraically, graphically, and numerically

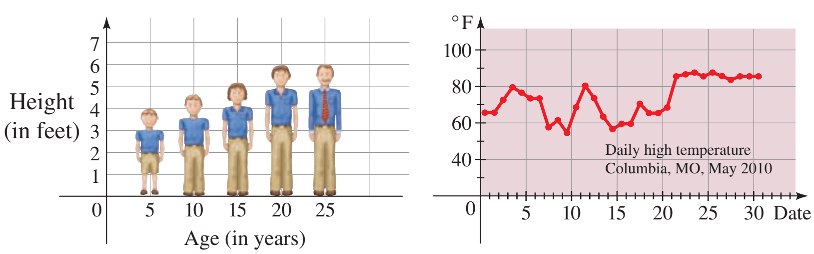
In this section we explore the idea of a function and then give the mathematical definition of function.

▼ Functions All Around Us

In nearly every physical phenomenon we observe that one quantity depends on another. For example, your height depends on your age, the temperature depends on the date, the cost of mailing a package depends on its weight (see Figure 1). We use the term *function* to describe this dependence of one quantity on another. That is, we say the following:

- Height is a function of age.
- Temperature is a function of date.
- Cost of mailing a package is a function of weight.

The U.S. Post Office uses a simple rule to determine the cost of mailing a first-class parcel on the basis of its weight. But it's not so easy to describe the rule that relates height to age or the rule that relates temperature to date.



w (ounces)	2011 Postage (dollars)
$0 < w \le 3$	1.71
$3 < w \le 4$	1.88
$4 < w \le 5$	2.05
$5 < w \le 6$	2.22
$6 < w \le 7$	2.39
$7 < w \le 8$	2.56

FIGURE 1 Height is a function of age. Temperature is a function of date. Postage is a function of weight.

Can you think of other functions? Here are some more examples:

- The area of a circle is a function of its radius.
- The number of bacteria in a culture is a function of time.
- The weight of an astronaut is a function of her elevation.
- The price of a commodity is a function of the demand for that commodity.

The rule that describes how the area A of a circle depends on its radius r is given by the formula $A = \pi r^2$. Even when a precise rule or formula describing a function is not available, we can still describe the function by a graph. For example, when you turn on a hot water faucet, the temperature of the water depends on how long the water has been running. So we can say:

- The temperature of water from the faucet is a function of time.

Figure 2 shows a rough graph of the temperature T of the water as a function of the time t that has elapsed since the faucet was turned on. The graph shows that the initial temperature of the water is close to room temperature. When the water from the hot water tank reaches the faucet, the water's temperature T increases quickly. In the next phase, T is constant at the temperature of the water in the tank. When the tank is drained, T decreases to the temperature of the cold water supply.

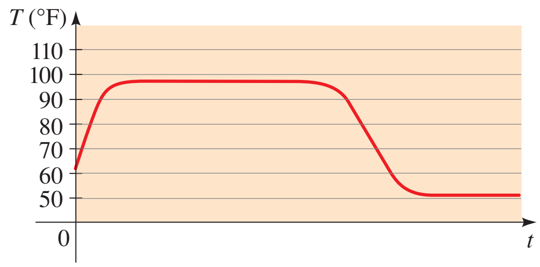


FIGURE 2 Graph of water temperature T as a function of time t

Definition of Function

We have previously used letters to stand for numbers. Here we do something quite different: We use letters to represent *rules*.

A function is a rule. To talk about a function, we need to give it a name. We will use letters such as $f, g, h, \ldots$ to represent functions. For example, we can use the letter f to represent a rule as follows:

"f" is the rule "square the number"

When we write $f(2)$, we mean "apply the rule f to the number 2." Applying the rule gives $f(2) = 2^2 = 4$. Similarly, $f(3) = 3^2 = 9$, $f(4) = 4^2 = 16$, and in general $f(x) = x^2$.

DEFINITION OF A FUNCTION

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

The $\boxed{\sqrt{\ }}$ key on your calculator is a good example of a function as a machine. First you input x into the display. Then you press the key labeled $\boxed{\sqrt{\ }}$. (On most *graphing* calculators the order of these operations is reversed.) If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input, and the calculator will indicate an error. If $x \geq 0$, then an approximation to $\sqrt{x}$ appears in the display, correct to a certain number of decimal places. (Thus, the $\boxed{\sqrt{\ }}$ key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$.)

We usually consider functions for which the sets A and B are sets of real numbers. The symbol $f(x)$ is read "f of x" or "f at x" and is called the **value of f at x**, or the **image of x under f**. The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain, that is,

$$\text{range of } f = \{f(x) \mid x \in A\}$$

The symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**. The symbol that represents a number in the range of f is called a **dependent variable**. So if we write $y = f(x)$, then x is the independent variable and y is the dependent variable.

It is helpful to think of a function as a **machine** (see Figure 3). If x is in the domain of the function f, then when x enters the machine, it is accepted as an **input** and the machine produces an **output** $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

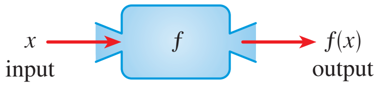


FIGURE 3 Machine diagram of f

Another way to picture a function is by an **arrow diagram** as in Figure 4. Each arrow connects an element of A to an element of B. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

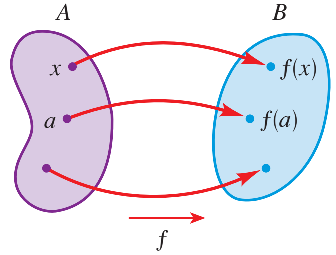


FIGURE 4 Arrow diagram of f

EXAMPLE 1 | Analyzing a Function

A function f is defined by the formula

$$f(x) = x^2 + 4$$

(a) Express in words how f acts on the input x to produce the output $f(x)$.
(b) Evaluate $f(3)$, $f(-2)$, and $f(\sqrt{5})$.
(c) Find the domain and range of f.
(d) Draw a machine diagram for f.

SOLUTION

(a) The formula tells us that f first squares the input x and then adds 4 to the result. So f is the function

"square, then add 4"

(b) The values of f are found by substituting for x in the formula $f(x) = x^2 + 4$.

$$f(3) = 3^2 + 4 = 13 \qquad \text{Replace } x \text{ by } 3$$

$$f(-2) = (-2)^2 + 4 = 8 \qquad \text{Replace } x \text{ by } -2$$

$$f(\sqrt{5}) = (\sqrt{5})^2 + 4 = 9 \qquad \text{Replace } x \text{ by } \sqrt{5}$$

(c) The domain of f consists of all possible inputs for f. Since we can evaluate the formula $f(x) = x^2 + 4$ for every real number x, the domain of f is the set $\mathbb{R}$ of all real numbers.

The range of f consists of all possible outputs of f. Because $x^2 \geq 0$ for all real numbers x, we have $x^2 + 4 \geq 4$, so for every output of f we have $f(x) \geq 4$. Thus the range of f is $\{y \mid y \geq 4\} = [4, \infty)$.

(d) A machine diagram for f is shown in Figure 5.

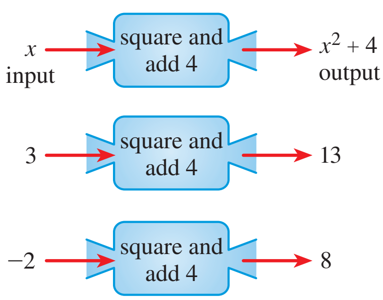


FIGURE 5 Machine diagram

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **9**, **13**, **17**, AND **49**

Evaluating a Function

In the definition of a function the independent variable x plays the role of a placeholder. For example, the function $f(x) = 3x^2 + x - 5$ can be thought of as

$$f(\square) = 3 \cdot \square^2 + \square - 5$$

To evaluate f at a number, we substitute the number for the placeholder.

EXAMPLE 2 | Evaluating a Function

Let $f(x) = 3x^2 + x - 5$. Evaluate each function value.

(a) $f(-2)$ **(b)** $f(0)$ **(c)** $f(4)$ **(d)** $f(\frac{1}{2})$

SOLUTION To evaluate f at a number, we substitute the number for x in the definition of f.

(a) $f(-2) = 3 \cdot (-2)^2 + (-2) - 5 = 5$

(b) $f(0) = 3 \cdot 0^2 + 0 - 5 = -5$

(c) $f(4) = 3 \cdot (4)^2 + 4 - 5 = 47$

(d) $f(\frac{1}{2}) = 3 \cdot (\frac{1}{2})^2 + \frac{1}{2} - 5 = -\frac{15}{4}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **19**

EXAMPLE 3 | A Piecewise Defined Function

A cell phone plan costs \$39 a month. The plan includes 400 free minutes and charges 20¢ for each additional minute of usage. The monthly charges are a function of the number of minutes used, given by

$$C(x) = \begin{cases} 39 & \text{if } 0 \le x \le 400 \\ 39 + 0.20(x - 400) & \text{if } x > 400 \end{cases}$$

Find $C(100)$, $C(400)$, and $C(480)$.

A **piecewise defined function** is defined by different formulas on different parts of its domain. The function C of Example 3 is piecewise defined.

SOLUTION Remember that a function is a rule. Here is how we apply the rule for this function. First we look at the value of the input x. If $0 \le x \le 400$, then the value of $C(x)$ is 39. On the other hand, if $x > 400$, then the value of $C(x)$ is $39 + 0.20(x - 400)$.

Since $100 \le 400$, we have $C(100) = 39$.

Since $400 \le 400$, we have $C(400) = 39$.

Since $480 > 400$, we have $C(480) = 39 + 0.20(480 - 400) = 55$.

Thus the plan charges \$39 for 100 minutes, \$39 for 400 minutes, and \$55 for 480 minutes.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **29** ■

From Examples 2 and 3 we see that the values of a function can change from one input to another. The **net change** in the value of a function f as the input changes from a to b (where $a \le b$) is given by

$$f(b) - f(a)$$

The next example illustrates this concept.

EXAMPLE 4 | Finding Net Change

Let $f(x) = x^2$. Find the net change in the value of f between the given inputs.

(a) From 1 to 3 **(b)** From -2 to 2

The values of the function in Example 4 decrease and then increase between -2 and 2, but the net change from -2 to 2 is 0 because $f(-2)$ and $f(2)$ have the same value.

SOLUTION

(a) The net change is $f(3) - f(1) = 9 - 1 = 8$.

(b) The net change is $f(2) - f(-2) = 4 - 4 = 0$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **37** ■

EXAMPLE 5 | Evaluating a Function

If $f(x) = 2x^2 + 3x - 1$, evaluate the following.

(a) $f(a)$ **(b)** $f(-a)$

(c) $f(a + h)$ **(d)** $\dfrac{f(a + h) - f(a)}{h}$, $h \neq 0$

Expressions like the one in part (d) of Example 5 occur frequently in calculus; they are called *difference quotients*, and they represent the average change in the value of f between $x = a$ and $x = a + h$.

SOLUTION

(a) $f(a) = 2a^2 + 3a - 1$

(b) $f(-a) = 2(-a)^2 + 3(-a) - 1 = 2a^2 - 3a - 1$

(c)
$$\begin{aligned} f(a + h) &= 2(a + h)^2 + 3(a + h) - 1 \\ &= 2(a^2 + 2ah + h^2) + 3(a + h) - 1 \\ &= 2a^2 + 4ah + 2h^2 + 3a + 3h - 1 \end{aligned}$$

(d) Using the results from parts (c) and (a), we have

$$\frac{f(a+h)-f(a)}{h} = \frac{(2a^2+4ah+2h^2+3a+3h-1)-(2a^2+3a-1)}{h}$$

$$= \frac{4ah+2h^2+3h}{h} = 4a+2h+3$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41**

EXAMPLE 6 | The Weight of an Astronaut

If an astronaut weighs 130 pounds on the surface of the earth, then her weight when she is h miles above the earth is given by the function

$$w(h) = 130\left(\frac{3960}{3960+h}\right)^2$$

(a) What is her weight when she is 100 mi above the earth?

(b) Construct a table of values for the function w that gives her weight at heights from 0 to 500 mi. What do you conclude from the table?

(c) Find the net change in the astronaut's weight from ground level to a height of 500 mi.

The weight of an object on or near the earth is the gravitational force that the earth exerts on it. When in orbit around the earth, an astronaut experiences the sensation of "weightlessness" because the centripetal force that keeps her in orbit is exactly the same as the gravitational pull of the earth.

SOLUTION

(a) We want the value of the function w when $h = 100$; that is, we must calculate $w(100)$:

$$w(100) = 130\left(\frac{3960}{3960+100}\right)^2 \approx 123.67$$

So at a height of 100 mi she weighs about 124 lb.

(b) The table gives the astronaut's weight, rounded to the nearest pound, at 100-mile increments. The values in the table are calculated as in part (a).

h	$w(h)$
0	130
100	124
200	118
300	112
400	107
500	102

The table indicates that the higher the astronaut travels, the less she weighs.

(c) The net change in the astronaut's weight from $h = 0$ to $h = 500$ is

$$w(500) - w(0) = 102 - 130 = -28$$

The negative sign indicates that the astronaut's weight *decreased* by 28 lb.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **77**

The Domain of a Function

Recall that the *domain* of a function is the set of all inputs for the function. The domain of a function may be stated explicitly. For example, if we write

$$f(x) = x^2 \qquad 0 \le x \le 5$$

then the domain is the set of all real numbers x for which $0 \le x \le 5$. If the function is given by an algebraic expression and the domain is not stated explicitly, then by convention *the domain of the function is the domain of the algebraic expression—that is, the set of all real numbers for which the expression is defined as a real number*. For example, consider the functions

Domains of algebraic expressions are discussed on page 44.

$$f(x) = \frac{1}{x - 4} \qquad g(x) = \sqrt{x}$$

The function f is not defined at $x = 4$, so its domain is $\{x \mid x \neq 4\}$. The function g is not defined for negative x, so its domain is $\{x \mid x \ge 0\}$.

EXAMPLE 7 | Finding Domains of Functions

Find the domain of each function.

(a) $f(x) = \dfrac{1}{x^2 - x}$ **(b)** $g(x) = \sqrt{9 - x^2}$ **(c)** $h(t) = \dfrac{t}{\sqrt{t + 1}}$

SOLUTION

(a) A rational expression is not defined when the denominator is 0. Since

$$f(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

we see that $f(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain of f is

$$\{x \mid x \neq 0, x \neq 1\}$$

The domain may also be written in interval notation as

$$(\infty, 0) \cup (0, 1) \cup (1, \infty)$$

(b) We can't take the square root of a negative number, so we must have $9 - x^2 \ge 0$. Using the methods of Section 1.8, we can solve this inequality to find that $-3 \le x \le 3$. Thus the domain of g is

$$\{x \mid -3 \le x \le 3\} = [-3, 3]$$

(c) We can't take the square root of a negative number, and we can't divide by 0, so we must have $t + 1 > 0$, that is, $t > -1$. So the domain of h is

$$\{t \mid t > -1\} = (-1, \infty)$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **53** AND **57**

▼ Four Ways to Represent a Function

To help us understand what a function is, we have used machine and arrow diagrams. We can describe a specific function in the following four ways:

- verbally (by a description in words)
- algebraically (by an explicit formula)
- visually (by a graph)
- numerically (by a table of values)

A single function may be represented in all four ways, and it is often useful to go from one representation to another to gain insight into the function. However, certain functions are described more naturally by one method than by the others. An example of a verbal description is the following rule for converting between temperature scales:

> "To find the Fahrenheit equivalent of a Celsius temperature, multiply the Celsius temperature by $\frac{9}{5}$, then add 32."

In Example 8 we see how to describe this verbal rule or function algebraically, graphically, and numerically. A useful representation of the area of a circle as a function of its radius is the algebraic formula

$$A(r) = \pi r^2$$

The graph produced by a seismograph (see the box below) is a visual representation of the vertical acceleration function $a(t)$ of the ground during an earthquake. As a final example, consider the function $C(w)$, which is described verbally as "the cost of mailing a first-class letter with weight w." The most convenient way of describing this function is numerically—that is, using a table of values.

We will be using all four representations of functions throughout this book. We summarize them in the following box.

FOUR WAYS TO REPRESENT A FUNCTION

Verbal

Using words:

"To convert from Celsius to Fahrenheit, multiply the Celsius temperature by $\frac{9}{5}$, then add 32."

Relation between Celsius and Fahrenheit temperature scales

Algebraic

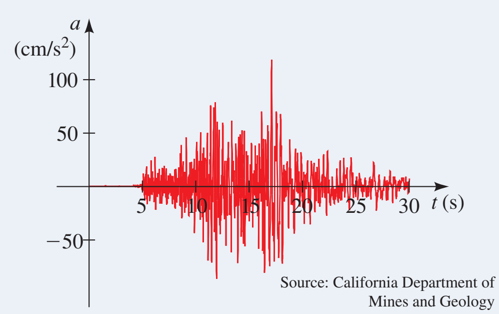
Using a formula:

$$A(r) = \pi r^2$$

Area of a circle

Visual

Using a graph:



Vertical acceleration during an earthquake

Numerical

Using a table of values:

w (ounces)	$C(w)$ (dollars)
$0 < w \le 3$	1.71
$3 < w \le 4$	1.88
$4 < w \le 5$	2.05
$5 < w \le 6$	2.22
$6 < w \le 7$	2.39
$\vdots$	$\vdots$

Cost of mailing a first-class parcel

EXAMPLE 8 | Representing a Function Verbally, Algebraically, Numerically, and Graphically

Let $F(C)$ be the Fahrenheit temperature corresponding to the Celsius temperature C. (Thus F is the function that converts Celsius inputs to Fahrenheit outputs.) The box above gives a verbal description of this function. Find ways to represent this function

(a) Algebraically (using a formula)

(b) Numerically (using a table of values)

(c) Visually (using a graph)

SOLUTION

(a) The verbal description tells us that we should first multiply the input C by $\frac{9}{5}$ and then add 32 to the result. So we get

$$F(C) = \tfrac{9}{5}C + 32$$

(b) We use the algebraic formula for F that we found in part (a) to construct a table of values:

C (Celsius)	F (Fahrenheit)
−10	14
0	32
10	50
20	68
30	86
40	104

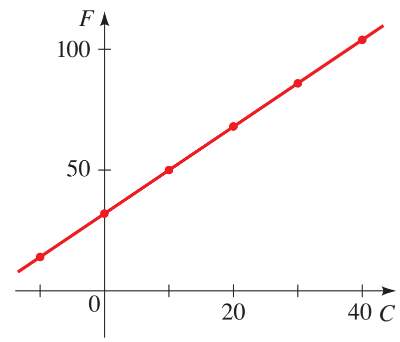


FIGURE 6 Celsius and Fahrenheit

(c) We use the points tabulated in part (b) to help us draw the graph of this function in Figure 6.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **71**

2.1 EXERCISES

CONCEPTS

1. If $f(x) = x^3 + 1$, then

(a) the value of f at $x = -1$ is $f(\_\_\_\_) =$ ________.

(b) the value of f at $x = 2$ is $f(\_\_\_\_) =$ ________.

(c) the net change in the value of f between $x = -1$ and $x = 2$ is $f(\_\_\_\_) - f(\_\_\_\_) =$ ________.

2. For a function f, the set of all possible inputs is called the ________ of f, and the set of all possible outputs is called the ________ of f.

3. (a) Which of the following functions have 5 in their domain?

$$f(x) = x^2 - 3x \qquad g(x) = \frac{x-5}{x} \qquad h(x) = \sqrt{x-10}$$

(b) For the functions from part (a) that *do* have 5 in their domain, find the value of the function at 5.

4. A function is given algebraically by the formula $f(x) = (x-4)^2 + 3$. Complete these other ways to represent f:

(a) *Verbal:* "Subtract 4, then ________ and ________.

(b) *Numerical:*

x	$f(x)$
0	19
2	
4	
6	

SKILLS

5–8 ■ Express the rule in function notation. (For example, the rule "square, then subtract 5" is expressed as the function $f(x) = x^2 - 5$.)

5. Subtract 2, then divide by 5

6. Add 6, then square

7. Multiply by 4, then subtract 1

8. Add 5, take the square root, then multiply by 2

9–12 ■ Express the function (or rule) in words.

9. $h(x) = x^2 + 2$

10. $k(x) = \sqrt{x+2}$

11. $f(x) = \dfrac{x-4}{3}$

12. $g(x) = \dfrac{x}{3} - 4$

13–14 ■ Draw a machine diagram for the function.

13. $f(x) = \sqrt{x-1}$

14. $f(x) = \dfrac{3}{x-2}$

15–16 ■ Complete the table.

15. $f(x) = 2(x-1)^2$

x	$f(x)$
−1	
0	
1	
2	
3	

16. $g(x) = |2x + 3|$

x	$g(x)$
−3	
−2	
0	
1	
3	

17–28 ■ Evaluate the function at the indicated values.

17. $f(x) = x^2 - 6;\quad f(-3), f(3), f(0), f(\frac{1}{2})$

18. $f(x) = x^3 + 2x;\quad f(-2), f(-1), f(0), f(\frac{1}{2})$

19. $f(x) = 2x + 1;$
$f(1), f(-2), f(\frac{1}{2}), f(a), f(-a), f(a-1)$

20. $h(x) = 3x - 2;$
$h(1), h(-1), h(a), h(-x), h(a-2), h(\sqrt{x})$

21. $f(x) = x^2 + 2x;$
$f(0), f(3), f(-3), f(a), f(-x), f\left(\dfrac{1}{a}\right)$

22. $h(t) = t + \dfrac{1}{t};$
$h(-1), h(2), h(\frac{1}{2}), h(x-1), h\left(\dfrac{1}{x}\right)$

23. $g(x) = \dfrac{1-x}{1+x};$
$g(2), g(-1), g(\frac{1}{2}), g(a), g(a-1), g(x^2-1)$

24. $g(t) = \dfrac{t+2}{t-2};$
$g(-2), g(2), g(0), g(a), g(a^2-2), g(a+1)$

25. $f(x) = 2x^2 + 3x - 4;$
$f(0), f(2), f(\sqrt{2}), f(x+1), f(-x), f(x^3)$

26. $f(x) = x^3 - 4x^2;$
$f(0), f(1), f(-1), f(\frac{3}{2}), f\left(\dfrac{x}{2}\right), f(x^2)$

27. $f(x) = 2|x-1|;$
$f(-2), f(0), f(\frac{1}{2}), f(2), f(x+1), f(x^2+2)$

28. $f(x) = \dfrac{|x|}{x};$
$f(-2), f(-1), f(0), f(5), f(x^2), f\left(\dfrac{1}{x}\right)$

29–32 ■ Evaluate the piecewise defined function at the indicated values.

29. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x+1 & \text{if } x \ge 0 \end{cases}$
$f(-2), f(-1), f(0), f(1), f(2)$

30. $f(x) = \begin{cases} 5 & \text{if } x \le 2 \\ 2x-3 & \text{if } x > 2 \end{cases}$
$f(-3), f(0), f(2), f(3), f(5)$

31. $f(x) = \begin{cases} x^2 + 2x & \text{if } x \le -1 \\ x & \text{if } -1 < x \le 1 \\ -1 & \text{if } x > 1 \end{cases}$
$f(-4), f(-\frac{3}{2}), f(-1), f(0), f(25)$

32. $f(x) = \begin{cases} 3x & \text{if } x < 0 \\ x+1 & \text{if } 0 \le x \le 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$
$f(-5), f(0), f(1), f(2), f(5)$

33–36 ■ Use the function to evaluate the indicated expressions and simplify.

33. $f(x) = x^2 + 1;\quad f(x+2), f(x) + f(2)$

34. $f(x) = 3x - 1;\quad f(2x), 2f(x)$

35. $f(x) = x + 4;\quad f(x^2), (f(x))^2$

36. $f(x) = 6x - 18;\quad f\left(\dfrac{x}{3}\right), \dfrac{f(x)}{3}$

37–40 ■ Find the net change in the value of the function between the given inputs.

37. $f(x) = 3x - 2;$ from 1 to 5

38. $f(x) = 4 - 5x;$ from 3 to 5

39. $g(t) = 1 - t^2;$ from -2 to 5

40. $h(t) = t^2 + 5;$ from -3 to 6

41–48 ■ Find $f(a)$, $f(a+h)$, and the difference quotient $\dfrac{f(a+h) - f(a)}{h}$, where $h \neq 0$.

41. $f(x) = 3x + 2$

42. $f(x) = x^2 + 1$

43. $f(x) = 5$

44. $f(x) = \dfrac{1}{x+1}$

45. $f(x) = \dfrac{x}{x+1}$

46. $f(x) = \dfrac{2x}{x-1}$

47. $f(x) = 3 - 5x + 4x^2$

48. $f(x) = x^3$

49–70 ■ Find the domain of the function.

49. $f(x) = 2x$

50. $f(x) = x^2 + 1$

51. $f(x) = 2x,\quad -1 \le x \le 5$

52. $f(x) = x^2 + 1,\quad 0 \le x \le 5$

53. $f(x) = \dfrac{1}{x-3}$

54. $f(x) = \dfrac{1}{3x-6}$

55. $f(x) = \dfrac{x+2}{x^2-1}$

56. $f(x) = \dfrac{x^4}{x^2+x-6}$

57. $f(x) = \sqrt{x-5}$

58. $f(x) = \sqrt[4]{x+9}$

59. $f(t) = \sqrt[3]{t-1}$

60. $g(x) = \sqrt{7-3x}$

61. $h(x) = \sqrt{2x-5}$

62. $G(x) = \sqrt{x^2-9}$

63. $g(x) = \dfrac{\sqrt{2+x}}{3-x}$

64. $g(x) = \dfrac{\sqrt{x}}{2x^2+x-1}$

65. $g(x) = \sqrt[4]{x^2-6x}$

66. $g(x) = \sqrt{x^2-2x-8}$

67. $f(x) = \dfrac{3}{\sqrt{x-4}}$

68. $f(x) = \dfrac{x^2}{\sqrt{6-x}}$

69. $f(x) = \dfrac{(x+1)^2}{\sqrt{2x-1}}$

70. $f(x) = \dfrac{x}{\sqrt[4]{9-x^2}}$

71–74 ■ A verbal description of a function is given. Find **(a)** algebraic, **(b)** numerical, and **(c)** graphical representations for the function.

71. To evaluate $f(x)$, divide the input by 3 and add $\frac{2}{3}$ to the result.

72. To evaluate $g(x)$, subtract 4 from the input and multiply the result by $\frac{3}{4}$.

73. Let $T(x)$ be the amount of sales tax charged in Lemon County on a purchase of x dollars. To find the tax, take 8% of the purchase price.

74. Let $V(d)$ be the volume of a sphere of diameter d. To find the volume, take the cube of the diameter, then multiply by π and divide by 6.

APPLICATIONS

75. Production Cost The cost C in dollars of producing x yards of a certain fabric is given by the function

$$C(x) = 1500 + 3x + 0.02x^2 + 0.0001x^3$$

(a) Find $C(10)$ and $C(100)$.
(b) What do your answers in part (a) represent?
(c) Find $C(0)$. (This number represents the *fixed costs*.)

76. Area of a Sphere The surface area S of a sphere is a function of its radius r given by

$$S(r) = 4\pi r^2$$

(a) Find $S(2)$ and $S(3)$.
(b) What do your answers in part (a) represent?

77. Torricelli's Law A tank holds 50 gallons of water, which drains from a leak at the bottom, causing the tank to empty in 20 minutes. The tank drains faster when it is nearly full because the pressure on the leak is greater. **Torricelli's Law** gives the volume of water remaining in the tank after t minutes as

$$V(t) = 50\left(1 - \frac{t}{20}\right)^2 \qquad 0 \le t \le 20$$

(a) Find $V(0)$ and $V(20)$.
(b) What do your answers to part (a) represent?
(c) Make a table of values of $V(t)$ for $t = 0, 5, 10, 15, 20$.
(d) Find the net change in the volume V as t changes from 0 min to 20 min.

78. How Far Can You See? Because of the curvature of the earth, the maximum distance D that you can see from the top of a tall building or from an airplane at height h is given by the function

$$D(h) = \sqrt{2rh + h^2}$$

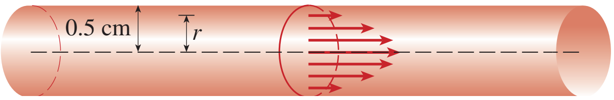
where $r = 3960$ mi is the radius of the earth and D and h are measured in miles.

(a) Find $D(0.1)$ and $D(0.2)$.
(b) How far can you see from the observation deck of Toronto's CN Tower, 1135 ft above the ground?
(c) Commercial aircraft fly at an altitude of about 7 mi. How far can the pilot see?
(d) Find the net change in the value of distance D as h changes from 1135 ft to 7 mi.

79. Blood Flow As blood moves through a vein or an artery, its velocity v is greatest along the central axis and decreases as the distance r from the central axis increases (see the figure). The formula that gives v as a function of r is called the **law of laminar flow**. For an artery with radius 0.5 cm, the relationship between v (in cm/s) and r (in cm) is given by the function

$$v(r) = 18{,}500(0.25 - r^2) \qquad 0 \le r \le 0.5$$

(a) Find $v(0.1)$ and $v(0.4)$.
(b) What do your answers to part (a) tell you about the flow of blood in this artery?
(c) Make a table of values of $v(r)$ for $r = 0, 0.1, 0.2, 0.3, 0.4, 0.5$.
(d) Find the net change in the velocity v as r changes from 0.1 cm to 0.5 cm.



80. Pupil Size When the brightness x of a light source is increased, the eye reacts by decreasing the radius R of the pupil. The dependence of R on x is given by the function

$$R(x) = \sqrt{\frac{13 + 7x^{0.4}}{1 + 4x^{0.4}}}$$

where R is measured in millimeters and x is measured in appropriate units of brightness.

(a) Find $R(1)$, $R(10)$, and $R(100)$.
(b) Make a table of values of $R(x)$.
(c) Find the net change in the radius R as x changes from 10 to 100.

81. Relativity According to the Theory of Relativity, the length L of an object is a function of its velocity v with respect to an observer. For an object whose length at rest is 10 m, the function is given by

$$L(v) = 10\sqrt{1 - \frac{v^2}{c^2}}$$

where c is the speed of light (300,000 km/s).

(a) Find $L(0.5c)$, $L(0.75c)$, and $L(0.9c)$.
(b) How does the length of an object change as its velocity increases?

82. Income Tax In a certain country, income tax T is assessed according to the following function of income x:

$$T(x) = \begin{cases} 0 & \text{if } 0 \le x \le 10{,}000 \\ 0.08x & \text{if } 10{,}000 < x \le 20{,}000 \\ 1600 + 0.15x & \text{if } 20{,}000 < x \end{cases}$$

(a) Find $T(5{,}000)$, $T(12{,}000)$, and $T(25{,}000)$.
(b) What do your answers in part (a) represent?

83. Internet Purchases An Internet bookstore charges $15 shipping for orders under $100 but provides free shipping for orders of $100 or more. The cost C of an order is a function of the total price x of the books purchased, given by

$$C(x) = \begin{cases} x + 15 & \text{if } x < 100 \\ x & \text{if } x \ge 100 \end{cases}$$

(a) Find $C(75)$, $C(90)$, $C(100)$, and $C(105)$.
(b) What do your answers in part (a) represent?

84. Cost of a Hotel Stay A hotel chain charges $75 each night for the first two nights and $50 for each additional night's stay. The total cost T is a function of the number of nights x that a guest stays.
(a) Complete the expressions in the following piecewise defined function.

$$T(x) = \begin{cases} \boxed{} & \text{if } 0 \le x \le 2 \\ \boxed{} & \text{if } x > 2 \end{cases}$$

(b) Find $T(2)$, $T(3)$, and $T(5)$.
(c) What do your answers in part (b) represent?

85. Speeding Tickets In a certain state the maximum speed permitted on freeways is 65 mi/h, and the minimum is 40. The fine F for violating these limits is $15 for every mile above the maximum or below the minimum.
(a) Complete the expressions in the following piecewise defined function, where x is the speed at which you are driving.

$$F(x) = \begin{cases} \boxed{} & \text{if } 0 < x < 40 \\ \boxed{} & \text{if } 40 \le x \le 65 \\ \boxed{} & \text{if } x > 65 \end{cases}$$

(b) Find $F(30)$, $F(50)$, and $F(75)$.
(c) What do your answers in part (b) represent?

86. Height of Grass A home owner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period beginning on a Sunday.

87. Temperature Change You place a frozen pie in an oven and bake it for an hour. Then you take the pie out and let it cool before eating it. Sketch a rough graph of the temperature of the pie as a function of time.

88. Daily Temperature Change Temperature readings T (in °F) were recorded every 2 hours from midnight to noon in Atlanta, Georgia, on March 18, 1996. The time t was measured in hours from midnight. Sketch a rough graph of T as a function of t.

t	0	2	4	6	8	10	12
T	58	57	53	50	51	57	61

89. Population Growth The population P (in thousands) of San Jose, California, from 1988 to 2000 is shown in the table. (Midyear estimates are given.) Draw a rough graph of P as a function of time t.

t	1988	1990	1992	1994	1996	1998	2000
P	733	782	800	817	838	861	895

DISCOVERY ▪ DISCUSSION ▪ WRITING

90. Examples of Functions At the beginning of this section we discussed three examples of everyday, ordinary functions: Height is a function of age, temperature is a function of date, and postage cost is a function of weight. Give three other examples of functions from everyday life.

91. Four Ways to Represent a Function In the box on page 180 we represented four different functions verbally, algebraically, visually, and numerically. Think of a function that can be represented in all four ways, and write the four representations.

2.2 Graphs of Functions

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Graph functions by plotting points ▶ Graph functions with a graphing calculator ▶ Graph piecewise defined functions ▶ Use the Vertical Line Test ▶ Determine whether an equation defines a function

The most important way to visualize a function is through its graph. In this section we investigate in more detail the concept of graphing functions.

Graphing Functions by Plotting Points

To graph a function f, we plot the points $(x, f(x))$ in a coordinate plane. In other words, we plot the points (x, y) whose x-coordinate is an input and whose y-coordinate is the corresponding output of the function.

THE GRAPH OF A FUNCTION

If f is a function with domain A, then the **graph** of f is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

plotted in a coordinate plane. In other words, the graph of f is the set of all points (x, y) such that $y = f(x)$; that is, the graph of f is the graph of the equation $y = f(x)$.

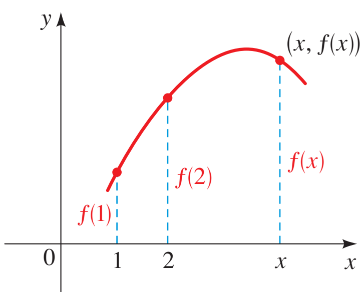
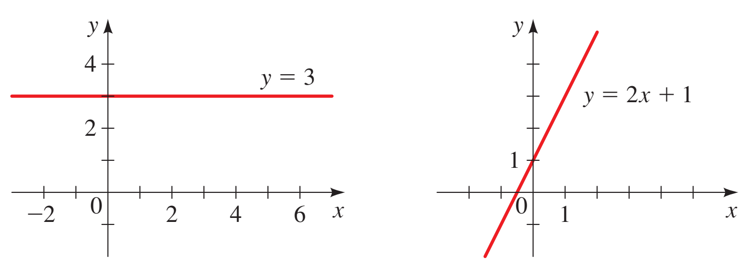


FIGURE 1 The height of the graph above the point x is the value of $f(x)$.

The graph of a function f gives a picture of the behavior or "life history" of the function. We can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 1).

A function f of the form $f(x) = mx + b$ is called a **linear function** because its graph is the graph of the equation $y = mx + b$, which represents a line with slope m and y-intercept b. A special case of a linear function occurs when the slope is $m = 0$. The function $f(x) = b$, where b is a given number, is called a **constant function** because all its values are the same number, namely, b. Its graph is the horizontal line $y = b$. Figure 2 shows the graphs of the constant function $f(x) = 3$ and the linear function $f(x) = 2x + 1$.



The constant function $f(x) = 3$ The linear function $f(x) = 2x + 1$

FIGURE 2

EXAMPLE 1 | Graphing Functions by Plotting Points

Sketch graphs of the following functions.

(a) $f(x) = x^2$ **(b)** $g(x) = x^3$ **(c)** $h(x) = \sqrt{x}$

SOLUTION We first make a table of values. Then we plot the points given by the table and join them by a smooth curve to obtain the graph. The graphs are sketched in Figure 3 on the next page.

x	$f(x) = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
±1	1
±2	4
±3	9

x	$g(x) = x^3$
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8
$-\frac{1}{2}$	$-\frac{1}{8}$
-1	-1
-2	-8

x	$h(x) = \sqrt{x}$
0	0
1	1
2	$\sqrt{2}$
3	$\sqrt{3}$
4	2
5	$\sqrt{5}$

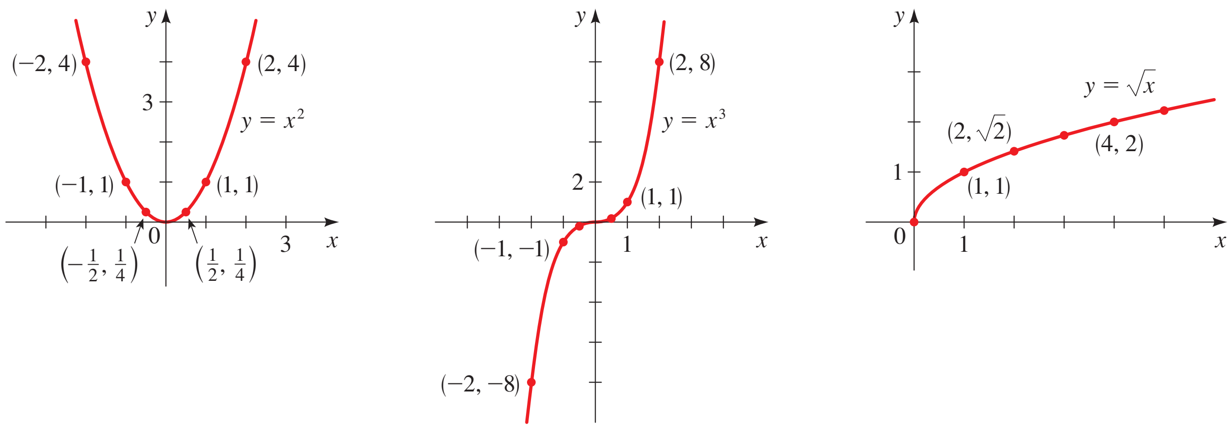


FIGURE 3 (a) $f(x) = x^2$ (b) $g(x) = x^3$ (c) $h(x) = \sqrt{x}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **11, 17,** AND **21**

Graphing Functions with a Graphing Calculator

See Appendix B, *Graphing with a Graphing Calculator,* for general guidelines on using a graphing calculator. See Appendix C, *Using the TI-83/84 Graphing Calculator,* for specific instructions.

A convenient way to graph a function is to use a graphing calculator. To graph the function f, we use a calculator to graph the equation $y = f(x)$.

EXAMPLE 2 | Graphing a Function with a Graphing Calculator

Use a graphing calculator to graph the function $f(x) = x^3 - 8x^2$ in an appropriate viewing rectangle.

SOLUTION To graph the function $f(x) = x^3 - 8x^2$, we must graph the equation $y = x^3 - 8x^2$. On the TI-83 graphing calculator the default viewing rectangle gives the graph in Figure 4(a). But this graph appears to spill over the top and bottom of the screen. We need to expand the vertical axis to get a better representation of the graph. The viewing rectangle $[-4, 10]$ by $[-100, 100]$ gives a more complete picture of the graph, as shown in Figure 4(b).

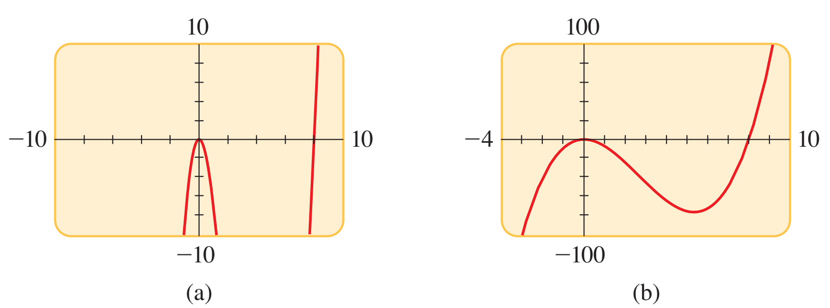


FIGURE 4 Graphing the function $f(x) = x^3 - 8x^2$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **31**

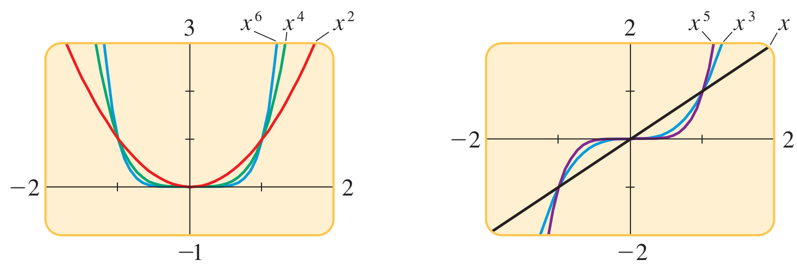
EXAMPLE 3 | A Family of Power Functions

(a) Graph the functions $f(x) = x^n$ for $n = 2$, 4, and 6 in the viewing rectangle $[-2, 2]$ by $[-1, 3]$.

(b) Graph the functions $f(x) = x^n$ for $n = 1$, 3, and 5 in the viewing rectangle $[-2, 2]$ by $[-2, 2]$.

(c) What conclusions can you draw from these graphs?

SOLUTION To graph the function $f(x) = x^n$, we graph the equation $y = x^n$. The graphs for parts (a) and (b) are shown in Figure 5.



(a) Even powers of x (b) Odd powers of x

FIGURE 5 A family of power functions $f(x) = x^n$

(c) We see that the general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.

If n is even, the graph of $f(x) = x^n$ is similar to the parabola $y = x^2$.

If n is odd, the graph of $f(x) = x^n$ is similar to that of $y = x^3$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **71**

Notice from Figure 5 that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $x > 1$. When $0 < x < 1$, the lower powers of x are the "bigger" functions. But when $x > 1$, the higher powers of x are the dominant functions.

Graphing Piecewise Defined Functions

A piecewise defined function is defined by different formulas on different parts of its domain. As you might expect, the graph of such a function consists of separate pieces.

EXAMPLE 4 | Graph of a Piecewise Defined Function

Sketch the graph of the function

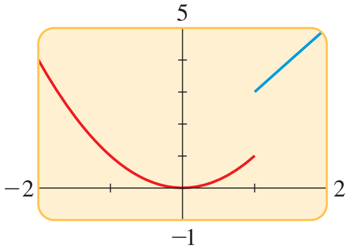
$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$$

SOLUTION If $x \le 1$, then $f(x) = x^2$, so the part of the graph to the left of $x = 1$ coincides with the graph of $y = x^2$, which we sketched in Figure 3. If $x > 1$, then $f(x) = 2x + 1$, so the part of the graph to the right of $x = 1$ coincides with the line $y = 2x + 1$, which we graphed in Figure 2. This enables us to sketch the graph in Figure 6.

The solid dot at $(1, 1)$ indicates that this point is included in the graph; the open dot at $(1, 3)$ indicates that this point is excluded from the graph.

On many graphing calculators the graph in Figure 6 can be produced by using the logical functions in the calculator. For example, on the TI-83 the following equation gives the required graph:

$Y_1 = (X \le 1)X^2 + (X > 1)(2X + 1)$



(To avoid the extraneous vertical line between the two parts of the graph, put the calculator in `Dot` mode.)

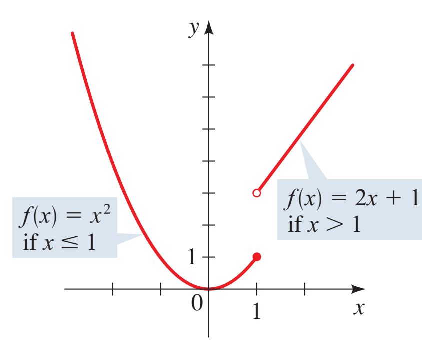


FIGURE 6

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **37**

EXAMPLE 5 | Graph of the Absolute Value Function

Sketch a graph of the absolute value function $f(x) = |x|$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 4, we note that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis (see Figure 7).

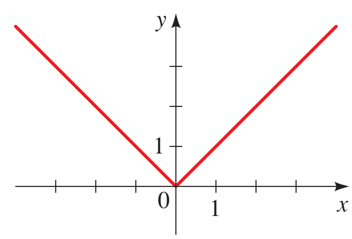


FIGURE 7 Graph of $f(x) = |x|$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **25**

The **greatest integer function** is defined by

$$[\![x]\!] = \text{greatest integer less than or equal to } x$$

For example, $[\![2]\!] = 2$, $[\![2.3]\!] = 2$, $[\![1.999]\!] = 1$, $[\![0.002]\!] = 0$, $[\![-3.5]\!] = -4$, and $[\![-0.5]\!] = -1$.

EXAMPLE 6 | Graph of the Greatest Integer Function

Sketch a graph of $f(x) = [\![x]\!]$.

SOLUTION The table shows the values of f for some values of x. Note that $f(x)$ is constant between consecutive integers, so the graph between integers is a horizontal line segment, as shown in Figure 8.

x	$[\![x]\!]$
$\vdots$	$\vdots$
$-2 \leq x < -1$	-2
$-1 \leq x < 0$	-1
$0 \leq x < 1$	0
$1 \leq x < 2$	1
$2 \leq x < 3$	2
$\vdots$	$\vdots$

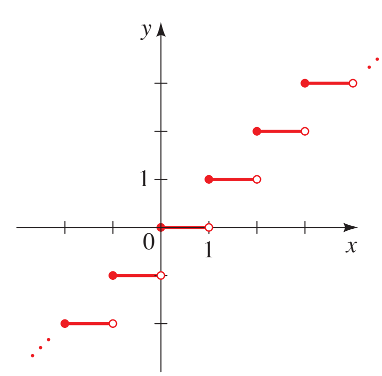


FIGURE 8 The greatest integer function, $y = [\![x]\!]$

The greatest integer function is an example of a **step function**. The next example gives a real-world example of a step function.

EXAMPLE 7 | The Cost Function for Long-Distance Phone Calls

The cost of a long-distance daytime phone call from Toronto, Canada, to Mumbai, India, is 69 cents for the first minute and 58 cents for each additional minute (or part of a minute). Draw the graph of the cost C (in dollars) of the phone call as a function of time t (in minutes).

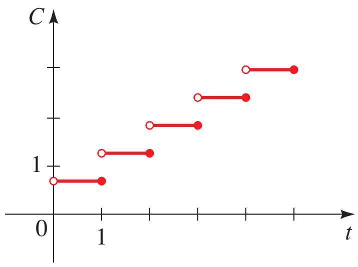


FIGURE 9 Cost of a long-distance call

SOLUTION Let $C(t)$ be the cost for t minutes. Since $t > 0$, the domain of the function is $(0, \infty)$. From the given information we have

$$\begin{aligned}
C(t) &= 0.69 && \text{if } 0 < t \le 1 \\
C(t) &= 0.69 + 0.58 = 1.27 && \text{if } 1 < t \le 2 \\
C(t) &= 0.69 + 2(0.58) = 1.85 && \text{if } 2 < t \le 3 \\
C(t) &= 0.69 + 3(0.58) = 2.43 && \text{if } 3 < t \le 4
\end{aligned}$$

and so on. The graph is shown in Figure 9.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **83**

A function is called **continuous** if its graph has no "breaks" or "holes." The functions in Examples 1, 2, 3, and 5 are continuous; the functions in Examples 4, 6, and 7 are not continuous.

The Vertical Line Test

The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions? This is answered by the following test.

THE VERTICAL LINE TEST

A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

We can see from Figure 10 why the Vertical Line Test is true. If each vertical line $x = a$ intersects a curve only once at (a, b), then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and at (a, c), then the curve cannot represent a function because a function cannot assign two different values to a.

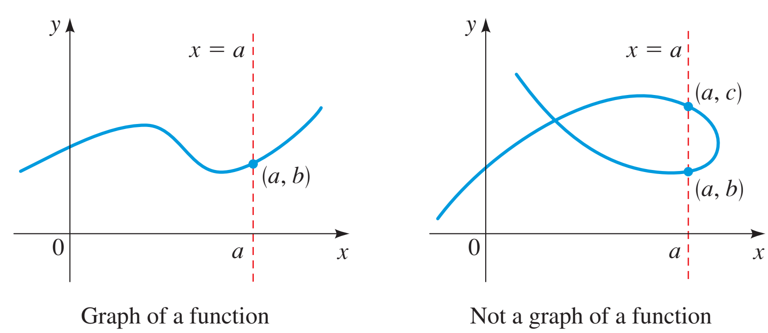


FIGURE 10 Vertical Line Test

EXAMPLE 8 | Using the Vertical Line Test

Using the Vertical Line Test, we see that the curves in parts (b) and (c) of Figure 11 represent functions, whereas those in parts (a) and (d) do not.

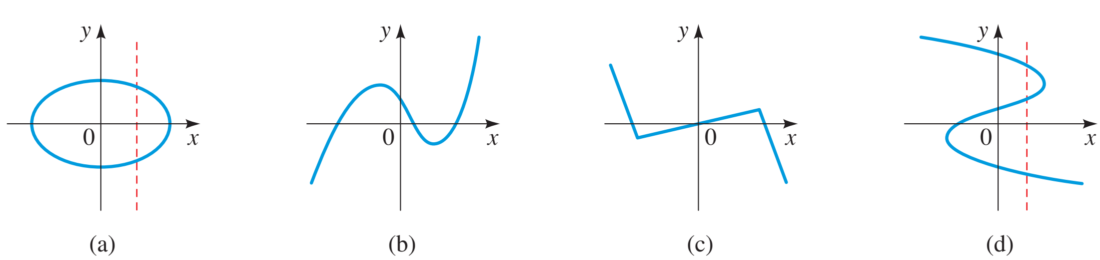


FIGURE 11

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **53**

Equations That Define Functions

Any equation in the variables x and y defines a relationship between these variables. For example, the equation

$$y - x^2 = 0$$

defines a relationship between y and x. Does this equation define y as a *function* of x? To find out, we solve for y and get

$$y = x^2 \qquad \text{Equation form}$$

We see that the equation defines a rule, or function, that gives one value of y for each value of x. We can express this rule in function notation as

$$f(x) = x^2 \qquad \text{Function form}$$

But not every equation defines y as a function of x, as the following example shows.

EXAMPLE 9 | Equations That Define Functions

Does the equation define y as a function of x?

(a) $y - x^2 = 2$ **(b)** $x^2 + y^2 = 4$

SOLUTION

(a) Solving for y in terms of x gives

$$\begin{aligned} y - x^2 &= 2 \\ y &= x^2 + 2 \qquad \text{Add } x^2 \end{aligned}$$

The last equation is a rule that gives one value of y for each value of x, so it defines y as a function of x. We can write the function as $f(x) = x^2 + 2$.

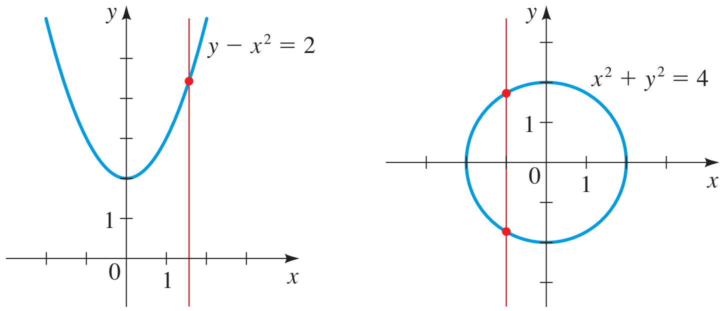
(b) We try to solve for y in terms of x:

$$\begin{aligned} x^2 + y^2 &= 4 \\ y^2 &= 4 - x^2 \qquad \text{Subtract } x^2 \\ y &= \pm\sqrt{4 - x^2} \qquad \text{Take square roots} \end{aligned}$$

The last equation gives two values of y for a given value of x. Thus the equation does not define y as a function of x.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **59** AND **63** ■

The graphs of the equations in Example 9 are shown in Figure 12. The Vertical Line Test shows graphically that the equation in Example 9(a) defines a function but the equation in Example 9(b) does not.



(a) (b)

FIGURE 12

NOTIMEX/FOTO/BBVA/COR/HUM/Newscom

DONALD KNUTH was born in Milwaukee in 1938 and is Professor Emeritus of Computer Science at Stanford University. When Knuth was a high school student, he became fascinated with graphs of functions and laboriously drew many hundreds of them because he wanted to see the behavior of a great variety of functions. (Today, of course, it is far easier to use computers and graphing calculators to do this.) While still a graduate student at Caltech, he started writing a monumental series of books entitled *The Art of Computer Programming*.

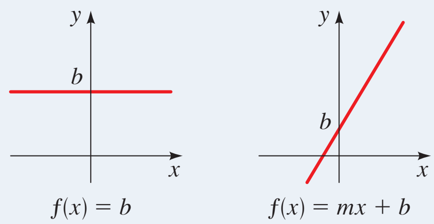
Knuth is famous for his invention of TEX, a system of computer-assisted typesetting. This system was used in the preparation of the manuscript for this textbook.

Knuth has received numerous honors, among them election as an associate of the French Academy of Sciences, and as a Fellow of the Royal Society. President Carter awarded him the National Medal of Science in 1979.

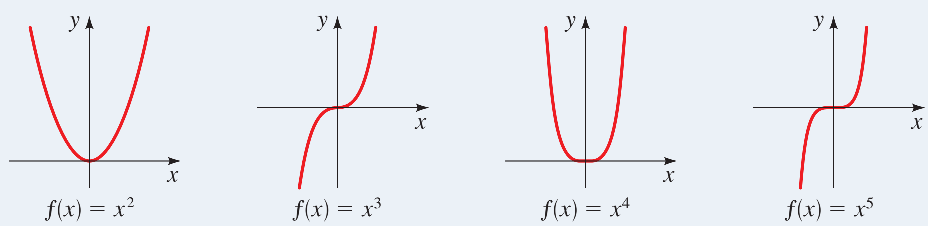
The following box shows the graphs of some functions that you will see frequently in this book.

SOME FUNCTIONS AND THEIR GRAPHS

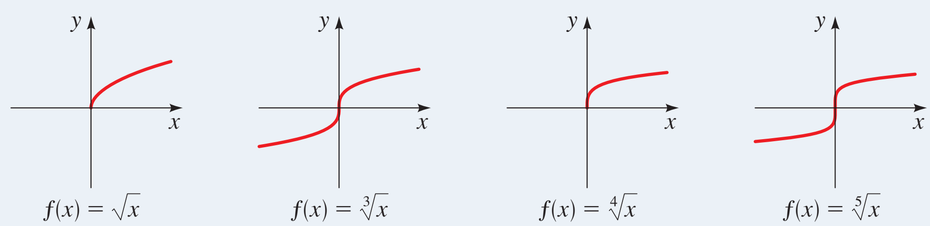
Linear functions
$f(x) = mx + b$



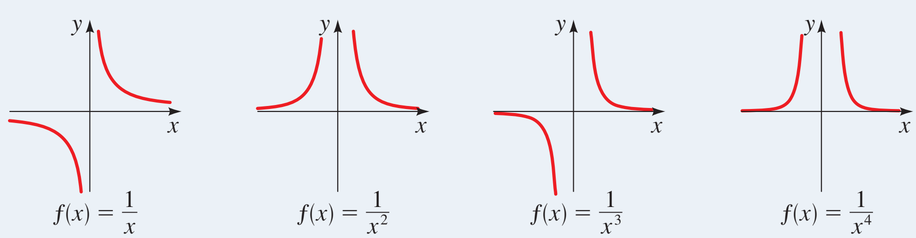
Power functions
$f(x) = x^n$



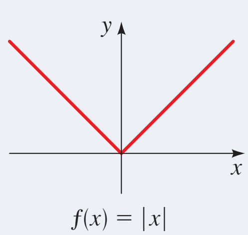
Root functions
$f(x) = \sqrt[n]{x}$



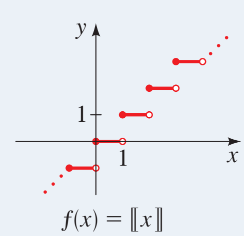
Reciprocal functions
$f(x) = \dfrac{1}{x^n}$



Absolute value function
$f(x) = |x|$



Greatest integer function
$f(x) = [\![x]\!]$



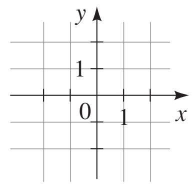
2.2 EXERCISES

CONCEPTS

1. To graph the function f, we plot the points $(x, \_\_\_\_\_\_)$ in a coordinate plane. To graph $f(x) = x^2 - 2$, we plot the points $(x, \_\_\_\_\_\_)$. So the point $(3, \_\_\_\_\_\_)$ is on the graph of f. The height of the graph of f above the x-axis when

$x = 3$ is ________. Complete the table, and sketch a graph of f.

x	$f(x)$	(x, y)
-2		
-1		
0		
1		
2		

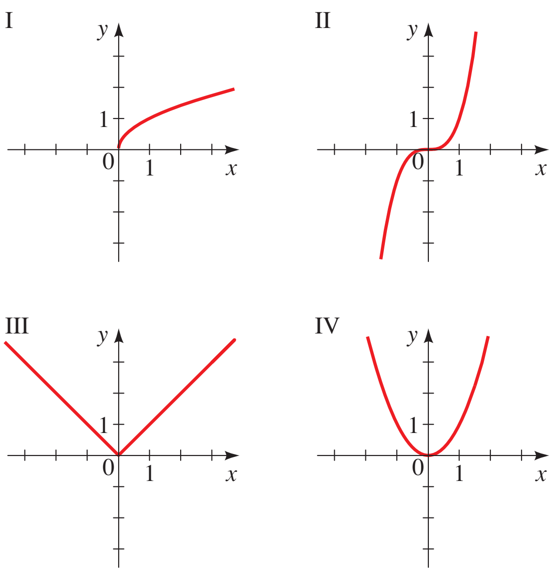


2. If $f(2) = 3$, then the point (2, ________) is on the graph of f.

3. If the point (2, 3) is on the graph of f, then $f(2) =$ ________.

4. Match the function with its graph.

(a) $f(x) = x^2$ **(b)** $f(x) = x^3$
(c) $f(x) = \sqrt{x}$ **(d)** $f(x) = |x|$



SKILLS

5–30 ■ Sketch the graph of the function by first making a table of values.

5. $f(x) = 3$ **6.** $f(x) = -2$

7. $f(x) = x + 2$ **8.** $f(x) = 4 - 2x$

9. $f(x) = -x + 3, \quad -3 \le x \le 3$

10. $f(x) = \dfrac{x - 3}{2}, \quad 0 \le x \le 5$

11. $f(x) = -x^2$ **12.** $f(x) = x^2 - 4$

13. $h(x) = 16 - x^2$ **14.** $g(x) = (x - 3)^2$

15. $W(x) = x^2 + 2x + 1$ **16.** $k(x) = -x^2 + 2x + 3$

17. $g(x) = x^3 - 8$ **18.** $g(x) = (x + 2)^3$

19. $g(x) = x^2 - 2x$ **20.** $h(x) = 4x^2 - x^4$

21. $f(x) = 1 + \sqrt{x}$ **22.** $f(x) = \sqrt{x + 4}$

23. $g(x) = -\sqrt{x}$ **24.** $g(x) = \sqrt{-x}$

25. $H(x) = |2x|$ **26.** $H(x) = |x + 1|$

27. $G(x) = |x| + x$ **28.** $G(x) = |x| - x$

29. $f(x) = |2x - 2|$ **30.** $f(x) = \dfrac{x}{|x|}$

31–34 ■ Graph the function in each of the given viewing rectangles, and select the one that produces the most appropriate graph of the function.

31. $f(x) = 8x - x^2$
(a) $[-5, 5]$ by $[-5, 5]$
(b) $[-10, 10]$ by $[-10, 10]$
(c) $[-2, 10]$ by $[-5, 20]$
(d) $[-10, 10]$ by $[-100, 100]$

32. $g(x) = x^2 - x - 20$
(a) $[-2, 2]$ by $[-5, 5]$
(b) $[-10, 10]$ by $[-10, 10]$
(c) $[-7, 7]$ by $[-25, 20]$
(d) $[-10, 10]$ by $[-100, 100]$

33. $h(x) = x^3 - 5x - 4$
(a) $[-2, 2]$ by $[-2, 2]$
(b) $[-3, 3]$ by $[-10, 10]$
(c) $[-3, 3]$ by $[-10, 5]$
(d) $[-10, 10]$ by $[-10, 10]$

34. $k(x) = \frac{1}{32}x^4 - x^2 + 2$
(a) $[-1, 1]$ by $[-1, 1]$
(b) $[-2, 2]$ by $[-2, 2]$
(c) $[-5, 5]$ by $[-5, 5]$
(d) $[-10, 10]$ by $[-10, 10]$

35–48 ■ Sketch the graph of the piecewise defined function.

35. $f(x) = \begin{cases} 0 & \text{if } x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$

36. $f(x) = \begin{cases} 1 & \text{if } x \le 1 \\ x + 1 & \text{if } x > 1 \end{cases}$

37. $f(x) = \begin{cases} 3 & \text{if } x < 2 \\ x - 1 & \text{if } x \ge 2 \end{cases}$

38. $f(x) = \begin{cases} 1 - x & \text{if } x < -2 \\ 5 & \text{if } x \ge -2 \end{cases}$

39. $f(x) = \begin{cases} x & \text{if } x \le 0 \\ x + 1 & \text{if } x > 0 \end{cases}$

40. $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \ge -1 \end{cases}$

41. $f(x) = \begin{cases} -1 & \text{if } x < -1 \\ 1 & \text{if } -1 \le x \le 1 \\ -1 & \text{if } x > 1 \end{cases}$

42. $f(x) = \begin{cases} -1 & \text{if } x < -1 \\ x & \text{if } -1 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

43. $f(x) = \begin{cases} 2 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$

44. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \le 2 \\ x & \text{if } x > 2 \end{cases}$

45. $f(x) = \begin{cases} 0 & \text{if } |x| \le 2 \\ 3 & \text{if } |x| > 2 \end{cases}$

46. $f(x) = \begin{cases} x^2 & \text{if } |x| \le 1 \\ 1 & \text{if } |x| > 1 \end{cases}$

47. $f(x) = \begin{cases} 4 & \text{if } x < -2 \\ x^2 & \text{if } -2 \le x \le 2 \\ -x + 6 & \text{if } x > 2 \end{cases}$

48. $f(x) = \begin{cases} -x & \text{if } x \le 0 \\ 9 - x^2 & \text{if } 0 < x \le 3 \\ x - 3 & \text{if } x > 3 \end{cases}$

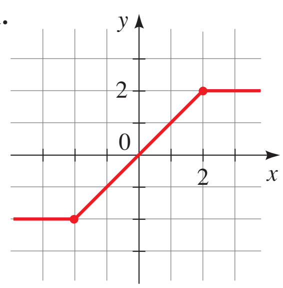
49–50 ■ Use a graphing device to draw the graph of the piecewise defined function. (See the margin note on page 187.)

49. $f(x) = \begin{cases} x + 2 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$

50. $f(x) = \begin{cases} 2x - x^2 & \text{if } x > 1 \\ (x - 1)^3 & \text{if } x \le 1 \end{cases}$

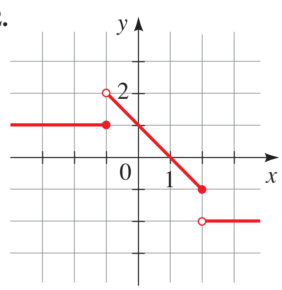
51–52 ■ The graph of a piecewise defined function is given. Find a formula for the function in the indicated form.

51.



$f(x) = \begin{cases} \boxed{} & \text{if } x < -2 \\ \boxed{} & \text{if } -2 \le x \le 2 \\ \boxed{} & \text{if } x > 2 \end{cases}$

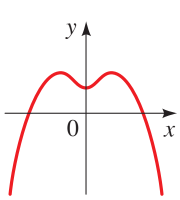
52.



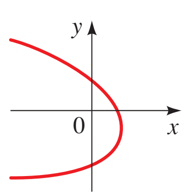
$f(x) = \begin{cases} \boxed{} & \text{if } x \le -1 \\ \boxed{} & \text{if } -1 < x \le 2 \\ \boxed{} & \text{if } x > 2 \end{cases}$

53–54 ■ Use the Vertical Line Test to determine whether the curve is the graph of a function of x.

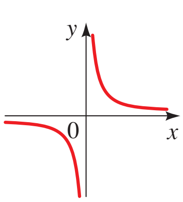
53. (a)



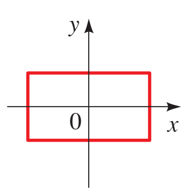
(b)



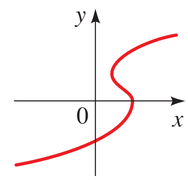
(c)



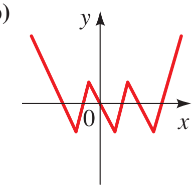
(d)



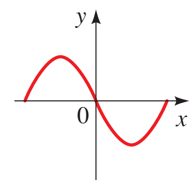
54. (a)



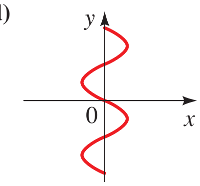
(b)



(c)

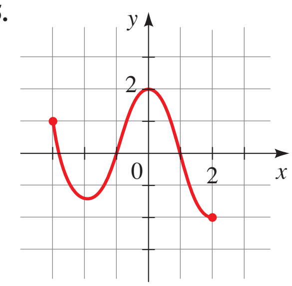


(d)

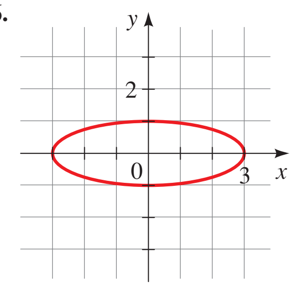


55–58 ■ Use the Vertical Line Test to determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

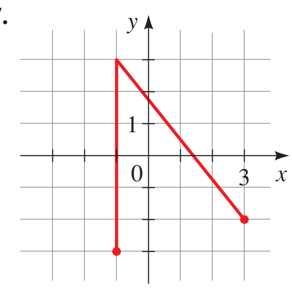
55.



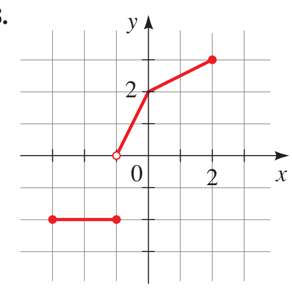
56.



57.



58.



59–70 ■ Determine whether the equation defines y as a function of x. (See Example 9.)

59. $x^2 + 2y = 4$

60. $3x + 7y = 21$

61. $x = y^2$

62. $x^2 + (y - 1)^2 = 4$

63. $x + y^2 = 9$

64. $x^2 + y = 9$

65. $x^2y + y = 1$

66. $\sqrt{x} + y = 12$

67. $2|x| + y = 0$

68. $2x + |y| = 0$

69. $x = y^3$

70. $x = y^4$

71–76 ■ A family of functions is given. In parts (a) and (b) graph all the given members of the family in the viewing rectangle indicated. In part (c) state the conclusions that you can make from your graphs.

71. $f(x) = x^2 + c$

(a) $c = 0, 2, 4, 6;\quad [-5, 5]$ by $[-10, 10]$

(b) $c = 0, -2, -4, -6;\quad [-5, 5]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

72. $f(x) = (x - c)^2$

(a) $c = 0, 1, 2, 3;\quad [-5, 5]$ by $[-10, 10]$

(b) $c = 0, -1, -2, -3;\quad [-5, 5]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

73. $f(x) = (x - c)^3$
(a) $c = 0, 2, 4, 6$; $[-10, 10]$ by $[-10, 10]$
(b) $c = 0, -2, -4, -6$; $[-10, 10]$ by $[-10, 10]$
(c) How does the value of c affect the graph?

74. $f(x) = cx^2$
(a) $c = 1, \frac{1}{2}, 2, 4$; $[-5, 5]$ by $[-10, 10]$
(b) $c = 1, -1, -\frac{1}{2}, -2$; $[-5, 5]$ by $[-10, 10]$
(c) How does the value of c affect the graph?

75. $f(x) = x^c$
(a) $c = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}$; $[-1, 4]$ by $[-1, 3]$
(b) $c = 1, \frac{1}{3}, \frac{1}{5}$; $[-3, 3]$ by $[-2, 2]$
(c) How does the value of c affect the graph?

76. $f(x) = \dfrac{1}{x^n}$
(a) $n = 1, 3$; $[-3, 3]$ by $[-3, 3]$
(b) $n = 2, 4$; $[-3, 3]$ by $[-3, 3]$
(c) How does the value of n affect the graph?

77–80 ■ Find a function whose graph is the given curve.

77. The line segment joining the points $(-2, 1)$ and $(4, -6)$

78. The line segment joining the points $(-3, -2)$ and $(6, 3)$

79. The top half of the circle $x^2 + y^2 = 9$

80. The bottom half of the circle $x^2 + y^2 = 9$

APPLICATIONS

81. Weather Balloon As a weather balloon is inflated, the thickness T of its rubber skin is related to the radius of the balloon by

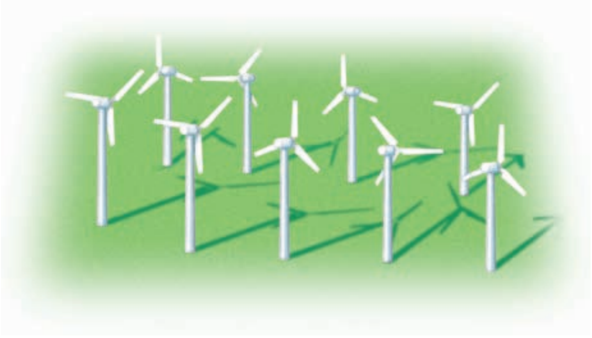
$$T(r) = \frac{0.5}{r^2}$$

where T and r are measured in centimeters. Graph the function T for values of r between 10 and 100.

82. Power from a Wind Turbine The power produced by a wind turbine depends on the speed of the wind. If a windmill has blades 3 meters long, then the power P produced by the turbine is modeled by

$$P(v) = 14.1v^3$$

where P is measured in watts (W) and v is measured in meters per second (m/s). Graph the function P for wind speeds between 1 m/s and 10 m/s.



83. Utility Rates Westside Energy charges its electric customers a base rate of \$6.00 per month, plus 10¢ per kilowatt-hour (kWh) for the first 300 kWh used and 6¢ per kWh for all usage over 300 kWh. Suppose a customer uses x kWh of electricity in one month.
(a) Express the monthly cost E as a piecewise-defined function of x.
(b) Graph the function E for $0 \le x \le 600$.

84. Taxicab Function A taxi company charges \$2.00 for the first mile (or part of a mile) and 20 cents for each succeeding tenth of a mile (or part). Express the cost C (in dollars) of a ride as a piecewise-defined function of the distance x traveled (in miles) for $0 < x < 2$, and sketch the graph of this function.

85. Postage Rates The 2011 domestic postage rate for first-class letters weighing 3.5 oz or less is 44 cents for the first ounce (or less), plus 20 cents for each additional ounce (or part of an ounce). Express the postage P as a piecewise-defined function of the weight x of a letter, with $0 < x \le 3.5$, and sketch the graph of this function.

DISCOVERY ■ DISCUSSION ■ WRITING

86. When Does a Graph Represent a Function? For every integer n, the graph of the equation $y = x^n$ is the graph of a function, namely $f(x) = x^n$. Explain why the graph of $x = y^2$ is *not* the graph of a function of x. Is the graph of $x = y^3$ the graph of a function of x? If so, of what function of x is it the graph? Determine for what integers n the graph of $x = y^n$ is the graph of a function of x.

87. Step Functions In Example 7 and Exercises 84 and 85 we are given functions whose graphs consist of horizontal line segments. Such functions are often called *step functions*, because their graphs look like stairs. Give some other examples of step functions that arise in everyday life.

88. Stretched Step Functions Sketch graphs of the functions $f(x) = [\![x]\!]$, $g(x) = [\![2x]\!]$, and $h(x) = [\![3x]\!]$ on separate graphs. How are the graphs related? If n is a positive integer, what does the graph of $k(x) = [\![nx]\!]$ look like?

89. Graph of the Absolute Value of a Function
(a) Draw the graphs of the functions

$$f(x) = x^2 + x - 6$$

and $$g(x) = |x^2 + x - 6|$$

How are the graphs of f and g related?
(b) Draw the graphs of the functions $f(x) = x^4 - 6x^2$ and $g(x) = |x^4 - 6x^2|$. How are the graphs of f and g related?
(c) In general, if $g(x) = |f(x)|$, how are the graphs of f and g related? Draw graphs to illustrate your answer.

DISCOVERY PROJECT **Relations and Functions**

In this project we explore the concept of function by comparing it with the concept of a *relation*. You can find the project at the book companion website: **www.stewartmath.com**

2.3 Getting Information from the Graph of a Function

LEARNING OBJECTIVES After completing this section, you will be able to:

Find function values from a graph ▶ Find domain and range from a graph ▶ Find where a function is increasing or decreasing from a graph ▶ Find local maxima and minima from a graph

Many properties of a function are more easily obtained from a graph than from the rule that describes the function. We will see in this section how a graph tells us whether the values of a function are increasing or decreasing and also where the maximum and minimum values of a function are.

▼ Values of a Function; Domain and Range

A complete graph of a function contains all the information about a function, because the graph tells us which input values correspond to which output values. To analyze the graph of a function, we must keep in mind that *the height of the graph is the value of the function*. So we can read off the values of a function from its graph.

EXAMPLE 1 | Finding the Values of a Function from a Graph

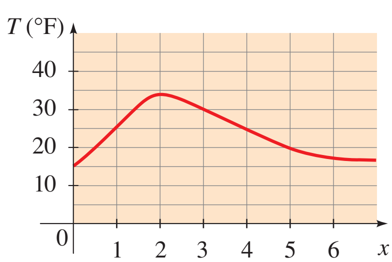


FIGURE 1 Temperature function

The function T graphed in Figure 1 gives the temperature between noon and 6:00 P.M. at a certain weather station.

(a) Find $T(1)$, $T(3)$, and $T(5)$.

(b) Which is larger, $T(2)$ or $T(4)$?

(c) Find the value(s) of x for which $T(x) = 25$.

(d) Find the value(s) of x for which $T(x) \geq 25$.

(e) Find the net change in temperature from 1 P.M. to 3 P.M.

SOLUTION

(a) $T(1)$ is the temperature at 1:00 P.M. It is represented by the height of the graph above the x-axis at $x = 1$. Thus, $T(1) = 25$. Similarly, $T(3) = 30$ and $T(5) = 20$.

(b) Since the graph is higher at $x = 2$ than at $x = 4$, it follows that $T(2)$ is larger than $T(4)$.

(c) The height of the graph is 25 when x is 1 and when x is 4. In other words, the temperature is 25 at 1:00 P.M. and 4:00 P.M.

(d) The graph is higher than 25 for x between 1 and 4. In other words, the temperature was 25 or greater between 1:00 P.M. and 4:00 P.M.

Net change is defined on page 177.

(e) The net change in temperature is

$$T(3) - T(1) = 30 - 25 = 5$$

So there was a net increase of 5°F from 1 P.M. to 3 P.M.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **5** AND **47** ■

The graph of a function helps us to picture the domain and range of the function on the x-axis and y-axis, as shown in Figure 2.

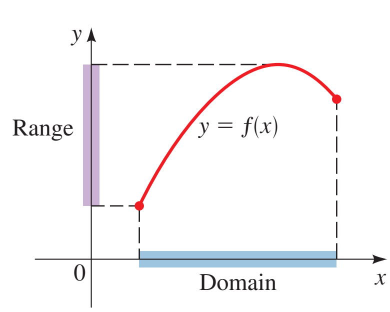


FIGURE 2 Domain and range of f

EXAMPLE 2 | Finding the Domain and Range from a Graph

See Appendix B, *Graphing with a Graphing Calculator,* for guidelines on using a graphing calculator. See Appendix C, *Using the TI-83/84 Graphing Calculator,* for specific graphing instructions.

(a) Use a graphing calculator to draw the graph of $f(x) = \sqrt{4 - x^2}$.

(b) Find the domain and range of f.

SOLUTION

(a) The graph is shown in Figure 3.

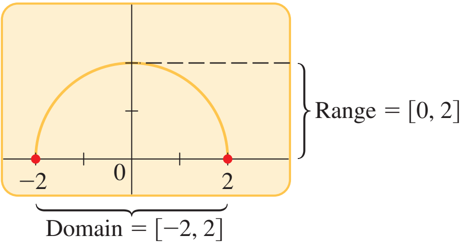


FIGURE 3 Graph of $f(x) = \sqrt{4 - x^2}$

(b) From the graph in Figure 3 we see that the domain is $[-2, 2]$ and the range is $[0, 2]$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **19**

Increasing and Decreasing Functions

It is very useful to know where the graph of a function rises and where it falls. The graph shown in Figure 4 rises, falls, then rises again as we move from left to right: It rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be *increasing* when its graph rises and *decreasing* when its graph falls.

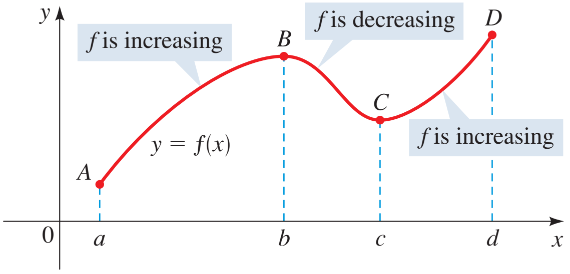


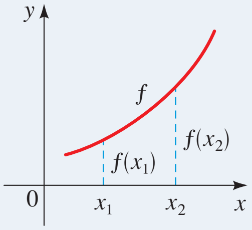
FIGURE 4 f is increasing on $[a, b]$ and $[c, d]$. f is decreasing on $[b, c]$.

We have the following definition.

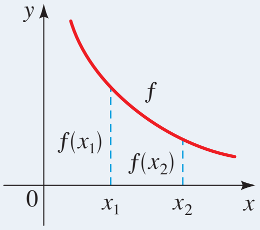
DEFINITION OF INCREASING AND DECREASING FUNCTIONS

f is **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.



f is increasing



f is decreasing

EXAMPLE 3 | Intervals on Which a Function Increases and Decreases

The graph in Figure 5 gives the weight W of a person at age x. Determine the intervals on which the function W is increasing and on which it is decreasing.

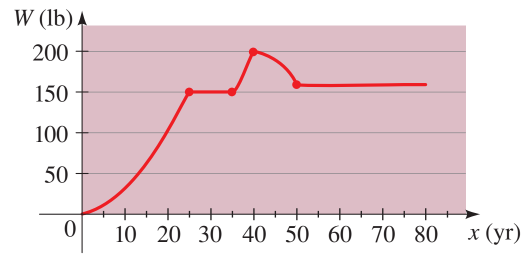


FIGURE 5 Weight as a function of age

SOLUTION The function W is increasing on $[0, 25]$ and $[35, 40]$. It is decreasing on $[40, 50]$. The function W is constant (neither increasing nor decreasing) on $[25, 30]$ and $[50, 80]$. This means that the person gained weight until age 25, then gained weight again between ages 35 and 40. He lost weight between ages 40 and 50.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **49**

EXAMPLE 4 | Finding Intervals Where a Function Increases and Decreases

(a) Sketch a graph of the function $f(x) = 12x^2 + 4x^3 - 3x^4$.

(b) Find the domain and range of f.

(c) Find the intervals on which f is increasing and on which f is decreasing.

SOLUTION

(a) We use a graphing calculator to sketch the graph in Figure 6.

(b) The domain of f is $\mathbb{R}$ because f is defined for all real numbers. Using the TRACE feature on the calculator, we find that the highest value is $f(2) = 32$. So the range of f is $(-\infty, 32]$.

(c) From the graph we see that f is increasing on the intervals $(-\infty, -1]$ and $[0, 2]$ and is decreasing on $[-1, 0]$ and $[2, \infty)$.

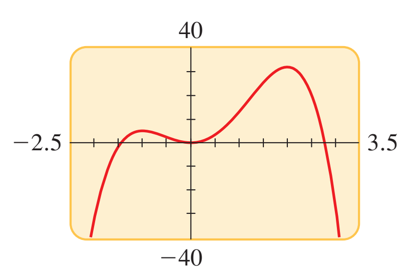


FIGURE 6 Graph of $f(x) = 12x^2 + 4x^3 - 3x^4$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **27**

EXAMPLE 5 | Finding Intervals Where a Function Increases and Decreases

(a) Sketch the graph of the function $f(x) = x^{2/3}$.

(b) Find the domain and range of the function.

(c) Find the intervals on which f is increasing and on which f is decreasing.

SOLUTION

(a) We use a graphing calculator to sketch the graph in Figure 7.

(b) From the graph we observe that the domain of f is $\mathbb{R}$ and the range is $[0, \infty)$.

(c) From the graph we see that f is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.

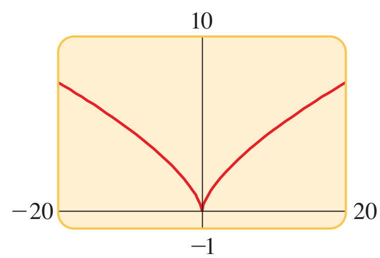


FIGURE 7 Graph of $f(x) = x^{2/3}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **33**

Local Maximum and Minimum Values of a Function

Finding the largest or smallest values of a function is important in many applications. For example, if a function represents revenue or profit, then we are interested in its maximum value. For a function that represents cost, we would want to find its minimum value. (See *Focus on Modeling: Modeling with Functions* on pages 247–256 for many such examples.) We can easily find these values from the graph of a function. We first define what we mean by a local maximum or minimum.

LOCAL MAXIMA AND MINIMA OF A FUNCTION

1. The function value $f(a)$ is a **local maximum value** of f if

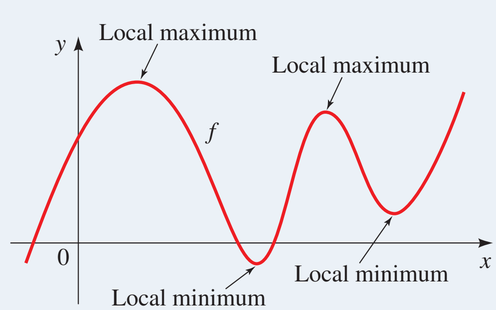
$$f(a) \geq f(x) \quad \text{when } x \text{ is near } a$$

(This means that $f(a) \geq f(x)$ for all x in some open interval containing a.) In this case we say that f has a **local maximum** at $x = a$.

2. The function value $f(a)$ is a **local minimum** of f if

$$f(a) \leq f(x) \quad \text{when } x \text{ is near } a$$

(This means that $f(a) \leq f(x)$ for all x in some open interval containing a.) In this case we say that f has a **local minimum** at $x = a$.



We can find the local maximum and minimum values of a function using a graphing calculator.

If there is a viewing rectangle such that the point $(a, f(a))$ is the highest point on the graph of f *within* the viewing rectangle (not on the edge), then the number $f(a)$ is a local maximum value of f (see Figure 8). Notice that $f(a) \geq f(x)$ for all numbers x that are close to a.

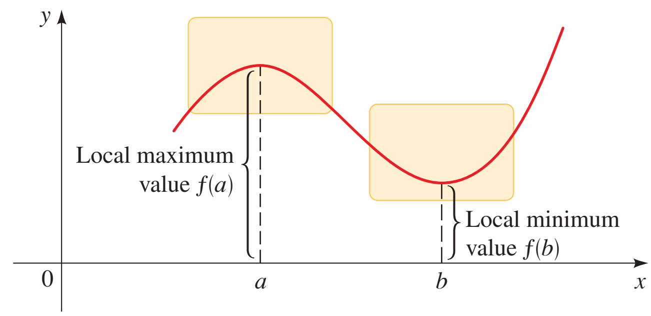


FIGURE 8

Similarly, if there is a viewing rectangle such that the point $(b, f(b))$ is the lowest point on the graph of f within the viewing rectangle, then the number $f(b)$ is a local minimum value of f. In this case $f(b) \leq f(x)$ for all numbers x that are close to b.

EXAMPLE 6 | Finding Local Maxima and Minima from a Graph

Find the local maximum and minimum values of the function $f(x) = x^3 - 8x + 1$, rounded to three decimal places.

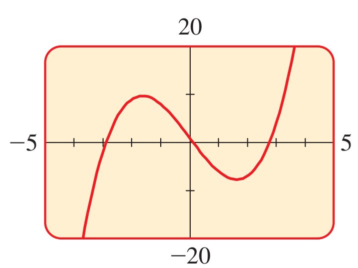


FIGURE 9 Graph of $f(x) = x^3 - 8x + 1$

SOLUTION The graph of f is shown in Figure 9. There appears to be one local maximum between $x = -2$ and $x = -1$, and one local minimum between $x = 1$ and $x = 2$.

Let's find the coordinates of the local maximum point first. We zoom in to enlarge the area near this point, as shown in Figure 10. Using the TRACE feature on the graphing device, we move the cursor along the curve and observe how the y-coordinates change. The local maximum value of y is 9.709, and this value occurs when x is -1.633, correct to three decimal places.

We locate the minimum value in a similar fashion. By zooming in to the viewing rectangle shown in Figure 11, we find that the local minimum value is about -7.709, and this value occurs when $x \approx 1.633$.

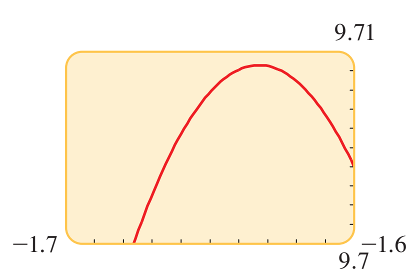


FIGURE 10

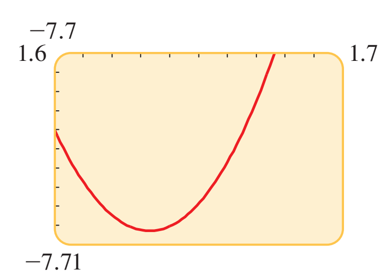


FIGURE 11

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **39** ■

The `maximum` and `minimum` commands on a TI-83 or TI-84 calculator provide another method for finding extreme values of functions. We use this method in the next example.

EXAMPLE 7 | A Model for the Food Price Index

A model for the food price index (the price of a representative "basket" of foods) between 1990 and 2000 is given by the function

$$I(t) = -0.0113t^3 + 0.0681t^2 + 0.198t + 99.1$$

where t is measured in years since midyear 1990, so $0 \le t \le 10$, and $I(t)$ is scaled so that $I(3) = 100$. Estimate the time when food was most expensive during the period 1990–2000.

SOLUTION The graph of I as a function of t is shown in Figure 12(a). There appears to be a maximum between $t = 4$ and $t = 7$. Using the `maximum` command, as shown in Figure 12(b), we see that the maximum value of I is about 100.38, and it occurs when $t \approx 5.15$, which corresponds to August 1995.

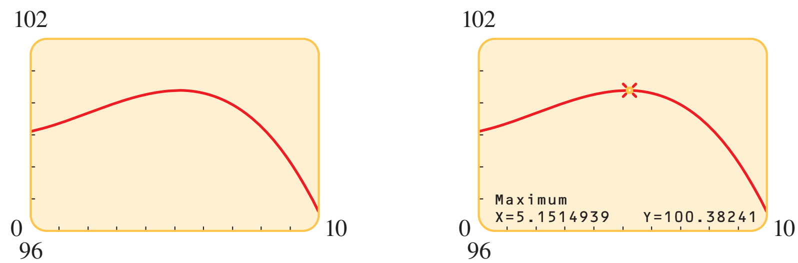


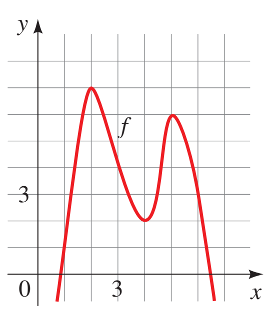
FIGURE 12 (a) (b)

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **57**

2.3 EXERCISES

CONCEPTS

1–4 ■ These exercises refer to the graph of the function f shown below.



1. To find a function value $f(a)$ from the graph of f, we find the height of the graph above the x-axis at $x =$ ________. From the graph of f we see that $f(3) =$ ________ and $f(1) =$ ____. The net change in f between $x = 1$ and $x = 3$ is $f($____$) - f($____$) =$ ____.

2. The domain of the function f is all the ________-values of the points on the graph, and the range is all the corresponding ________-values. From the graph of f we see that the domain of f is the interval ________ and the range of f is the interval ________.

3. (a) If f is increasing on an interval, then the y-values of the points on the graph ________ as the x-values increase. From the graph of f we see that f is increasing on the intervals ________ and ________.

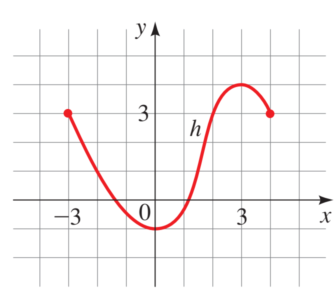
(b) If f is decreasing on an interval, then y-values of the points on the graph ________ as the x-values increase. From the graph of f we see that f is decreasing on the intervals ________ and ________.

4. (a) A function value $f(a)$ is a local maximum value of f if $f(a)$ is the ________ value of f on some open interval containing a. From the graph of f we see that a local maximum value of f is ________ and that this value occurs when x is ________.

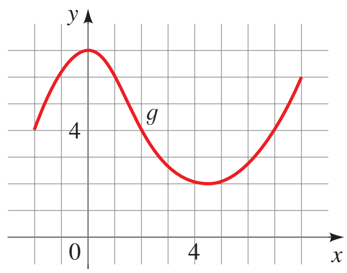
(b) The function value $f(a)$ is a local minimum value of f if $f(a)$ is the ________ value of f on some open interval containing a. From the graph of f we see that a local minimum value of f is ________ and that this value occurs when x is ________.

SKILLS

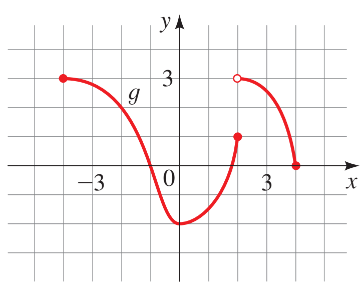
5. The graph of a function h is given.
(a) Find $h(-2)$, $h(0)$, $h(2)$, and $h(3)$.
(b) Find the domain and range of h.
(c) Find the values of x for which $h(x) = 3$.
(d) Find the values of x for which $h(x) \le 3$.
(e) Find the net change in h between $x = -3$ and $x = 3$.



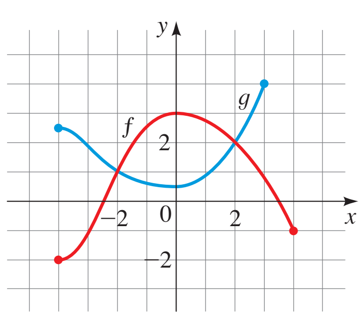
6. The graph of a function g is given.
  (a) Find $g(-2)$, $g(0)$, and $g(7)$.
  (b) Find the domain and range of g.
  (c) Find the values of x for which $g(x) = 4$.
  (d) Find the values of x for which $g(x) > 4$.
  (e) Find the net change in g between $x = 0$ and $x = 7$.



7. The graph of a function g is given.
  (a) Find $g(-4)$, $g(-2)$, $g(0)$, $g(2)$, and $g(4)$.
  (b) Find the domain and range of g.



8. Graphs of the functions f and g are given.
  (a) Which is larger, $f(0)$ or $g(0)$?
  (b) Which is larger, $f(-3)$ or $g(-3)$?
  (c) For which values of x is $f(x) = g(x)$?



9–22 ■ A function f is given. **(a)** Use a graphing calculator to draw the graph of f. **(b)** Find the domain and range of f from the graph.

9. $f(x) = x - 1$ 　 **10.** $f(x) = 2(x + 1)$

11. $f(x) = x - 1, \quad -2 \le x \le 2$

12. $f(x) = 2(x + 1), \quad -3 \le x \le 3$

13. $f(x) = 4, \quad 1 \le x \le 3$ 　 **14.** $f(x) = x^2, \quad -2 \le x \le 5$

15. $f(x) = 4 - x^2$ 　 **16.** $f(x) = x^2 + 4$

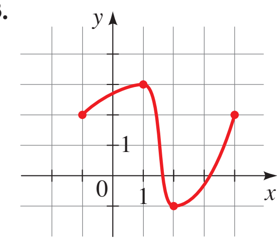
17. $f(x) = x^2 + 4x + 3$ 　 **18.** $f(x) = -x^2 + 2x + 1$

19. $f(x) = \sqrt{16 - x^2}$ 　 **20.** $f(x) = -\sqrt{25 - x^2}$

21. $f(x) = \sqrt{x - 1}$ 　 **22.** $f(x) = \sqrt{x + 2}$

23–26 ■ The graph of a function is given. **(a)** Find the domain and range of f. **(b)** Find the intervals on which f is increasing and on which f is decreasing.

23.



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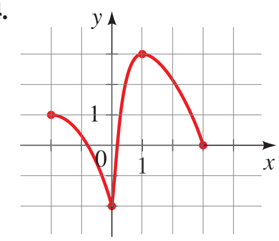

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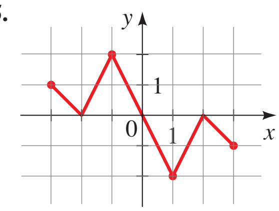

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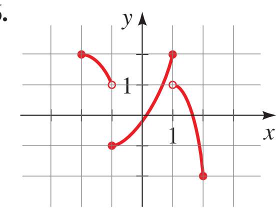


27–34 ■ A function f is given. **(a)** Use a graphing calculator to draw the graph of f. **(b)** Find the domain and range of f. **(c)** State approximately the intervals on which f is increasing and on which f is decreasing.

27. $f(x) = x^2 - 5x$ 　 **28.** $f(x) = x^3 - 4x$

29. $f(x) = 2x^3 - 3x^2 - 12x$ 　 **30.** $f(x) = x^4 - 16x^2$

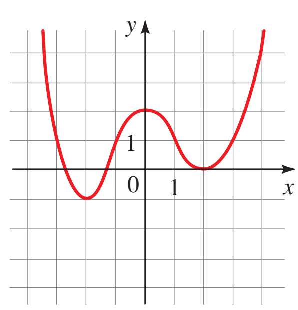
31. $f(x) = x^3 + 2x^2 - x - 2$

32. $f(x) = x^4 - 4x^3 + 2x^2 + 4x - 3$

33. $f(x) = x^{2/5}$ 　 **34.** $f(x) = 4 - x^{2/3}$

35–38 ■ The graph of a function is given. **(a)** Find all the local maximum and minimum values of the function and the value of x at which each occurs. **(b)** Find the intervals on which the function is increasing and on which the function is decreasing.

35.



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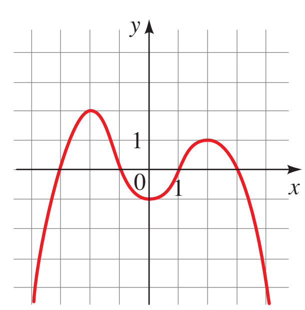

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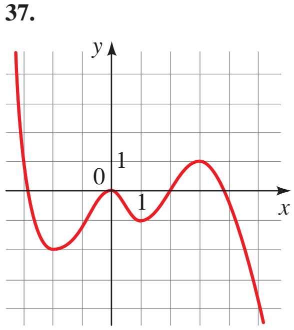

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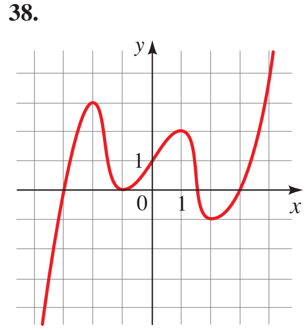


39–46 ■ A function is given. **(a)** Find all the local maximum and minimum values of the function and the value of x at which each occurs. State each answer rounded to two decimal places. **(b)** Find the intervals on which the function is increasing and on which the function is decreasing. State each answer rounded to two decimal places.

39. $f(x) = x^3 - x$

40. $f(x) = 3 + x + x^2 - x^3$

41. $g(x) = x^4 - 2x^3 - 11x^2$

42. $g(x) = x^5 - 8x^3 + 20x$

43. $U(x) = x\sqrt{6 - x}$

44. $U(x) = x\sqrt{x - x^2}$

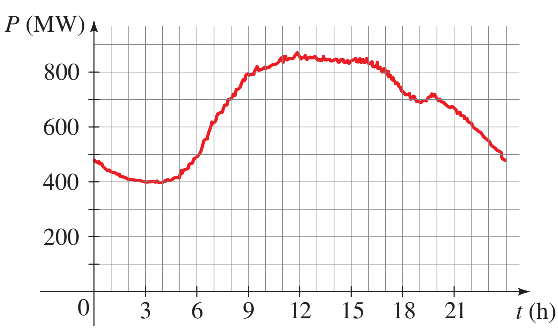
45. $V(x) = \dfrac{1 - x^2}{x^3}$

46. $V(x) = \dfrac{1}{x^2 + x + 1}$

APPLICATIONS

47. Power Consumption The figure shows the power consumption in San Francisco for a day in September (P is measured in megawatts; t is measured in hours starting at midnight).

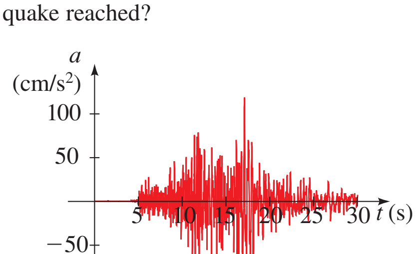
(a) What was the power consumption at 6:00 A.M.? At 6:00 P.M.?
(b) When was the power consumption the lowest?
(c) When was the power consumption the highest?
(d) Find the net change in the power consumption from 9:00 A.M. to 7:00 P.M.



Source: Pacific Gas & Electric

48. Earthquake The graph shows the vertical acceleration of the ground from the 1994 Northridge earthquake in Los Angeles, as measured by a seismograph. (Here t represents the time in seconds.)

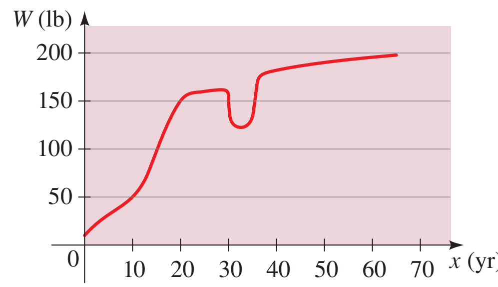
(a) At what time t did the earthquake first make noticeable movements of the earth?
(b) At what time t did the earthquake seem to end?
(c) At what time t was the maximum intensity of the earthquake reached?



Source: California Department of Mines and Geology

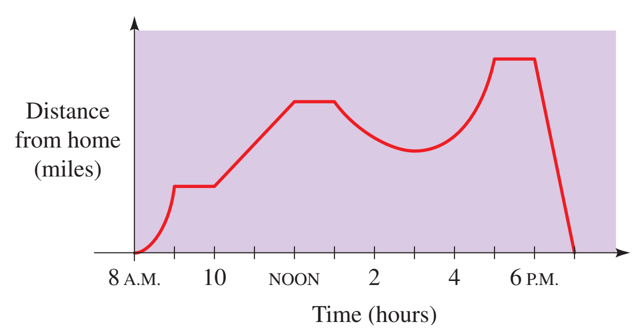
49. Weight Function The graph gives the weight W of a person at age x.

(a) Determine the intervals on which the function W is increasing and those on which it is decreasing.
(b) What do you think happened when this person was 30 years old?
(c) Find the net change in the person's weight W from age 10 to age 20.



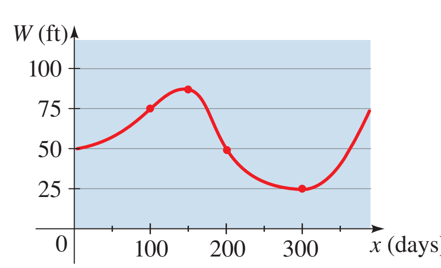
50. Distance Function The graph gives a sales representative's distance from his home as a function of time on a certain day.

(a) Determine the time intervals on which his distance from home was increasing and those on which it was decreasing.
(b) Describe in words what the graph indicates about his travels on this day.
(c) Find the net change in his distance from home between noon and 1:00 P.M.



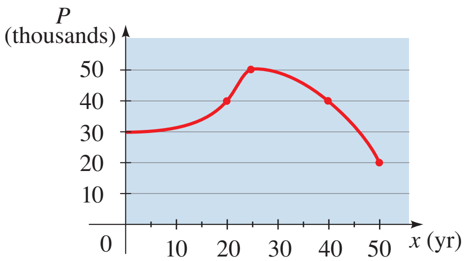
51. Changing Water Levels The graph shows the depth of water W in a reservoir over a one-year period as a function of the number of days x since the beginning of the year.

(a) Determine the intervals on which the function W is increasing and on which it is decreasing.
(b) At what value of x does W achieve a local maximum? A local minimum?
(c) Find the net change in the depth W from 100 days to 300 days.

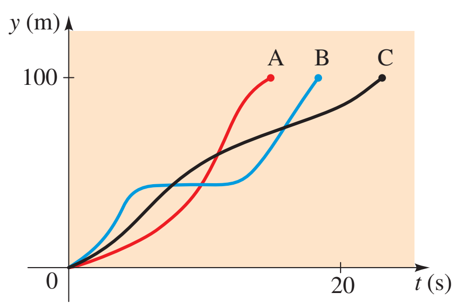


52. Population Growth and Decline The graph shows the population P in a small industrial city from 1950 to 2000. The variable x represents the number of years since 1950.

(a) Determine the intervals on which the function P is increasing and on which it is decreasing.
(b) What was the maximum population, and in what year was it attained?
(c) Find the net change in the population P from 1970 to 1990.



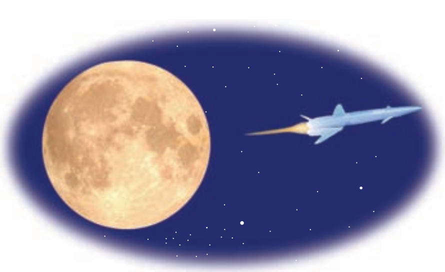
53. Hurdle Race Three runners compete in a 100-meter hurdle race. The graph depicts the distance run as a function of time for each runner. Describe in words what the graph tells you about this race. Who won the race? Did each runner finish the race? What do you think happened to runner B?



54. Gravity Near the Moon We can use Newton's Law of Gravity to measure the gravitational attraction between the moon and an algebra student in a space ship located a distance x above the moon's surface:

$$F(x) = \frac{350}{x^2}$$

Here F is measured in newtons (N), and x is measured in millions of meters.
(a) Graph the function F for values of x between 0 and 10.
(b) Use the graph to describe the behavior of the gravitational attraction F as the distance x increases.



55. Radii of Stars Astronomers infer the radii of stars using the Stefan Boltzmann Law:

$$E(T) = (5.67 \times 10^{-8})T^4$$

where E is the energy radiated per unit of surface area measured in watts (W) and T is the absolute temperature measured in kelvins (K).

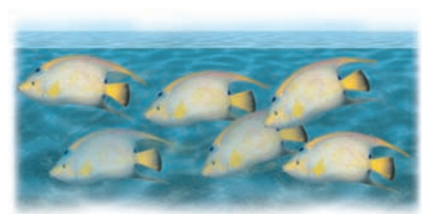
(a) Graph the function E for temperatures T between 100 K and 300 K.
(b) Use the graph to describe the change in energy E as the temperature T increases.

56. Migrating Fish A fish swims at a speed v relative to the water, against a current of 5 mi/h. Using a mathematical model of energy expenditure, it can be shown that the total energy E required to swim a distance of 10 mi is given by

$$E(v) = 2.73v^3\frac{10}{v - 5}$$

Biologists believe that migrating fish try to minimize the total energy required to swim a fixed distance. Find the value of v that minimizes energy required.

NOTE: This result has been verified; migrating fish swim against a current at a speed 50% greater than the speed of the current.



57. Highway Engineering A highway engineer wants to estimate the maximum number of cars that can safely travel a particular highway at a given speed. She assumes that each car is 17 ft long, travels at a speed s, and follows the car in front of it at the "safe following distance" for that speed. She finds that the number N of cars that can pass a given point per minute is modeled by the function

$$N(s) = \frac{88s}{17 + 17\left(\frac{s}{20}\right)^2}$$

At what speed can the greatest number of cars travel the highway safely?

58. Volume of Water Between 0°C and 30°C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

Find the temperature at which the volume of 1 kg of water is a minimum.

59. Coughing When a foreign object that is lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward, causing an increase in pressure in the lungs. At the same time, the trachea contracts, causing the expelled air to move faster and increasing the pressure on the foreign object. According to a mathematical model of coughing, the velocity v (in cm/s) of the airstream through an average-sized person's trachea is related to the radius r of the trachea (in cm) by the function

$$v(r) = 3.2(1 - r)r^2 \qquad \tfrac{1}{2} \le r \le 1$$

Determine the value of r for which v is a maximum.

DISCOVERY ▪ DISCUSSION ▪ WRITING

60. Functions That Are Always Increasing or Decreasing Sketch rough graphs of functions that are defined for all real numbers and that exhibit the indicated behavior (or explain why the behavior is impossible).

(a) f is always increasing, and $f(x) > 0$ for all x
(b) f is always decreasing, and $f(x) > 0$ for all x
(c) f is always increasing, and $f(x) < 0$ for all x
(d) f is always decreasing, and $f(x) < 0$ for all x

61. Maxima and Minima In Example 7 we saw a real-world situation in which the maximum value of a function is important. Name several other everyday situations in which a maximum or minimum value is important.

62. Minimizing a Distance When we seek a minimum or maximum value of a function, it is sometimes easier to work with a simpler function instead.

(a) Suppose

$$g(x) = \sqrt{f(x)}$$

where $f(x) \geq 0$ for all x. Explain why the local minima and maxima of f and g occur at the same values of x.

(b) Let $g(x)$ be the distance between the point $(3, 0)$ and the point (x, x^2) on the graph of the parabola $y = x^2$. Express g as a function of x.

(c) Find the minimum value of the function g that you found in part (b). Use the principle described in part (a) to simplify your work.

2.4 AVERAGE RATE OF CHANGE OF A FUNCTION

LEARNING OBJECTIVES After completing this section, you will be able to:

Find average rates of change ▶ Interpret average rates of change in real-world situations ▶ Recognize that a function with constant average rate of change is linear

Functions are often used to model changing quantities. In this section we learn how to find the rate at which the values of a function change as the input variable changes.

▼ Average Rate of Change

We are all familiar with the concept of speed: If you drive a distance of 120 miles in 2 hours, then your average speed, or rate of travel, is $\frac{120 \text{ mi}}{2 \text{ h}} = 60$ mi/h. Now suppose you take a car trip and record the distance that you travel every few minutes. The distance s you have traveled is a function of the time t:

$$s(t) = \text{total distance traveled at time } t$$

We graph the function s as shown in Figure 1. The graph shows that you have traveled a total of 50 miles after 1 hour, 75 miles after 2 hours, 140 miles after 3 hours, and so on. To find your *average* speed between any two points on the trip, we divide the distance traveled by the time elapsed.

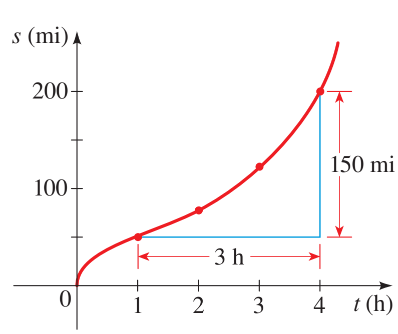


FIGURE 1
Average speed

Let's calculate your average speed between 1:00 P.M. and 4:00 P.M. The time elapsed is $4 - 1 = 3$ hours. To find the distance you traveled, we subtract the distance at 1:00 P.M. from the distance at 4:00 P.M., that is, $200 - 50 = 150$ mi. Thus your average speed is

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{150 \text{ mi}}{3 \text{ h}} = 50 \text{ mi/h}$$

The average speed that we have just calculated can be expressed by using function notation:

$$\text{average speed} = \frac{s(4) - s(1)}{4 - 1} = \frac{200 - 50}{3} = 50 \text{ mi/h}$$

Note that the average speed is different over different time intervals. For example, between 2:00 P.M. and 3:00 P.M. we find that

$$\text{average speed} = \frac{s(3) - s(2)}{3 - 2} = \frac{140 - 75}{1} = 65 \text{ mi/h}$$

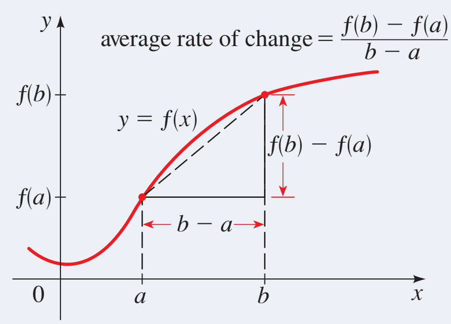
Finding average rates of change is important in many contexts. For instance, we might be interested in knowing how quickly the air temperature is dropping as a storm approaches or how fast revenues are increasing from the sale of a new product. So we need to know how to determine the average rate of change of the functions that model these quantities. In fact, the concept of average rate of change can be defined for any function.

AVERAGE RATE OF CHANGE

The **average rate of change** of the function $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

The average rate of change is the slope of the **secant line** between $x = a$ and $x = b$ on the graph of f, that is, the line that passes through $(a, f(a))$ and $(b, f(b))$.



In the expression for average rate of change, the numerator $f(b) - f(a)$ is the net change in the value of f between $x = a$ and $x = b$ (see page 177).

EXAMPLE 1 | Calculating the Average Rate of Change

For the function $f(x) = (x - 3)^2$, whose graph is shown in Figure 2, find the net change and the average rate of change between the following points:

(a) $x = 1$ and $x = 3$ **(b)** $x = 4$ and $x = 7$

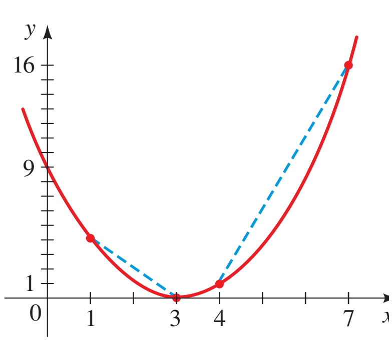


FIGURE 2 $f(x) = (x - 3)^2$

SOLUTION

(a) Net change $= f(3) - f(1)$ — Definition

$= (3 - 3)^2 - (1 - 3)^2$ — Use $f(x) = (x - 3)^2$

$= -4$ — Calculate

Average rate of change $= \dfrac{f(3) - f(1)}{3 - 1}$ — Definition

$= \dfrac{-4}{2} = -2$ — Calculate

(b) $$\begin{aligned} \text{Net change} &= f(7) - f(4) && \text{Definition} \\ &= (7-3)^2 - (4-3)^2 && \text{Use } f(x) = (x-3)^2 \\ &= 15 && \text{Calculate} \end{aligned}$$

$$\begin{aligned} \text{Average rate of change} &= \frac{f(7) - f(4)}{7-4} && \text{Definition} \\ &= \frac{15}{3} = 5 && \text{Calculate} \end{aligned}$$

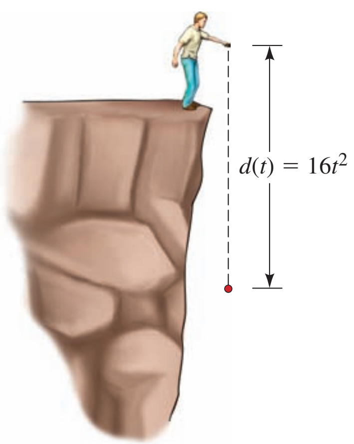
PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **13**

EXAMPLE 2 | Average Speed of a Falling Object

If an object is dropped from a high cliff or a tall building, then the distance it has fallen after t seconds is given by the function $d(t) = 16t^2$. Find its average speed (average rate of change) over the following intervals:

(a) Between 1 s and 5 s **(b)** Between $t = a$ and $t = a + h$

SOLUTION



Function: In t seconds the stone falls $16t^2$ ft.

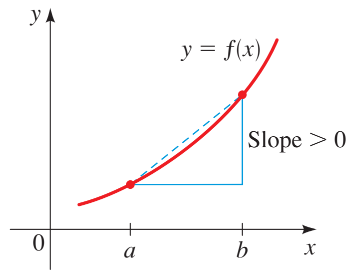
(a) $$\begin{aligned} \text{Average rate of change} &= \frac{d(5) - d(1)}{5-1} && \text{Definition} \\ &= \frac{16(5)^2 - 16(1)^2}{5-1} && \text{Use } d(t) = 16t^2 \\ &= \frac{400-16}{4} && \text{Calculate} \\ &= 96 \text{ ft/s} && \text{Calculate} \end{aligned}$$

(b) $$\begin{aligned} \text{Average rate of change} &= \frac{d(a+h) - d(a)}{(a+h) - a} && \text{Definition} \\ &= \frac{16(a+h)^2 - 16(a)^2}{(a+h) - a} && \text{Use } d(t) = 16t^2 \\ &= \frac{16(a^2 + 2ah + h^2 - a^2)}{h} && \text{Expand and factor 16} \\ &= \frac{16(2ah + h^2)}{h} && \text{Simplify numerator} \\ &= \frac{16h(2a+h)}{h} && \text{Factor } h \\ &= 16(2a + h) && \text{Simplify} \end{aligned}$$

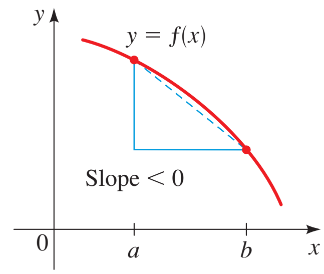
PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17**

The average rate of change calculated in Example 2(b) is known as a *difference quotient*. In calculus we use difference quotients to calculate *instantaneous* rates of change. An example of an instantaneous rate of change is the speed shown on the speedometer of your car. This changes from one instant to the next as your car's speed changes.

The graphs in Figure 3 show that if a function is increasing on an interval, then the average rate of change between any two points is positive, whereas if a function is decreasing on an interval, then the average rate of change between any two points is negative.



f increasing
Average rate of change positive



f decreasing
Average rate of change negative

FIGURE 3

EXAMPLE 3 | Average Rate of Temperature Change

Time	Temperature (°F)
8:00 A.M.	38
9:00 A.M.	40
10:00 A.M.	44
11:00 A.M.	50
12:00 NOON	56
1:00 P.M.	62
2:00 P.M.	66
3:00 P.M.	67
4:00 P.M.	64
5:00 P.M.	58
6:00 P.M.	55
7:00 P.M.	51

The table in the margin gives the outdoor temperatures observed by a science student on a spring day. Draw a graph of the data, and find the average rate of change of temperature between the following times:

(a) 8:00 A.M. and 9:00 A.M.

(b) 1:00 P.M. and 3:00 P.M.

(c) 4:00 P.M. and 7:00 P.M.

SOLUTION A graph of the temperature data is shown in Figure 4. Let t represent time, measured in hours since midnight (so, for example, 2:00 P.M. corresponds to $t = 14$). Define the function F by

$$F(t) = \text{temperature at time } t$$

Temperature at 9:00 A.M. Temperature at 8:00 A.M.

(a) $$\text{Average rate of change} = \frac{F(9) - F(8)}{9 - 8} = \frac{40 - 38}{9 - 8} = 2$$

The average rate of change was 2°F per hour.

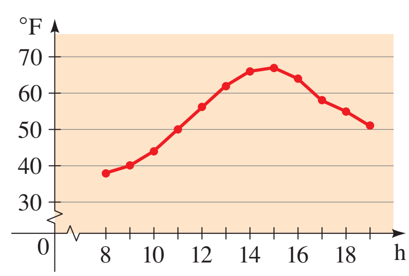


FIGURE 4

(b) $$\text{Average rate of change} = \frac{F(15) - F(13)}{15 - 13} = \frac{67 - 62}{2} = 2.5$$

The average rate of change was 2.5°F per hour.

(c) $$\text{Average rate of change} = \frac{F(19) - F(16)}{19 - 16} = \frac{51 - 64}{3} \approx -4.3$$

The average rate of change was about −4.3°F per hour during this time interval. The negative sign indicates that the temperature was dropping.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **27**

Linear Functions Have Constant Rate of Change

For a linear function $f(x) = mx + b$ the average rate of change between any two points is the same constant m. This agrees with what we learned in Section 1.3: that the slope of a line $y = mx + b$ is the average rate of change of y with respect to x. On the other hand, if a function f has constant average rate of change, then it must be a linear function. You are asked to prove this fact in Exercise 35. In the next example we find the average rate of change for a particular linear function.

EXAMPLE 4 | Linear Functions Have Constant Rate of Change

Let $f(x) = 3x - 5$. Find the average rate of change of f between the following points.

(a) $x = 0$ and $x = 1$

(b) $x = 3$ and $x = 7$

(c) $x = a$ and $x = a + h$

What conclusion can you draw from your answers?

SOLUTION

(a) $$\text{Average rate of change} = \frac{f(1) - f(0)}{1 - 0} = \frac{(3 \cdot 1 - 5) - (3 \cdot 0 - 5)}{1}$$
$$= \frac{(-2) - (-5)}{1} = 3$$

(b) $$\text{Average rate of change} = \frac{f(7) - f(3)}{7 - 3} = \frac{(3 \cdot 7 - 5) - (3 \cdot 3 - 5)}{4}$$
$$= \frac{16 - 4}{4} = 3$$

(c) $$\text{Average rate of change} = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{[3(a + h) - 5] - [3a - 5]}{h}$$
$$= \frac{3a + 3h - 5 - 3a + 5}{h} = \frac{3h}{h} = 3$$

It appears that the average rate of change is always 3 for this function. In fact, part (c) proves that the rate of change between any two arbitrary points $x = a$ and $x = a + h$ is 3.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **23**

2.4 EXERCISES

CONCEPTS

1. If you travel 100 miles in two hours, then your average speed for the trip is

$$\text{average speed} = \frac{\boxed{}}{\boxed{}} = \underline{\qquad}$$

2. The average rate of change of a function f between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\boxed{}}{\boxed{}}$$

3. The average rate of change of the function $f(x) = x^2$ between $x = 1$ and $x = 5$ is

$$\text{average rate of change} = \frac{\boxed{}}{\boxed{}} = \underline{\qquad}$$

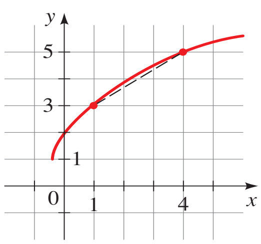
4. (a) The average rate of change of a function f between $x = a$ and $x = b$ is the slope of the ________ line between $(a, f(a))$ and $(b, f(b))$.

(b) The average rate of change of the linear function $f(x) = 3x + 5$ between any two points is ________.

SKILLS

5–8 ■ The graph of a function is given. Determine **(a)** the net change and **(b)** the average rate of change of the function between the indicated points on the graph.

5.



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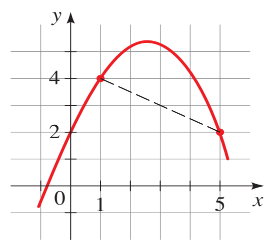

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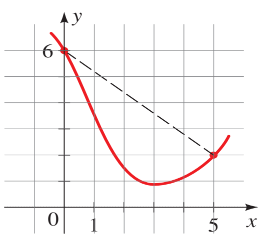

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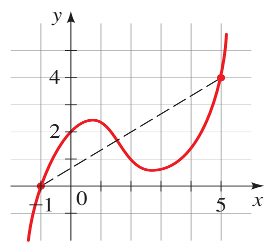


9–22 ■ A function is given. Determine **(a)** the net change and **(b)** the average rate of change of the function between the given values of the variable.

9. $f(x) = 3x - 2; \quad x = 2, x = 3$

10. $r(t) = 3 - \frac{1}{3}t; \quad t = 3, t = 6$

11. $h(t) = -t + \frac{3}{2}; \quad t = -4, t = 1$

12. $g(x) = 5 + \frac{1}{2}x; \quad x = 1, x = 5$

13. $h(t) = t^2 + 2t; \quad t = -1, t = 4$

14. $f(z) = 1 - 3z^2; \quad z = -2, z = 0$

15. $f(x) = x^3 - 4x^2; \quad x = 0, x = 10$

16. $f(x) = x + x^4; \quad x = -1, x = 3$

17. $f(x) = 3x^2; \quad x = 2, x = 2 + h$

18. $f(x) = 4 - x^2; \quad x = 1, x = 1 + h$

19. $g(x) = \dfrac{1}{x}; \quad x = 1, x = a$

20. $g(x) = \dfrac{2}{x + 1}; \quad x = 0, x = h$

21. $f(t) = \dfrac{2}{t}; \quad t = a, t = a + h$

22. $f(t) = \sqrt{t}; \quad t = a, t = a + h$

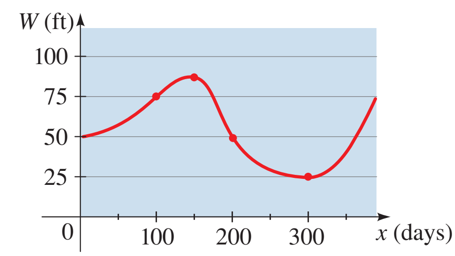
23–24 ■ A linear function is given. **(a)** Find the average rate of change of the function between $x = a$ and $x = a + h$. **(b)** Show that the average rate of change is the same as the slope of the line.

23. $f(x) = \frac{1}{2}x + 3$

24. $g(x) = -4x + 2$

APPLICATIONS

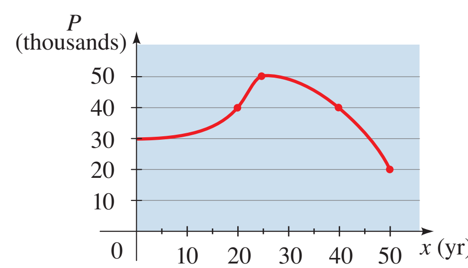
25. Changing Water Levels The graph shows the depth of water W in a reservoir over a one-year period as a function of the number of days x since the beginning of the year. What was the average rate of change of W between $x = 100$ and $x = 200$?



26. Population Growth and Decline The graph shows the population P in a small industrial city from 1950 to 2000. The variable x represents the number of years since 1950.

(a) What was the average rate of change of P between $x = 20$ and $x = 40$?

(b) Interpret the value of the average rate of change that you found in part (a).



27. Population Growth and Decline The table gives the population in a small coastal community for the period 1997–2006. Figures shown are for January 1 in each year.

(a) What was the average rate of change of population between 1998 and 2001?

(b) What was the average rate of change of population between 2002 and 2004?

(c) For what period of time was the population increasing?

(d) For what period of time was the population decreasing?

Year	Population
1997	624
1998	856
1999	1,336
2000	1,578
2001	1,591
2002	1,483
2003	994
2004	826
2005	801
2006	745

28. Running Speed A man is running around a circular track that is 200 m in circumference. An observer uses a stopwatch to record the runner's time at the end of each lap, obtaining the data in the following table.

(a) What was the man's average speed (rate) between 68 s and 152 s?

(b) What was the man's average speed between 263 s and 412 s?

(c) Calculate the man's speed for each lap. Is he slowing down, speeding up, or neither?

Time (s)	Distance (m)
32	200
68	400
108	600
152	800
203	1000
263	1200
335	1400
412	1600

29. CD Player Sales The table shows the number of CD players sold in a small electronics store in the years 1993–2003.

(a) What was the average rate of change of sales between 1993 and 2003?

(b) What was the average rate of change of sales between 1993 and 1994?

(c) What was the average rate of change of sales between 1994 and 1996?

(d) Between which two successive years did CD player sales *increase* most quickly? *Decrease* most quickly?

Year	CD players sold
1993	512
1994	520
1995	413
1996	410
1997	468
1998	510
1999	590
2000	607
2001	732
2002	612
2003	584

30. Book Collection Between 1980 and 2000 a rare book collector purchased books for his collection at the rate of 40 books per year. Use this information to complete the following table. (Note that not every year is given in the table.)

Year	Number of books
1980	420
1981	460
1982	
1985	
1990	
1992	
1995	
1997	
1998	
1999	
2000	1220

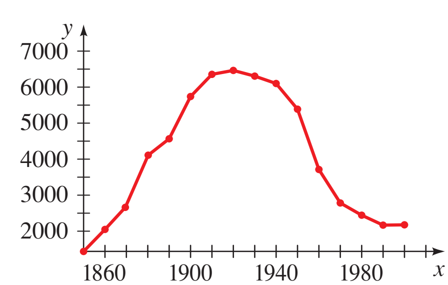
31. Cooling Soup When a bowl of hot soup is left in a room, the soup eventually cools down to room temperature. The temperature T of the soup is a function of time t. The table below gives the temperature (in °F) of a bowl of soup t minutes after it was set on the table. Find the average rate of change of the temperature of the soup over the first 20 minutes and over the next 20 minutes. During which interval did the soup cool off more quickly?

t (min)	T (°F)	t (min)	T (°F)
0	200	35	94
5	172	40	89
10	150	50	81
15	133	60	77
20	119	90	72
25	108	120	70
30	100	150	70

32. Farms in the United States The graph gives the number of farms in the United States from 1850 to 2000.

(a) Estimate the average rate of change in the number of farms between (i) 1860 and 1890 and (ii) 1950 and 1970.

(b) In which decade did the number of farms experience the greatest average rate of decline?

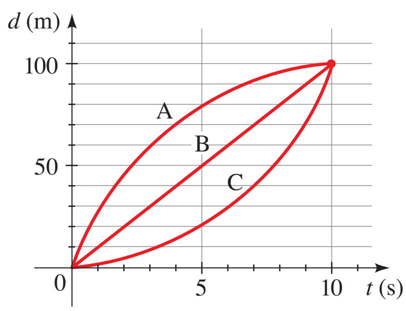


DISCOVERY ■ DISCUSSION ■ WRITING

33. 100-Meter Race A 100-m race ends in a three-way tie for first place. The graph at the top of the next column shows distance as a function of time for each of the three winners.

(a) Find the average speed for each winner.

(b) Describe the differences between the ways in which the three runners ran the race.



34. Linear Functions Have Constant Rate of Change If $f(x) = mx + b$ is a linear function, then the average rate of change of f between any two real numbers x_1 and x_2 is

$$\text{average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Calculate this average rate of change to show that it is the same as the slope m.

35. Functions with Constant Rate of Change Are Linear If the function f has the same average rate of change c between any two points, then for the points a and x we have

$$c = \frac{f(x) - f(a)}{x - a}$$

Rearrange this expression to show that

$$f(x) = cx + (f(a) - ca)$$

and conclude that f is a linear function.

2.5 Transformations of Functions

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Shift graphs vertically ▶ Shift graphs horizontally ▶ Stretch or shrink graphs vertically ▶ Stretch or shrink graphs horizontally ▶ Reflect graphs ▶ Determine whether a function is odd or even

In this section we study how certain transformations of a function affect its graph. This will give us a better understanding of how to graph functions. The transformations that we study are shifting, reflecting, and stretching.

▼ Vertical Shifting

Adding a constant to a function shifts its graph vertically: upward if the constant is positive and downward if it is negative.

In general, suppose we know the graph of $y = f(x)$. How do we obtain from it the graphs of

Recall that the graph of the function f is the same as the graph of the equation $y = f(x)$.

$$y = f(x) + c \qquad \text{and} \qquad y = f(x) - c \qquad (c > 0)$$

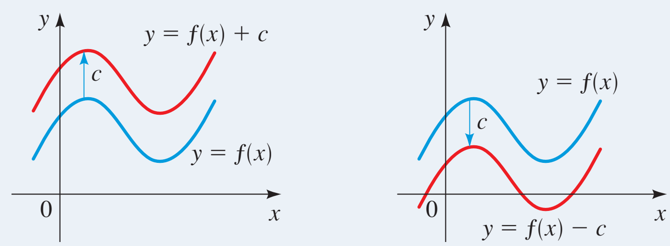
The y-coordinate of each point on the graph of $y = f(x) + c$ is c units above the y-coordinate of the corresponding point on the graph of $y = f(x)$. So we obtain the graph of $y = f(x) + c$ simply by shifting the graph of $y = f(x)$ upward c units. Similarly, we obtain the graph of $y = f(x) - c$ by shifting the graph of $y = f(x)$ downward c units.

VERTICAL SHIFTS OF GRAPHS

Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.

To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.



EXAMPLE 1 | Vertical Shifts of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = x^2 + 3$ **(b)** $h(x) = x^2 - 2$

SOLUTION The function $f(x) = x^2$ was graphed in Example 1(a), Section 2.2. It is sketched again in Figure 1.

(a) Observe that

$$g(x) = x^2 + 3 = f(x) + 3$$

So the y-coordinate of each point on the graph of g is 3 units above the corresponding point on the graph of f. This means that to graph g, we shift the graph of f upward 3 units, as in Figure 1.

(b) Similarly, to graph h, we shift the graph of f downward 2 units, as shown in Figure 1.

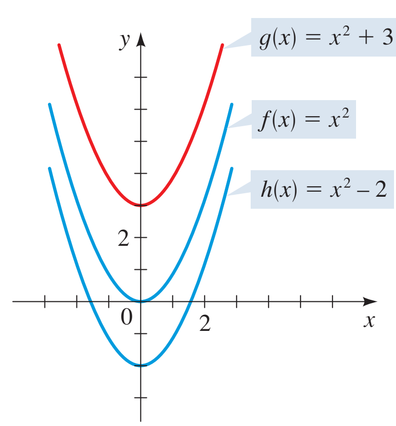


FIGURE 1

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **27** AND **29**

Horizontal Shifting

Suppose that we know the graph of $y = f(x)$. How do we use it to obtain the graphs of

$$y = f(x + c) \quad \text{and} \quad y = f(x - c) \quad (c > 0)$$

The value of $f(x - c)$ at x is the same as the value of $f(x)$ at $x - c$. Since $x - c$ is c units to the left of x, it follows that the graph of $y = f(x - c)$ is just the graph of $y = f(x)$

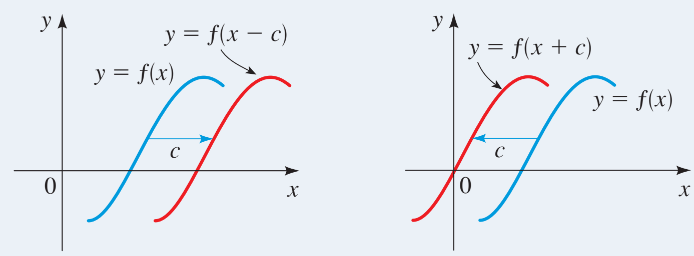
shifted to the right c units. Similar reasoning shows that the graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left c units. The following box summarizes these facts.

HORIZONTAL SHIFTS OF GRAPHS

Suppose $c > 0$.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.

To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.



EXAMPLE 2 | Horizontal Shifts of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = (x + 4)^2$ **(b)** $h(x) = (x - 2)^2$

SOLUTION

(a) To graph g, we shift the graph of f to the left 4 units.

(b) To graph h, we shift the graph of f to the right 2 units.

The graphs of g and h are sketched in Figure 2.

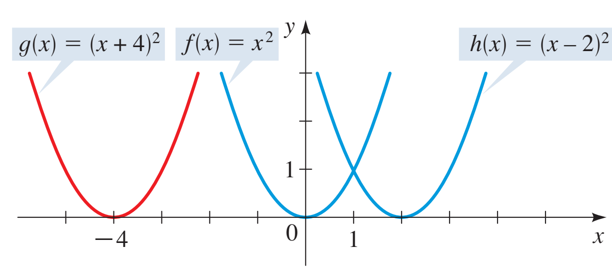


FIGURE 2

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **31** AND **33**

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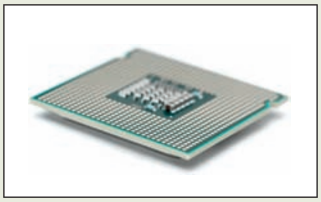
RENÉ DESCARTES (1596–1650) was born in the town of La Haye in southern France. From an early age Descartes liked mathematics because of "the certainty of its results and the clarity of its reasoning." He believed that to arrive at truth, one must begin by doubting everything, including one's own existence; this led him to formulate perhaps the best-known sentence in all of philosophy: "I think, therefore I am." In his book *Discourse on Method* he described what is now called the Cartesian plane. This idea of combining algebra and geometry enabled mathematicians for the first time to graph functions and thus "see" the equations they were studying. The philosopher John Stuart Mill called this invention "the greatest single step ever made in the progress of the exact sciences." Descartes liked to get up late and spend the morning in bed thinking and writing. He invented the coordinate plane while lying in bed watching a fly crawl on the ceiling, reasoning that he could describe the exact location of the fly by knowing its distance from two perpendicular walls. In 1649 Descartes became the tutor of Queen Christina of Sweden. She liked her lessons at 5 o'clock in the morning, when, she said, her mind was sharpest. However, the change from his usual habits and the ice-cold library where they studied proved too much for Descartes. In February 1650, after just two months of this, he caught pneumonia and died.

MATHEMATICS IN THE MODERN WORLD



Péter Gudella/Shutterstock.com

Computers

For centuries machines have been designed to perform specific tasks. For example, a washing machine washes clothes, a weaving machine weaves cloth, an adding machine adds numbers, and so on. The computer has changed all that.

The computer is a machine that does nothing—until it is given instructions on what to do. So your computer can play games, draw pictures, or calculate π to a million decimal places; it all depends on what program (or instructions) you give the computer. The computer can do all this because it is able to accept instructions and logically change those instructions based on incoming data. This versatility makes computers useful in nearly every aspect of human endeavor.

The idea of a computer was described theoretically in the 1940s by the mathematician Allan Turing (see page 105) in what he called a *universal machine*. In 1945 the mathematician John Von Neumann, extending Turing's ideas, built one of the first electronic computers.

Mathematicians continue to develop new theoretical bases for the design of computers. The heart of the computer is the "chip," which is capable of processing logical instructions. To get an idea of the chip's complexity, consider that the Pentium chip has over 3.5 million logic circuits!

EXAMPLE 3 | Combining Horizontal and Vertical Shifts

Sketch the graph of $f(x) = \sqrt{x - 3} + 4$.

SOLUTION We start with the graph of $y = \sqrt{x}$ (Example 1(c), Section 2.2) and shift it to the right 3 units to obtain the graph of $y = \sqrt{x - 3}$. Then we shift the resulting graph upward 4 units to obtain the graph of $f(x) = \sqrt{x - 3} + 4$ shown in Figure 3.

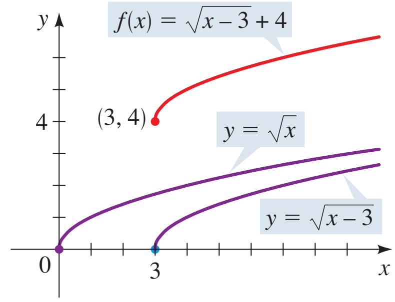


FIGURE 3

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **43**

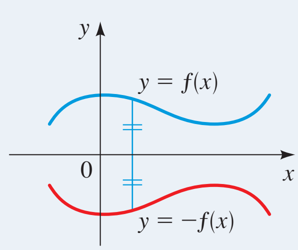
Reflecting Graphs

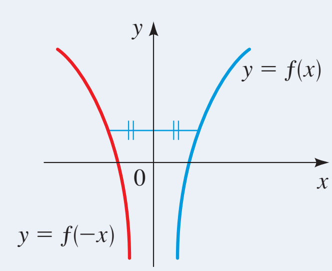
Suppose we know the graph of $y = f(x)$. How do we use it to obtain the graphs of $y = -f(x)$ and $y = f(-x)$? The y-coordinate of each point on the graph of $y = -f(x)$ is simply the negative of the y-coordinate of the corresponding point on the graph of $y = f(x)$. So the desired graph is the reflection of the graph of $y = f(x)$ in the x-axis. On the other hand, the value of $y = f(-x)$ at x is the same as the value of $y = f(x)$ at $-x$, so the desired graph here is the reflection of the graph of $y = f(x)$ in the y-axis. The following box summarizes these observations.

REFLECTING GRAPHS

To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.

To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.





EXAMPLE 4 | Reflecting Graphs

Sketch the graph of each function.

(a) $f(x) = -x^2$ **(b)** $g(x) = \sqrt{-x}$

SOLUTION

(a) We start with the graph of $y = x^2$. The graph of $f(x) = -x^2$ is the graph of $y = x^2$ reflected in the x-axis (see Figure 4).

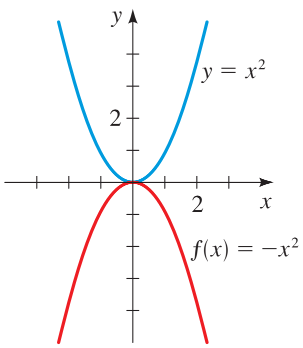


FIGURE 4

(b) We start with the graph of $y = \sqrt{x}$ (Example 1(c) in Section 2.2). The graph of $g(x) = \sqrt{-x}$ is the graph of $y = \sqrt{x}$ reflected in the y-axis (see Figure 5). Note that the domain of the function $g(x) = \sqrt{-x}$ is $\{x \mid x \leq 0\}$.

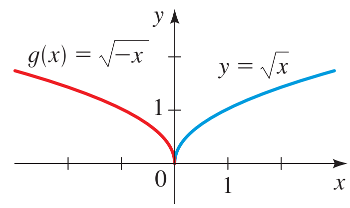


FIGURE 5

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **35** AND **37** ■

Vertical Stretching and Shrinking

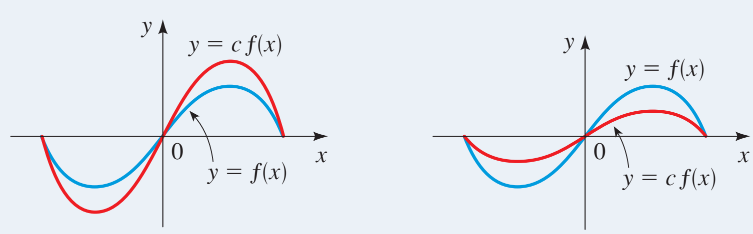
Suppose we know the graph of $y = f(x)$. How do we use it to obtain the graph of $y = cf(x)$? The y-coordinate of $y = cf(x)$ at x is the same as the corresponding y-coordinate of $y = f(x)$ multiplied by c. Multiplying the y-coordinates by c has the effect of vertically stretching or shrinking the graph by a factor of c.

VERTICAL STRETCHING AND SHRINKING OF GRAPHS

To graph $y = cf(x)$:

If $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.

If $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.



$c > 1$ $\qquad\qquad$ $0 < c < 1$

EXAMPLE 5 | Vertical Stretching and Shrinking of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = 3x^2$ $\qquad$ **(b)** $h(x) = \frac{1}{3}x^2$

SOLUTION

(a) The graph of g is obtained by multiplying the y-coordinate of each point on the graph of f by 3. That is, to obtain the graph of g, we stretch the graph of f vertically by a factor of 3. The result is the narrower parabola in Figure 6.

(b) The graph of h is obtained by multiplying the y-coordinate of each point on the graph of f by $\frac{1}{3}$. That is, to obtain the graph of h, we shrink the graph of f vertically by a factor of $\frac{1}{3}$. The result is the wider parabola in Figure 6.

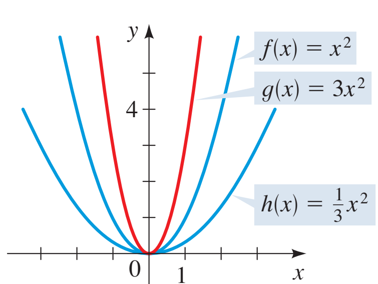


FIGURE 6

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **39** AND **41** ■

We illustrate the effect of combining shifts, reflections, and stretching in the following example.

EXAMPLE 6 | Combining Shifting, Stretching, and Reflecting

Sketch the graph of the function $f(x) = 1 - 2(x - 3)^2$.

SOLUTION Starting with the graph of $y = x^2$, we first shift to the right 3 units to get the graph of $y = (x - 3)^2$. Then we reflect in the x-axis and stretch by a factor of 2 to get the graph of $y = -2(x - 3)^2$. Finally, we shift upward 1 unit to get the graph of $f(x) = 1 - 2(x - 3)^2$ shown in Figure 7.

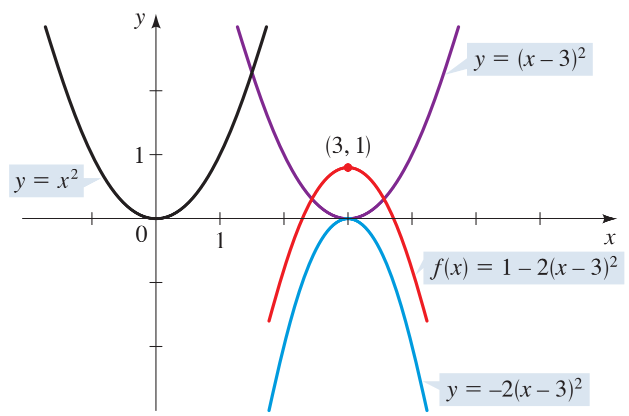


FIGURE 7

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45**

Horizontal Stretching and Shrinking

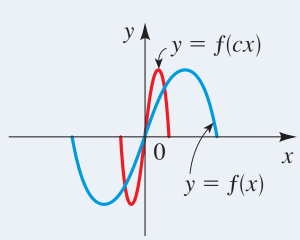
Now we consider horizontal shrinking and stretching of graphs. If we know the graph of $y = f(x)$, then how is the graph of $y = f(cx)$ related to it? The y-coordinate of $y = f(cx)$ at x is the same as the y-coordinate of $y = f(x)$ at cx. Thus the x-coordinates in the graph of $y = f(x)$ correspond to the x-coordinates in the graph of $y = f(cx)$ multiplied by c. Looking at this the other way around, we see that the x-coordinates in the graph of $y = f(cx)$ are the x-coordinates in the graph of $y = f(x)$ multiplied by $1/c$. In other words, to change the graph of $y = f(x)$ to the graph of $y = f(cx)$, we must shrink (or stretch) the graph horizontally by a factor of $1/c$, as summarized in the following box.

HORIZONTAL SHRINKING AND STRETCHING OF GRAPHS

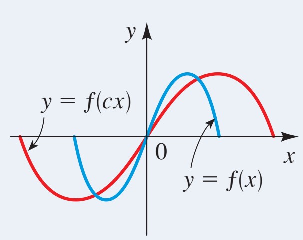
To graph $y = f(cx)$:

If $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/c$.

If $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $1/c$.



$c > 1$



$0 < c < 1$

EXAMPLE 7 | Horizontal Stretching and Shrinking of Graphs

The graph of $y = f(x)$ is shown in Figure 8 on the next page. Sketch the graph of each function.

(a) $y = f(2x)$ **(b)** $y = f(\frac{1}{2}x)$

SONYA KOVALEVSKY (1850–1891) is considered the most important woman mathematician of the 19th century. She was born in Moscow to an aristocratic family. While a child, she was exposed to the principles of calculus in a very unusual fashion: Her bedroom was temporarily wallpapered with the pages of a calculus book. She later wrote that she "spent many hours in front of that wall, trying to understand it." Since Russian law forbade women from studying in universities, she entered a marriage of convenience, which allowed her to travel to Germany and obtain a doctorate in mathematics from the University of Göttingen. She eventually was awarded a full professorship at the University of Stockholm, where she taught for eight years before dying in an influenza epidemic at the age of 41. Her research was instrumental in helping to put the ideas and applications of functions and calculus on a sound and logical foundation. She received many accolades and prizes for her research work.

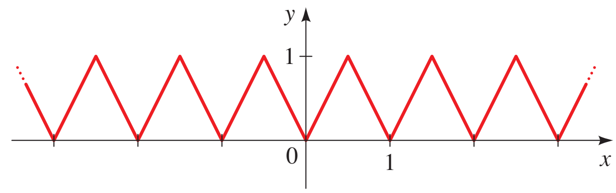


FIGURE 8 $y = f(x)$

SOLUTION Using the principles described in the preceding box, we obtain the graphs shown in Figures 9 and 10.

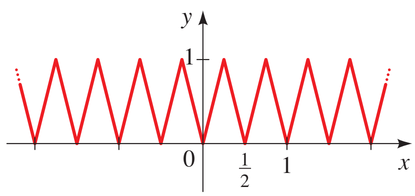


FIGURE 9 $y = f(2x)$

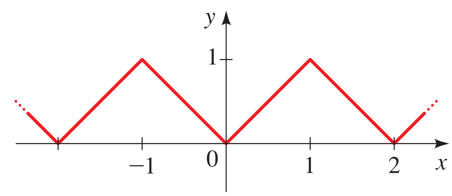


FIGURE 10 $y = f(\frac{1}{2}x)$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **69**

Even and Odd Functions

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = (-1)^2x^2 = x^2 = f(x)$$

The graph of an even function is symmetric with respect to the y-axis (see Figure 11). This means that if we have plotted the graph of f for $x \geq 0$, then we can obtain the entire graph simply by reflecting this portion in the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = (-1)^3x^3 = -x^3 = -f(x)$$

The graph of an odd function is symmetric about the origin (see Figure 12). If we have plotted the graph of f for $x \geq 0$, then we can obtain the entire graph by rotating this portion through 180° about the origin. (This is equivalent to reflecting first in the x-axis and then in the y-axis.)

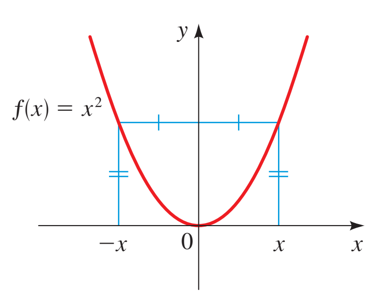


FIGURE 11 $f(x) = x^2$ is an even function.

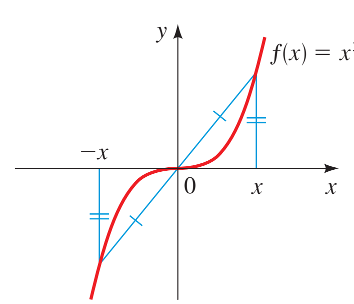


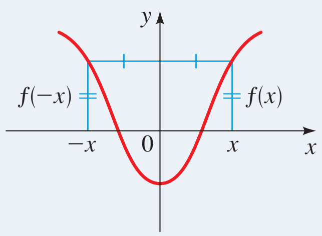
FIGURE 12 $f(x) = x^3$ is an odd function.

EVEN AND ODD FUNCTIONS

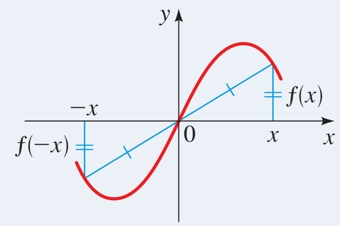
Let f be a function.

f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.



The graph of an even function is symmetric with respect to the y-axis.



The graph of an odd function is symmetric with respect to the origin.

EXAMPLE 8 | Even and Odd Functions

Determine whether the functions are even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$

(b) $g(x) = 1 - x^4$

(c) $h(x) = 2x - x^2$

SOLUTION

(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore, f is an odd function.

(b) $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

So g is even.

(c) $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **81**, **83**, AND **85**

The graphs of the functions in Example 8 are shown in Figure 13. The graph of f is symmetric about the origin, and the graph of g is symmetric about the y-axis. The graph of h is not symmetric either about the y-axis or the origin.

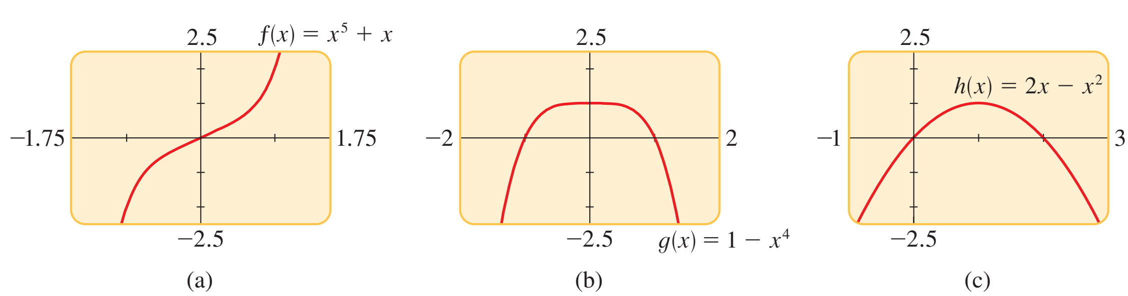


FIGURE 13

2.5 EXERCISES

CONCEPTS

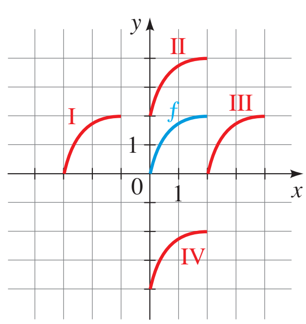
1–2 ■ Fill in the blank with the appropriate direction (left, right, up, or down).

1. (a) The graph of $y = f(x) + 3$ is obtained from the graph of $y = f(x)$ by shifting ________ 3 units.
(b) The graph of $y = f(x + 3)$ is obtained from the graph of $y = f(x)$ by shifting ________ 3 units.

2. (a) The graph of $y = f(x) - 3$ is obtained from the graph of $y = f(x)$ by shifting ________ 3 units.
(b) The graph of $y = f(x - 3)$ is obtained from the graph of $y = f(x)$ by shifting ________ 3 units.

3. Fill in the blank with the appropriate axis (x-axis or y-axis).
(a) The graph of $y = -f(x)$ is obtained from the graph of $y = f(x)$ by reflecting in the ________.
(b) The graph of $y = f(-x)$ is obtained from the graph of $y = f(x)$ by reflecting in the ________.

4. A graph of a function f is given. Match each equation with one of the graphs labeled I–IV.
(a) $f(x) + 2$ **(b)** $f(x + 3)$
(c) $f(x - 2)$ **(d)** $f(x) - 4$



SKILLS

5–16 ■ Suppose the graph of f is given. Describe how the graph of each function can be obtained from the graph of f.

5. (a) $y = f(x) + 3$ **(b)** $y = f(x + 3)$

6. (a) $y = f(x - 4)$ **(b)** $y = f(x) - 4$

7. (a) $y = -f(x)$ **(b)** $y = f(-x)$

8. (a) $y = -2f(x)$ **(b)** $y = -\frac{1}{2}f(x)$

9. (a) $y = f(x - 5) + 2$ **(b)** $y = f(x + 1) - 1$

10. (a) $y = f(x + 3) + 2$ **(b)** $y = f(x - 7) - 3$

11. (a) $y = -f(x) + 5$ **(b)** $y = 3f(x) - 5$

12. (a) $y = f(x - 4) + \frac{3}{4}$ **(b)** $y = f(x + 4) - \frac{3}{4}$

13. (a) $y = 2f(x + 1) - 3$ **(b)** $y = 2f(x - 1) + 3$

14. (a) $y = 3 - 2f(x)$ **(b)** $y = 2 - f(-x)$

15. (a) $y = f(4x)$ **(b)** $y = f(\frac{1}{4}x)$

16. (a) $y = f(2x) - 1$ **(b)** $y = 2f(\frac{1}{2}x)$

17–20 ■ Explain how the graph of g is obtained from the graph of f.

17. (a) $f(x) = x^2, \quad g(x) = (x + 2)^2$
(b) $f(x) = x^2, \quad g(x) = x^2 + 2$

18. (a) $f(x) = x^3, \quad g(x) = (x - 4)^3$
(b) $f(x) = x^3, \quad g(x) = x^3 - 4$

19. (a) $f(x) = |x|, \quad g(x) = |x + 2| - 2$
(b) $f(x) = |x|, \quad g(x) = |x - 2| + 2$

20. (a) $f(x) = \sqrt{x}, \quad g(x) = -\sqrt{x} + 1$
(b) $f(x) = \sqrt{x}, \quad g(x) = \sqrt{-x} + 1$

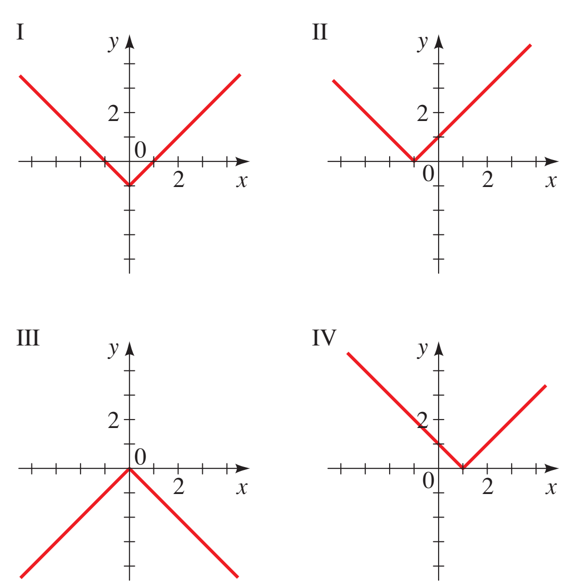
21. Use the graph of $y = x^2$ in Figure 4 to graph the following.
(a) $g(x) = x^2 + 1$ **(b)** $g(x) = (x - 1)^2$
(c) $g(x) = -x^2$ **(d)** $g(x) = (x - 1)^2 + 3$

22. Use the graph of $y = \sqrt{x}$ in Figure 5 to graph the following.
(a) $g(x) = \sqrt{x - 2}$ **(b)** $g(x) = \sqrt{x} + 1$
(c) $g(x) = \sqrt{x + 2} + 2$ **(d)** $g(x) = -\sqrt{x} + 1$

23–26 ■ Match the graph with the function. (See the graph of $y = |x|$ on page 191.)

23. $y = |x + 1|$ **24.** $y = |x - 1|$

25. $y = |x| - 1$ **26.** $y = -|x|$



27–50 ■ Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations.

27. $f(x) = x^2 - 1$ **28.** $f(x) = x^2 + 5$

29. $f(x) = \sqrt{x} + 1$ **30.** $f(x) = |x| - 1$

31. $f(x) = (x - 5)^2$ **32.** $f(x) = (x + 1)^2$

33. $f(x) = \sqrt{x+4}$

34. $f(x) = |x-3|$

35. $f(x) = -x^3$

36. $f(x) = -|x|$

37. $y = \sqrt[4]{-x}$

38. $y = \sqrt[3]{-x}$

39. $y = \frac{1}{4}x^2$

40. $y = -5\sqrt{x}$

41. $y = 3|x|$

42. $y = \frac{1}{2}|x|$

43. $y = (x-3)^2 + 5$

44. $y = \sqrt{x+4} - 3$

45. $y = 3 - \frac{1}{2}(x-1)^2$

46. $y = 2 - \sqrt{x+1}$

47. $y = |x+2| + 2$

48. $y = 2 - |x|$

49. $y = \frac{1}{2}\sqrt{x+4} - 3$

50. $y = 3 - 2(x-1)^2$

51–60 ■ A function f is given, and the indicated transformations are applied to its graph (in the given order). Write the equation for the final transformed graph.

51. $f(x) = x^2$; shift upward 3 units

52. $f(x) = x^3$; shift downward 1 unit

53. $f(x) = \sqrt{x}$; shift 2 units to the left

54. $f(x) = \sqrt[3]{x}$; shift 1 unit to the right

55. $f(x) = |x|$; shift 3 units to the right and shift upward 1 unit

56. $f(x) = |x|$; shift 4 units to the left and shift downward 2 units

57. $f(x) = \sqrt[4]{x}$; reflect in the y-axis and shift upward 1 unit

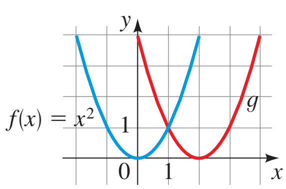
58. $f(x) = x^2$; shift 2 units to the left and reflect in the x-axis

59. $f(x) = x^2$; stretch vertically by a factor of 2, shift downward 2 units, and shift 3 units to the right

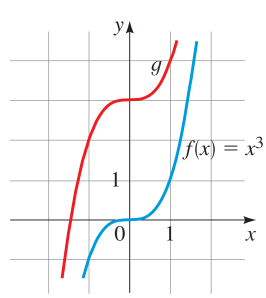
60. $f(x) = |x|$; shrink vertically by a factor of $\frac{1}{2}$, shift to the left 1 unit, and shift upward 3 units

61–66 ■ The graphs of f and g are given. Find a formula for the function g.

61.



62.



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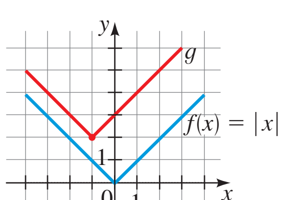

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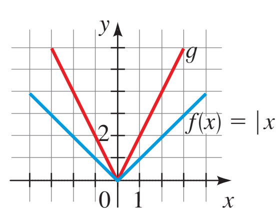

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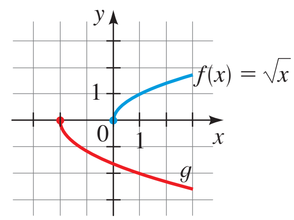

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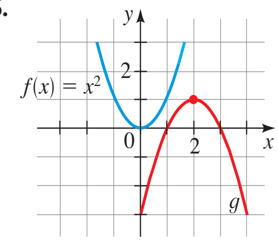
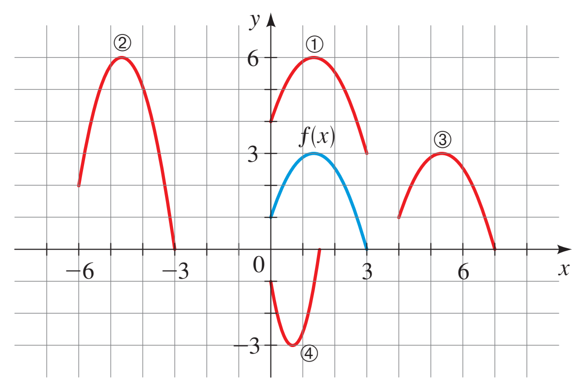
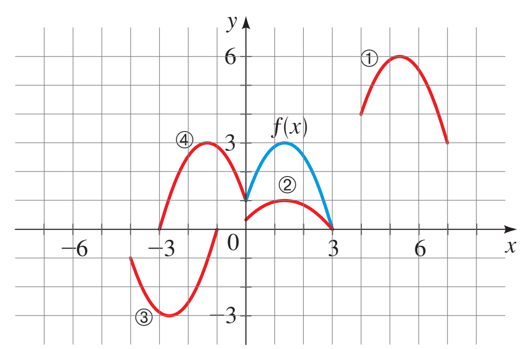


67–68 ■ The graph of $y = f(x)$ is given. Match each equation with its graph.

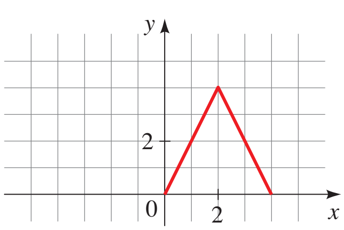
67. **(a)** $y = f(x-4)$ **(b)** $y = f(x) + 3$
(c) $y = 2f(x+6)$ **(d)** $y = -f(2x)$



68. **(a)** $y = \frac{1}{3}f(x)$ **(b)** $y = -f(x+4)$
(c) $y = f(x-4) + 3$ **(d)** $y = f(-x)$

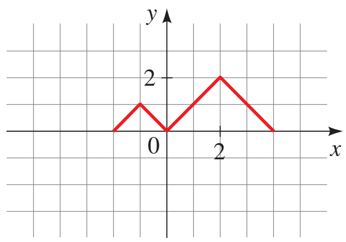


69. The graph of f is given. Sketch the graphs of the following functions.
(a) $y = f(x-2)$ **(b)** $y = f(x) - 2$
(c) $y = 2f(x)$ **(d)** $y = -f(x) + 3$
(e) $y = f(-x)$ **(f)** $y = \frac{1}{2}f(x-1)$



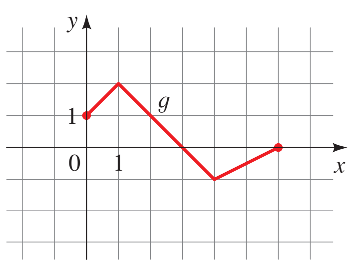
70. The graph of g is given. Sketch the graphs of the following functions.

(a) $y = g(x + 1)$ **(b)** $y = g(-x)$
(c) $y = g(x - 2)$ **(d)** $y = g(x) - 2$
(e) $y = -g(x)$ **(f)** $y = 2g(x)$



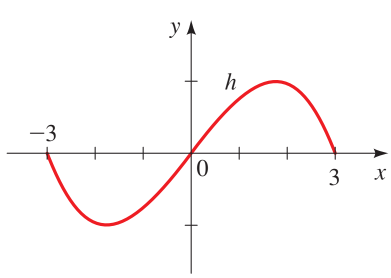
71. The graph of g is given. Use it to graph each of the following functions.

(a) $y = g(2x)$ **(b)** $y = g(\frac{1}{2}x)$



72. The graph of h is given. Use it to graph each of the following functions.

(a) $y = h(3x)$ **(b)** $y = h(\frac{1}{3}x)$



73–74 ■ Use the graph of $f(x) = [\![x]\!]$ described on page 188 to graph the indicated function.

73. $y = [\![2x]\!]$ **74.** $y = [\![\frac{1}{4}x]\!]$

75–78 ■ Graph the functions on the same screen using the given viewing rectangle. How is each graph related to the graph in part (a)?

75. Viewing rectangle $[-8, 8]$ by $[-2, 8]$

(a) $y = \sqrt[4]{x}$ **(b)** $y = \sqrt[4]{x + 5}$
(c) $y = 2\sqrt[4]{x + 5}$ **(d)** $y = 4 + 2\sqrt[4]{x + 5}$

76. Viewing rectangle $[-8, 8]$ by $[-6, 6]$

(a) $y = |x|$ **(b)** $y = -|x|$
(c) $y = -3|x|$ **(d)** $y = -3|x - 5|$

77. Viewing rectangle $[-4, 6]$ by $[-4, 4]$

(a) $y = x^6$ **(b)** $y = \frac{1}{3}x^6$
(c) $y = -\frac{1}{3}x^6$ **(d)** $y = -\frac{1}{3}(x - 4)^6$

78. Viewing rectangle $[-6, 6]$ by $[-4, 4]$

(a) $y = \dfrac{1}{\sqrt{x}}$ **(b)** $y = \dfrac{1}{\sqrt{x + 3}}$

(c) $y = \dfrac{1}{2\sqrt{x + 3}}$ **(d)** $y = \dfrac{1}{2\sqrt{x + 3}} - 3$

79. If $f(x) = \sqrt{2x - x^2}$, graph the following functions in the viewing rectangle $[-5, 5]$ by $[-4, 4]$. How is each graph related to the graph in part (a)?

(a) $y = f(x)$ **(b)** $y = f(2x)$ **(c)** $y = f(\frac{1}{2}x)$

80. If $f(x) = \sqrt{2x - x^2}$, graph the following functions in the viewing rectangle $[-5, 5]$ by $[-4, 4]$. How is each graph related to the graph in part (a)?

(a) $y = f(x)$ **(b)** $y = f(-x)$
(c) $y = -f(-x)$ **(d)** $y = f(-2x)$
(e) $y = f(-\frac{1}{2}x)$

81–88 ■ Determine whether the function f is even, odd, or neither. If f is even or odd, use symmetry to sketch its graph.

81. $f(x) = x^4$ **82.** $f(x) = x^3$

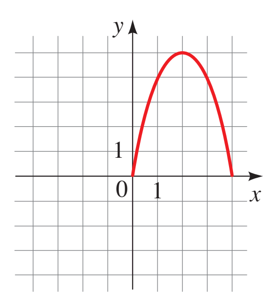
83. $f(x) = x^2 + x$ **84.** $f(x) = x^4 - 4x^2$

85. $f(x) = x^3 - x$ **86.** $f(x) = 3x^3 + 2x^2 + 1$

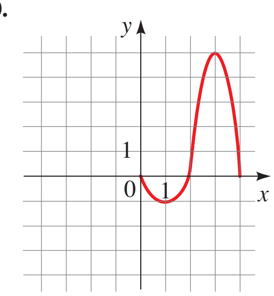
87. $f(x) = 1 - \sqrt[3]{x}$ **88.** $f(x) = x + \dfrac{1}{x}$

89–90 ■ The graph of a function defined for $x \geq 0$ is given. Complete the graph for $x < 0$ to make **(a)** an even function and **(b)** an odd function.

89.

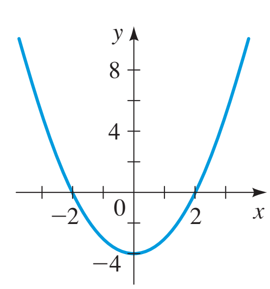


90.

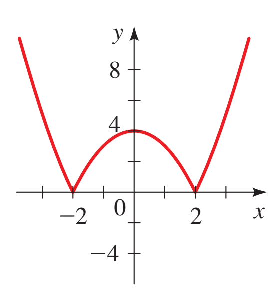


91–92 ■ These exercises show how the graph of $y = |f(x)|$ is obtained from the graph of $y = f(x)$.

91. The graphs of $f(x) = x^2 - 4$ and $g(x) = |x^2 - 4|$ are shown. Explain how the graph of g is obtained from the graph of f.

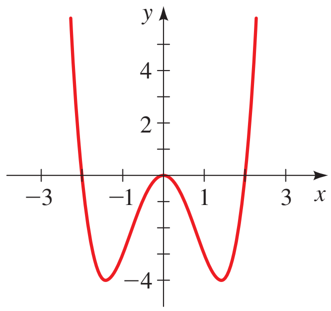


$f(x) = x^2 - 4$



$g(x) = |x^2 - 4|$

92. The graph of $f(x) = x^4 - 4x^2$ is shown. Use this graph to sketch the graph of $g(x) = |x^4 - 4x^2|$.



93–94 ■ Sketch the graph of each function.

93. **(a)** $f(x) = 4x - x^2$ **(b)** $g(x) = |4x - x^2|$

94. **(a)** $f(x) = x^3$ **(b)** $g(x) = |x^3|$

APPLICATIONS

95. Sales Growth The annual sales of a certain company can be modeled by the function $f(t) = 4 + 0.01t^2$, where t represents years since 1990 and $f(t)$ is measured in millions of dollars.

(a) What shifting and shrinking operations must be performed on the function $y = t^2$ to obtain the function $y = f(t)$?

(b) Suppose you want t to represent years since 2000 instead of 1990. What transformation would you have to apply to the function $y = f(t)$ to accomplish this? Write the new function $y = g(t)$ that results from this transformation.

96. Changing Temperature Scales The temperature on a certain afternoon is modeled by the function

$$C(t) = \tfrac{1}{2}t^2 + 2$$

where t represents hours after 12 noon $(0 \le t \le 6)$ and C is measured in °C.

(a) What shifting and shrinking operations must be performed on the function $y = t^2$ to obtain the function $y = C(t)$?

(b) Suppose you want to measure the temperature in °F instead. What transformation would you have to apply to the function $y = C(t)$ to accomplish this? (Use the fact that the relationship between Celsius and Fahrenheit degrees is given by $F = \frac{9}{5}C + 32$.) Write the new function $y = F(t)$ that results from this transformation.

DISCOVERY ■ DISCUSSION ■ WRITING

97. Sums of Even and Odd Functions If f and g are both even functions, is $f + g$ necessarily even? If both are odd, is their sum necessarily odd? What can you say about the sum if one is odd and one is even? In each case, prove your answer.

98. Products of Even and Odd Functions Answer the same questions as in Exercise 97, except this time consider the product of f and g instead of the sum.

99. Even and Odd Power Functions What must be true about the integer n if the function

$$f(x) = x^n$$

is an even function? If it is an odd function? Why do you think the names "even" and "odd" were chosen for these function properties?

2.6 Combining Functions

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find sums, differences, products, and quotients of functions ▶ Add functions graphically ▶ Find the composition of two functions ▶ Express a given function as a composite function

In this section we study different ways to combine functions to make new functions.

▼ Sums, Differences, Products, and Quotients

The sum of f and g is defined by

$$(f + g)(x) = f(x) + g(x)$$

The name of the new function is "$f + g$." So this + sign stands for the operation of addition of *functions*. The + sign on the right side, however, stands for addition of the *numbers* $f(x)$ and $g(x)$.

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. For example, we define the function $f + g$ by

$$(f + g)(x) = f(x) + g(x)$$

The new function $f + g$ is called the **sum** of the functions f and g; its value at x is $f(x) + g(x)$. Of course, the sum on the right-hand side makes sense only if both $f(x)$ and $g(x)$ are defined, that is, if x belongs to the domain of f and also to the domain of g. So if the domain of f is A and the domain of g is B, then the domain of $f + g$ is the intersection of these domains, that is, $A \cap B$. Similarly, we can define the **difference** $f - g$, the **product** fg, and the **quotient** f/g of the functions f and g. Their domains are $A \cap B$, but in the case of the quotient we must remember not to divide by 0.

ALGEBRA OF FUNCTIONS

Let f and g be functions with domains A and B. Then the functions $f + g$, $f - g$, fg, and f/g are defined as follows.

$$(f + g)(x) = f(x) + g(x) \qquad \text{Domain } A \cap B$$

$$(f - g)(x) = f(x) - g(x) \qquad \text{Domain } A \cap B$$

$$(fg)(x) = f(x)g(x) \qquad \text{Domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{Domain } \{x \in A \cap B \mid g(x) \neq 0\}$$

EXAMPLE 1 | Combinations of Functions and Their Domains

Let $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x}$.

(a) Find the functions $f + g$, $f - g$, fg, and f/g and their domains.

(b) Find $(f + g)(4)$, $(f - g)(4)$, $(fg)(4)$, and $(f/g)(4)$.

SOLUTION

(a) The domain of f is $\{x \mid x \neq 2\}$, and the domain of g is $\{x \mid x \geq 0\}$. The intersection of the domains of f and g is

$$\{x \mid x \geq 0 \text{ and } x \neq 2\} = [0, 2) \cup (2, \infty)$$

Thus we have

$$(f + g)(x) = f(x) + g(x) = \frac{1}{x-2} + \sqrt{x} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$(f - g)(x) = f(x) - g(x) = \frac{1}{x-2} - \sqrt{x} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$(fg)(x) = f(x)g(x) = \frac{\sqrt{x}}{x-2} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1}{(x-2)\sqrt{x}} \qquad \text{Domain } \{x \mid x > 0 \text{ and } x \neq 2\}$$

To divide fractions, invert the denominator and multiply:

$$\frac{1/(x-2)}{\sqrt{x}} = \frac{1/(x-2)}{\sqrt{x}/1} = \frac{1}{x-2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{(x-2)\sqrt{x}}$$

Note that in the domain of f/g we exclude 0 because $g(0) = 0$.

(b) Each of these values exist because $x = 4$ is in the domain of each function:

$$(f + g)(4) = f(4) + g(4) = \frac{1}{4-2} + \sqrt{4} = \frac{5}{2}$$

$$(f - g)(4) = f(4) - g(4) = \frac{1}{4-2} - \sqrt{4} = -\frac{3}{2}$$

$$(fg)(4) = f(4)g(4) = \left(\frac{1}{4-2}\right)\sqrt{4} = 1$$

$$\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{1}{(4-2)\sqrt{4}} = \frac{1}{4}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **7**

The graph of the function $f + g$ can be obtained from the graphs of f and g by **graphical addition**. This means that we add corresponding y-coordinates, as illustrated in the next example.

EXAMPLE 2 | Using Graphical Addition

The graphs of f and g are shown in Figure 1. Use graphical addition to graph the function $f + g$.

SOLUTION We obtain the graph of $f + g$ by "graphically adding" the value of $f(x)$ to $g(x)$ as shown in Figure 2. This is implemented by copying the line segment PQ on top of PR to obtain the point S on the graph of $f + g$.

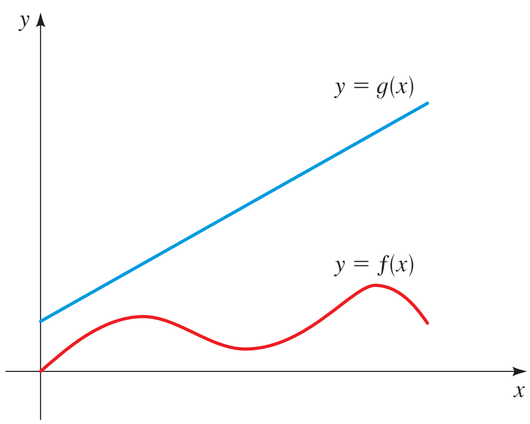


FIGURE 1

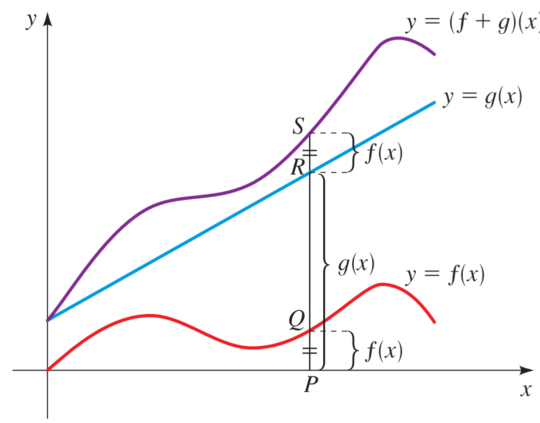


FIGURE 2 Graphical addition

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **19**

Composition of Functions

Now let's consider a very important way of combining two functions to get a new function. Suppose $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$. We may define a new function h as

$$h(x) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The function h is made up of the functions f and g in an interesting way: Given a number x, we first apply the function g to it, then apply f to the result. In this case, f is the rule "take the square root," g is the rule "square, then add 1," and h is the rule "square, then add 1, then take the square root." In other words, we get the rule h by applying the rule g and then the rule f. Figure 3 shows a machine diagram for h.

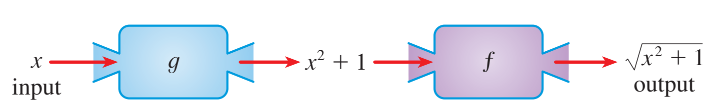


FIGURE 3 The h machine is composed of the g machine (first) and then the f machine.

In general, given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, we can then calculate the value of $f(g(x))$. The result is a new function $h(x) = f(g(x))$ that is obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ ("f composed with g").

COMPOSITION OF FUNCTIONS

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. We can picture $f \circ g$ using an arrow diagram (Figure 4).

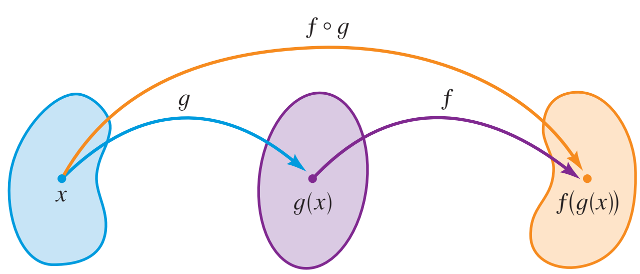


FIGURE 4 Arrow diagram for $f \circ g$

EXAMPLE 3 | Finding the Composition of Functions

Let $f(x) = x^2$ and $g(x) = x - 3$.

(a) Find the functions $f \circ g$ and $g \circ f$ and their domains.

(b) Find $(f \circ g)(5)$ and $(g \circ f)(7)$.

In Example 3, f is the rule "square" and g is the rule "subtract 3." The function $f \circ g$ *first* subtracts 3 and *then* squares; the function $g \circ f$ *first* squares and *then* subtracts 3.

SOLUTION

(a) We have

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\ &= f(x - 3) && \text{Definition of } g \\ &= (x - 3)^2 && \text{Definition of } f \end{aligned}$$

and

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f \\ &= g(x^2) && \text{Definition of } f \\ &= x^2 - 3 && \text{Definition of } g \end{aligned}$$

The domains of both $f \circ g$ and $g \circ f$ are $\mathbb{R}$.

(b) We have

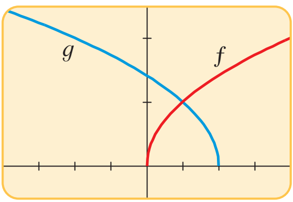
$$(f \circ g)(5) = f(g(5)) = f(2) = 2^2 = 4$$

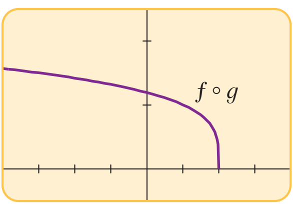
$$(g \circ f)(7) = g(f(7)) = g(49) = 49 - 3 = 46$$

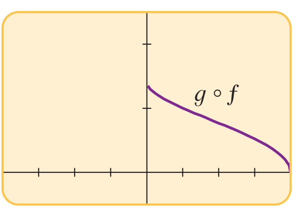
PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **25** AND **39**

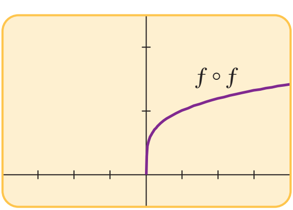
You can see from Example 3 that, in general, $f \circ g \neq g \circ f$. Remember that the notation $f \circ g$ means that the function g is applied first and then f is applied second.

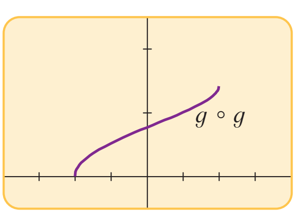
The graphs of f and g of Example 4, as well as those of $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$, are shown below. These graphs indicate that the operation of composition can produce functions that are quite different from the original functions.











EXAMPLE 4 | Finding the Composition of Functions

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find the following functions and their domains.

(a) $f \circ g$ **(b)** $g \circ f$ **(c)** $f \circ f$ **(d)** $g \circ g$

SOLUTION

(a)
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(\sqrt{2 - x}) && \text{Definition of } g\\ &= \sqrt{\sqrt{2 - x}} && \text{Definition of } f\\ &= \sqrt[4]{2 - x}\end{aligned}$$

The domain of $f \circ g$ is $\{x \mid 2 - x \ge 0\} = \{x \mid x \le 2\} = (-\infty, 2]$.

(b)
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f\\ &= g(\sqrt{x}) && \text{Definition of } f\\ &= \sqrt{2 - \sqrt{x}} && \text{Definition of } g\end{aligned}$$

For $\sqrt{x}$ to be defined, we must have $x \ge 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined, we must have $2 - \sqrt{x} \ge 0$, that is, $\sqrt{x} \le 2$, or $x \le 4$. Thus we have $0 \le x \le 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

(c)
$$\begin{aligned}(f \circ f)(x) &= f(f(x)) && \text{Definition of } f \circ f\\ &= f(\sqrt{x}) && \text{Definition of } f\\ &= \sqrt{\sqrt{x}} && \text{Definition of } f\\ &= \sqrt[4]{x}\end{aligned}$$

The domain of $f \circ f$ is $[0, \infty)$.

(d)
$$\begin{aligned}(g \circ g)(x) &= g(g(x)) && \text{Definition of } g \circ g\\ &= g(\sqrt{2 - x}) && \text{Definition of } g\\ &= \sqrt{2 - \sqrt{2 - x}} && \text{Definition of } g\end{aligned}$$

This expression is defined when both $2 - x \ge 0$ and $2 - \sqrt{2 - x} \ge 0$. The first inequality means $x \le 2$, and the second is equivalent to $\sqrt{2 - x} \le 2$, or $2 - x \le 4$, or $x \ge -2$. Thus $-2 \le x \le 2$, so the domain of $g \circ g$ is $[-2, 2]$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45**

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

EXAMPLE 5 | A Composition of Three Functions

Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) && \text{Definition of } f \circ g \circ h\\ &= f(g(x + 3)) && \text{Definition of } h\\ &= f((x + 3)^{10}) && \text{Definition of } g\\ &= \frac{(x + 3)^{10}}{(x + 3)^{10} + 1} && \text{Definition of } f\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **49**

So far, we have used composition to build complicated functions from simpler ones. But in calculus it is useful to be able to "decompose" a complicated function into simpler ones, as shown in the following example.

EXAMPLE 6 | Recognizing a Composition of Functions

Given $F(x) = \sqrt[4]{x + 9}$, find functions f and g such that $F = f \circ g$.

SOLUTION Since the formula for F says to first add 9 and then take the fourth root, we let

$$g(x) = x + 9 \qquad \text{and} \qquad f(x) = \sqrt[4]{x}$$

Then

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\ &= f(x + 9) && \text{Definition of } g \\ &= \sqrt[4]{x + 9} && \text{Definition of } f \\ &= F(x) \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **53**

EXAMPLE 7 | An Application of Composition of Functions

A ship is traveling at 20 mi/h parallel to a straight shoreline. The ship is 5 mi from shore. It passes a lighthouse at noon.

(a) Express the distance s between the lighthouse and the ship as a function of d, the distance the ship has traveled since noon; that is, find f so that $s = f(d)$.

(b) Express d as a function of t, the time elapsed since noon; that is, find g so that $d = g(t)$.

(c) Find $f \circ g$. What does this function represent?

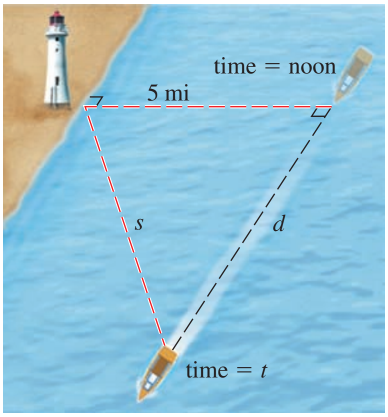


FIGURE 5

SOLUTION We first draw a diagram as in Figure 5.

(a) We can relate the distances s and d by the Pythagorean Theorem. Thus s can be expressed as a function of d by

$$s = f(d) = \sqrt{25 + d^2}$$

(b) Since the ship is traveling at 20 mi/h, the distance d it has traveled is a function of t as follows:

distance = rate × time

$$d = g(t) = 20t$$

(c) We have

$$\begin{aligned} (f \circ g)(t) &= f(g(t)) && \text{Definition of } f \circ g \\ &= f(20t) && \text{Definition of } g \\ &= \sqrt{25 + (20t)^2} && \text{Definition of } f \end{aligned}$$

The function $f \circ g$ gives the distance of the ship from the lighthouse as a function of time.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **65**

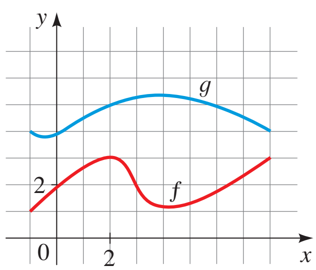
2.6 EXERCISES

CONCEPTS

1. From the graphs of f and g in the figure, we find

$(f+g)(2) =$ ________ $\qquad (f-g)(2) =$ ________

$(fg)(2) =$ ________ $\qquad \left(\dfrac{f}{g}\right)(2) =$ ________



2. By definition, $f \circ g(x) =$ ________. So if $g(2) = 5$ and $f(5) = 12$, then $f \circ g(2) =$ ________.

3. If the rule of the function f is "add one" and the rule of the function g is "multiply by 2," then the rule of $f \circ g$ is

"____,"

and the rule of $g \circ f$ is

"____."

4. We can express the functions in Exercise 3 algebraically as

$f(x) =$ ________ $\qquad g(x) =$ ________

$f \circ g(x) =$ ________ $\qquad g \circ f(x) =$ ________

SKILLS

5–14 ■ Find $f+g$, $f-g$, fg, and f/g and their domains.

5. $f(x) = x, \quad g(x) = 2x$

6. $f(x) = x, \quad g(x) = \sqrt{x}$

7. $f(x) = x, \quad g(x) = x^2$

8. $f(x) = x, \quad g(x) = x^3$

9. $f(x) = x - 3, \quad g(x) = x^2$

10. $f(x) = x^2 + 2x, \quad g(x) = 3x^2 - 1$

11. $f(x) = \sqrt{4 - x^2}, \quad g(x) = \sqrt{1 + x}$

12. $f(x) = \sqrt{9 - x^2}, \quad g(x) = \sqrt{x^2 - 4}$

13. $f(x) = \dfrac{2}{x}, \quad g(x) = \dfrac{4}{x + 4}$

14. $f(x) = \dfrac{2}{x + 1}, \quad g(x) = \dfrac{x}{x + 1}$

15–18 ■ Find the domain of the function.

15. $f(x) = \sqrt{x} + \sqrt{1 - x}$

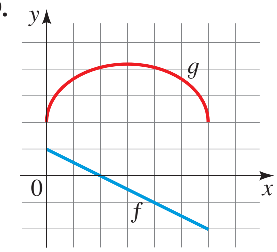
16. $g(x) = \sqrt{x + 1} - \dfrac{1}{x}$

17. $h(x) = (x - 3)^{-1/4}$

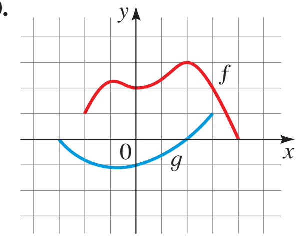
18. $k(x) = \dfrac{\sqrt{x + 3}}{x - 1}$

19–20 ■ Use graphical addition to sketch the graph of $f + g$.

19.



20.



21–24 ■ Draw the graphs of f, g, and $f + g$ on a common screen to illustrate graphical addition.

21. $f(x) = \sqrt{1 + x}, \quad g(x) = \sqrt{1 - x}$

22. $f(x) = x^2, \quad g(x) = \sqrt{x}$

23. $f(x) = x^2, \quad g(x) = \frac{1}{3}x^3$

24. $f(x) = \sqrt[4]{1 - x}, \quad g(x) = \sqrt{1 - \dfrac{x^2}{9}}$

25–30 ■ Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate the expression.

25. **(a)** $f(g(0))$ **(b)** $g(f(0))$

26. **(a)** $f(f(4))$ **(b)** $g(g(3))$

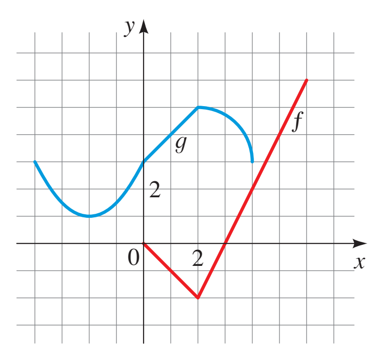
27. **(a)** $(f \circ g)(-2)$ **(b)** $(g \circ f)(-2)$

28. **(a)** $(f \circ f)(-1)$ **(b)** $(g \circ g)(2)$

29. **(a)** $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$

30. **(a)** $(f \circ f)(x)$ **(b)** $(g \circ g)(x)$

31–36 ■ Use the given graphs of f and g to evaluate the expression.



31. $f(g(2))$

32. $g(f(0))$

33. $(g \circ f)(4)$

34. $(f \circ g)(0)$

35. $(g \circ g)(-2)$

36. $(f \circ f)(4)$

37–48 ■ Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

37. $f(x) = 2x + 3, \quad g(x) = 4x - 1$

38. $f(x) = 6x - 5, \quad g(x) = \dfrac{x}{2}$

39. $f(x) = x^2, \quad g(x) = x + 1$

40. $f(x) = x^3 + 2, \quad g(x) = \sqrt[3]{x}$

41. $f(x) = \dfrac{1}{x}, \quad g(x) = 2x + 4$

42. $f(x) = x^2, \quad g(x) = \sqrt{x - 3}$

43. $f(x) = |x|, \quad g(x) = 2x + 3$

44. $f(x) = x - 4, \quad g(x) = |x + 4|$

45. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = 2x - 1$

46. $f(x) = \dfrac{1}{\sqrt{x}}, \quad g(x) = x^2 - 4x$

47. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = \dfrac{1}{x}$

48. $f(x) = \dfrac{2}{x}, \quad g(x) = \dfrac{x}{x + 2}$

49–52 ■ Find $f \circ g \circ h$.

49. $f(x) = x - 1, \quad g(x) = \sqrt{x}, \quad h(x) = x - 1$

50. $f(x) = \dfrac{1}{x}, \quad g(x) = x^3, \quad h(x) = x^2 + 2$

51. $f(x) = x^4 + 1, \quad g(x) = x - 5, \quad h(x) = \sqrt{x}$

52. $f(x) = \sqrt{x}, \quad g(x) = \dfrac{x}{x - 1}, \quad h(x) = \sqrt[3]{x}$

53–58 ■ Express the function in the form $f \circ g$.

53. $F(x) = (x - 9)^5$

54. $F(x) = \sqrt{x} + 1$

55. $G(x) = \dfrac{x^2}{x^2 + 4}$

56. $G(x) = \dfrac{1}{x + 3}$

57. $H(x) = |1 - x^3|$

58. $H(x) = \sqrt{1 + \sqrt{x}}$

59–62 ■ Express the function in the form $f \circ g \circ h$.

59. $F(x) = \dfrac{1}{x^2 + 1}$

60. $F(x) = \sqrt[3]{\sqrt{x} - 1}$

61. $G(x) = (4 + \sqrt[3]{x})^9$

62. $G(x) = \dfrac{2}{(3 + \sqrt{x})^2}$

APPLICATIONS

63–64 ■ **Revenue, Cost, and Profit** A print shop makes bumper stickers for election campaigns. If x stickers are ordered (where $x < 10{,}000$), then the price per bumper sticker is $0.15 - 0.000002x$ dollars, and the total cost of producing the order is $0.095x - 0.0000005x^2$ dollars.

63. Use the fact that

revenue = price per item × number of items sold

to express $R(x)$, the revenue from an order of x stickers, as a product of two functions of x.

64. Use the fact that

profit = revenue − cost

to express $P(x)$, the profit on an order of x stickers, as a difference of two functions of x.

65. Area of a Ripple A stone is dropped in a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
(a) Find a function g that models the radius as a function of time.
(b) Find a function f that models the area of the circle as a function of the radius.
(c) Find $f \circ g$. What does this function represent?

66. Inflating a Balloon A spherical balloon is being inflated. The radius of the balloon is increasing at the rate of 1 cm/s.
(a) Find a function f that models the radius as a function of time.
(b) Find a function g that models the volume as a function of the radius.
(c) Find $g \circ f$. What does this function represent?

67. Area of a Balloon A spherical weather balloon is being inflated. The radius of the balloon is increasing at the rate of 2 cm/s. Express the surface area of the balloon as a function of time t (in seconds).

68. Multiple Discounts You have a \$50 coupon from the manufacturer good for the purchase of a cell phone. The store where you are purchasing your cell phone is offering a 20% discount on all cell phones. Let x represent the regular price of the cell phone.
(a) Suppose only the 20% discount applies. Find a function f that models the purchase price of the cell phone as a function of the regular price x.
(b) Suppose only the \$50 coupon applies. Find a function g that models the purchase price of the cell phone as a function of the sticker price x.
(c) If you can use the coupon and the discount, then the purchase price is either $f \circ g(x)$ or $g \circ f(x)$, depending on the order in which they are applied to the price. Find both $f \circ g(x)$ and $g \circ f(x)$. Which composition gives the lower price?

69. Multiple Discounts An appliance dealer advertises a 10% discount on all his washing machines. In addition, the manufacturer offers a \$100 rebate on the purchase of a washing machine. Let x represent the sticker price of the washing machine.

(a) Suppose only the 10% discount applies. Find a function f that models the purchase price of the washer as a function of the sticker price x.

(b) Suppose only the \$100 rebate applies. Find a function g that models the purchase price of the washer as a function of the sticker price x.

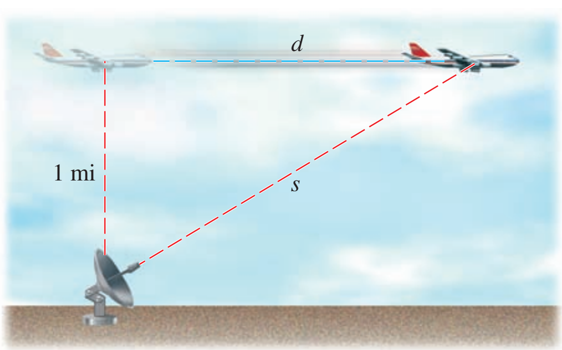
(c) Find $f \circ g$ and $g \circ f$. What do these functions represent? Which is the better deal?

70. Airplane Trajectory An airplane is flying at a speed of 350 mi/h at an altitude of one mile. The plane passes directly above a radar station at time $t = 0$.

(a) Express the distance s (in miles) between the plane and the radar station as a function of the horizontal distance d (in miles) that the plane has flown.

(b) Express d as a function of the time t (in hours) that the plane has flown.

(c) Use composition to express s as a function of t.



DISCOVERY ■ DISCUSSION ■ WRITING

71. Compound Interest A savings account earns 5% interest compounded annually. If you invest x dollars in such an account, then the amount $A(x)$ of the investment after one year is the initial investment plus 5%; that is,

$$A(x) = x + 0.05x = 1.05x$$

Find

$$A \circ A$$

$$A \circ A \circ A$$

$$A \circ A \circ A \circ A$$

What do these compositions represent? Find a formula for what you get when you compose n copies of A.

72. Composing Linear Functions The graphs of the functions

$$f(x) = m_1x + b_1$$

$$g(x) = m_2x + b_2$$

are lines with slopes m_1 and m_2, respectively. Is the graph of $f \circ g$ a line? If so, what is its slope?

73. Solving an Equation for an Unknown Function Suppose that

$$g(x) = 2x + 1$$

$$h(x) = 4x^2 + 4x + 7$$

Find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.) Now suppose that

$$f(x) = 3x + 5$$

$$h(x) = 3x^2 + 3x + 2$$

Use the same sort of reasoning to find a function g such that $f \circ g = h$.

74. Compositions of Odd and Even Functions Suppose that

$$h = f \circ g$$

If g is an even function, is h necessarily even? If g is odd, is h odd? What if g is odd and f is odd? What if g is odd and f is even?

DISCOVERY PROJECT **Iteration and Chaos**

In this project we explore the process of repeatedly composing a function with itself; the result can be regular or chaotic. You can find the project at the book companion website: **www.stewartmath.com**

2.7 One-to-One Functions and Their Inverses

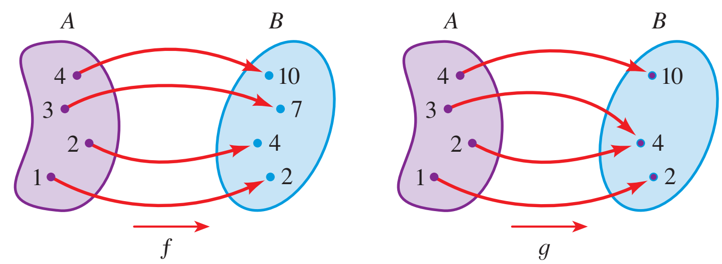
LEARNING OBJECTIVES **After completing this section, you will be able to:**

Determine whether a function is one-to-one ▶ Find the inverse of a one-to-one function ▶ Draw the graph of the inverse of a function

The *inverse* of a function is a rule that acts on the output of the function and produces the corresponding input. So the inverse "undoes" or reverses what the function has done. Not all functions have inverses; those that do are called *one-to-one*.

One-to-One Functions

Let's compare the functions f and g whose arrow diagrams are shown in Figure 1. Note that f never takes on the same value twice (any two numbers in A have different images), whereas g does take on the same value twice (both 2 and 3 have the same image, 4). In symbols, $g(2) = g(3)$ but $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. Functions that have this latter property are called *one-to-one*.



f is one-to-one $\qquad$ g is not one-to-one

FIGURE 1

DEFINITION OF A ONE-TO-ONE FUNCTION

A function with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

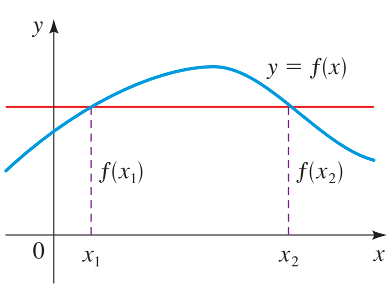


FIGURE 2 This function is not one-to-one because $f(x_1) = f(x_2)$.

An equivalent way of writing the condition for a one-to-one function is this:

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

If a horizontal line intersects the graph of f at more than one point, then we see from Figure 2 that there are numbers $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. This means that f is not one-to-one. Therefore, we have the following geometric method for determining whether a function is one-to-one.

HORIZONTAL LINE TEST

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

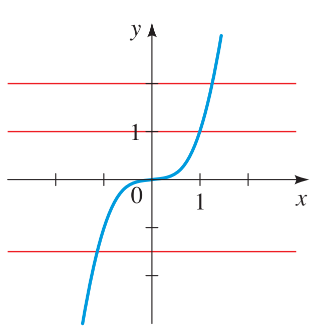


FIGURE 3 $f(x) = x^3$ is one-to-one.

EXAMPLE 1 | Deciding Whether a Function Is One-to-One

Is the function $f(x) = x^3$ one-to-one?

SOLUTION 1 If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers cannot have the same cube). Therefore, $f(x) = x^3$ is one-to-one.

SOLUTION 2 From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **15**

Notice that the function f of Example 1 is increasing and is also one-to-one. In fact, it can be proved that *every increasing function and every decreasing function is one-to-one*.

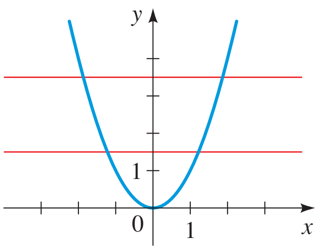


FIGURE 4 $g(x) = x^2$ is not one-to-one.

EXAMPLE 2 | Deciding Whether a Function Is One-to-One

Is the function $g(x) = x^2$ one-to-one?

SOLUTION 1 This function is not one-to-one because, for instance,

$$g(1) = 1 \qquad \text{and} \qquad g(-1) = 1$$

so 1 and -1 have the same image.

SOLUTION 2 From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17**

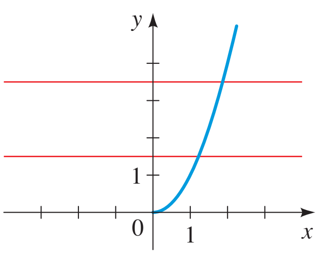


FIGURE 5 $h(x) = x^2\ (x \geq 0)$ is one-to-one.

Although the function g in Example 2 is not one-to-one, it is possible to restrict its domain so that the resulting function is one-to-one. In fact, if we define

$$h(x) = x^2 \qquad x \geq 0$$

then h is one-to-one, as you can see from Figure 5 and the Horizontal Line Test.

EXAMPLE 3 | Showing That a Function Is One-to-One

Show that the function $f(x) = 3x + 4$ is one-to-one.

SOLUTION Suppose there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. Then

$$\begin{aligned} 3x_1 + 4 &= 3x_2 + 4 && \text{Suppose } f(x_1) = f(x_2) \\ 3x_1 &= 3x_2 && \text{Subtract 4} \\ x_1 &= x_2 && \text{Divide by 3} \end{aligned}$$

Therefore f is one-to-one.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **13**

The Inverse of a Function

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

> **DEFINITION OF THE INVERSE OF A FUNCTION**
>
> Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by
>
> $$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$
>
> for any y in B.

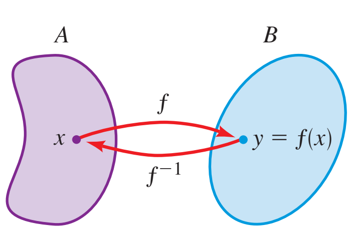


FIGURE 6

This definition says that if f takes x to y, then f^{-1} takes y back to x. (If f were not one-to-one, then f^{-1} would not be defined uniquely.) The arrow diagram in Figure 6 indicates that f^{-1} reverses the effect of f. From the definition we have

$$\text{domain of } f^{-1} = \text{range of } f$$
$$\text{range of } f^{-1} = \text{domain of } f$$

EXAMPLE 4 | Finding f^{-1} for Specific Values

Don't mistake the -1 in f^{-1} for an exponent.

$$f^{-1}(x) \quad \textit{does not mean} \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ is written as $(f(x))^{-1}$.

If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(5)$, $f^{-1}(7)$, and $f^{-1}(-10)$.

SOLUTION From the definition of f^{-1} we have

$$\begin{aligned} f^{-1}(5) &= 1 \quad \text{because} \quad f(1) = 5 \\ f^{-1}(7) &= 3 \quad \text{because} \quad f(3) = 7 \\ f^{-1}(-10) &= 8 \quad \text{because} \quad f(8) = -10 \end{aligned}$$

Figure 7 shows how f^{-1} reverses the effect of f in this case.

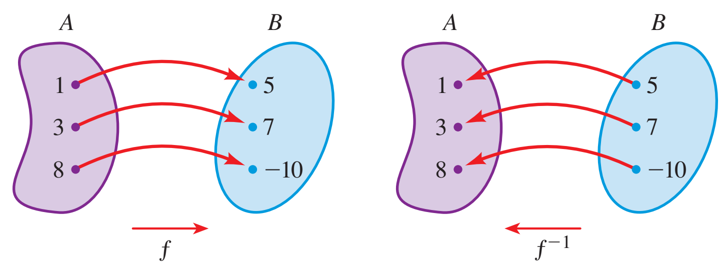


FIGURE 7

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **23**

EXAMPLE 5 | Finding Values of f^{-1} Graphically

The graph of a function f is given in Figure 8. Use the graph to find

(a) $f^{-1}(3)$ **(b)** $f^{-1}(5)$

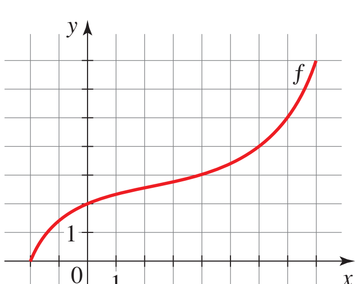


FIGURE 8 Graph of f

SOLUTION

(a) From the graph we see that $f(4) = 3$, so $f^{-1}(3) = 4$. (See Figure 9.)

(b) From the graph we see that $f(7) = 5$, so $f^{-1}(5) = 7$. (See Figure 9.)

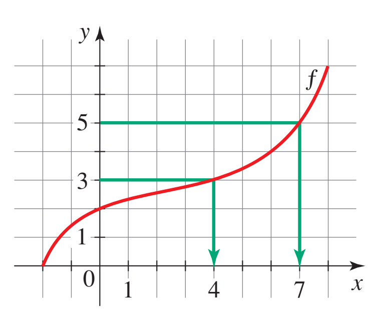


FIGURE 9 Finding values of f^{-1} from the graph of f

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **27**

By definition the inverse function f^{-1} undoes what f does: If we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started. Similarly, f undoes what f^{-1} does. In general, any function that reverses the effect of f in this way must be the inverse of f. These observations are expressed precisely as follows.

INVERSE FUNCTION PROPERTY

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

These properties indicate that f is the inverse function of f^{-1}, so we say that f and f^{-1} are *inverses of each other*.

EXAMPLE 6 | Verifying That Two Functions Are Inverses

Show that $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses of each other.

SOLUTION Note that the domain and range of both f and g is $\mathbb{R}$. We have

$$g(f(x)) = g(x^3) = (x^3)^{1/3} = x$$

$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$$

So by the Property of Inverse Functions, f and g are inverses of each other. These equations simply say that the cube function and the cube root function, when composed, cancel each other.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **31**

▼ Finding the Inverse of a Function

Now let's examine how we compute inverse functions. We first observe from the definition of f^{-1} that

$$y = f(x) \quad \Leftrightarrow \quad f^{-1}(y) = x$$

So if $y = f(x)$ and if we are able to solve this equation for x in terms of y, then we must have $x = f^{-1}(y)$. If we then interchange x and y, we have $y = f^{-1}(x)$, which is the desired equation.

HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Note that Steps 2 and 3 can be reversed. In other words, we can interchange x and y first and then solve for y in terms of x.

In Example 7 note how f^{-1} reverses the effect of f. The function f is the rule "Multiply by 3, then subtract 2," whereas f^{-1} is the rule "Add 2, then divide by 3."

EXAMPLE 7 | Finding the Inverse of a Function

Find the inverse of the function $f(x) = 3x - 2$.

SOLUTION First we write $y = f(x)$.

$$y = 3x - 2$$

CHECK YOUR ANSWER

We use the Inverse Function Property:

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(3x-2) \\ &= \frac{(3x-2)+2}{3} \\ &= \frac{3x}{3} = x \\ f(f^{-1}(x)) &= f\left(\frac{x+2}{3}\right) \\ &= 3\left(\frac{x+2}{3}\right) - 2 \\ &= x + 2 - 2 = x \quad ✓ \end{aligned}$$

Then we solve this equation for x:

$$\begin{aligned} 3x &= y + 2 && \text{Add 2} \\ x &= \frac{y+2}{3} && \text{Divide by 3} \end{aligned}$$

Finally, we interchange x and y:

$$y = \frac{x+2}{3}$$

Therefore, the inverse function is $f^{-1}(x) = \dfrac{x+2}{3}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41** ■

In Example 8 note how f^{-1} reverses the effect of f. The function f is the rule "Take the fifth power, subtract 3, then divide by 2," whereas f^{-1} is the rule "Multiply by 2, add 3, then take the fifth root."

EXAMPLE 8 | Finding the Inverse of a Function

Find the inverse of the function $f(x) = \dfrac{x^5 - 3}{2}$.

SOLUTION We first write $y = (x^5 - 3)/2$ and solve for x:

$$\begin{aligned} y &= \frac{x^5-3}{2} && \text{Equation defining function} \\ 2y &= x^5 - 3 && \text{Multiply by 2} \\ x^5 &= 2y + 3 && \text{Add 3 (and switch sides)} \\ x &= (2y+3)^{1/5} && \text{Take fifth root of each side} \end{aligned}$$

Then we interchange x and y to get $y = (2x+3)^{1/5}$. Therefore the inverse function is $f^{-1}(x) = (2x+3)^{1/5}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **57** ■

CHECK YOUR ANSWER

We use the Inverse Function Property:

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(\frac{x^5-3}{2}\right) \\ &= \left[2\left(\frac{x^5-3}{2}\right) + 3\right]^{1/5} \\ &= (x^5 - 3 + 3)^{1/5} \\ &= (x^5)^{1/5} = x \\ f(f^{-1}(x)) &= f((2x+3)^{1/5}) \\ &= \frac{[(2x+3)^{1/5}]^5 - 3}{2} \\ &= \frac{2x+3-3}{2} \\ &= \frac{2x}{2} = x \quad ✓ \end{aligned}$$

A **rational function** is a function defined by a rational expression. In the next example we find the inverse of a rational function.

EXAMPLE 9 | Finding the Inverse of a Rational Function

Find the inverse of the function $f(x) = \dfrac{2x+3}{x-1}$.

SOLUTION We first write $y = (2x+3)/(x-1)$ and solve for x:

$$\begin{aligned} y &= \frac{2x+3}{x-1} && \text{Equation defining function} \\ y(x-1) &= 2x + 3 && \text{Multiply by } x - 1 \\ yx - y &= 2x + 3 && \text{Expand} \\ yx - 2x &= y + 3 && \text{Bring } x\text{-terms to LHS} \\ x(y-2) &= y + 3 && \text{Factor } x \\ x &= \frac{y+3}{y-2} && \text{Divide by } y - 2 \end{aligned}$$

Rational functions are studied in Section 3.7.

Therefore the inverse function is $f^{-1}(x) = \dfrac{x+3}{x-2}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **49** ■

▼ Graphing the Inverse of a Function

The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f. If $f(a) = b$, then $f^{-1}(b) = a$. Thus the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from the point (a, b) by reflecting in the line $y = x$ (see Figure 10). Therefore, as Figure 11 illustrates, the following is true.

> The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

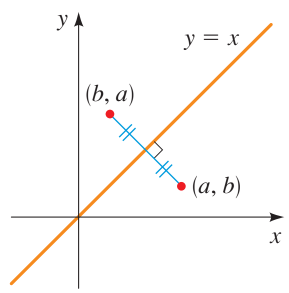


FIGURE 10

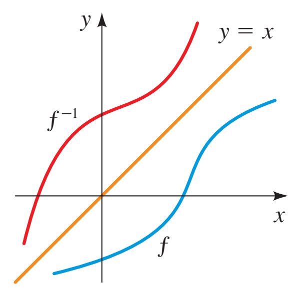


FIGURE 11

EXAMPLE 10 | Graphing the Inverse of a Function

(a) Sketch the graph of $f(x) = \sqrt{x - 2}$.

(b) Use the graph of f to sketch the graph of f^{-1}.

(c) Find an equation for f^{-1}.

SOLUTION

(a) Using the transformations from Section 2.5, we sketch the graph of $y = \sqrt{x - 2}$ by plotting the graph of the function $y = \sqrt{x}$ (Example 1(c) in Section 2.2) and shifting it to the right 2 units.

(b) The graph of f^{-1} is obtained from the graph of f in part (a) by reflecting it in the line $y = x$, as shown in Figure 12.

(c) Solve $y = \sqrt{x - 2}$ for x, noting that $y \geq 0$:

$$\sqrt{x - 2} = y$$

$$x - 2 = y^2 \qquad \text{Square each side}$$

$$x = y^2 + 2 \qquad y \geq 0 \qquad \text{Add 2}$$

Interchange x and y:

$$y = x^2 + 2 \qquad x \geq 0$$

Thus $$f^{-1}(x) = x^2 + 2 \qquad x \geq 0$$

This expression shows that the graph of f^{-1} is the right half of the parabola $y = x^2 + 2$, and from the graph shown in Figure 12 this seems reasonable.

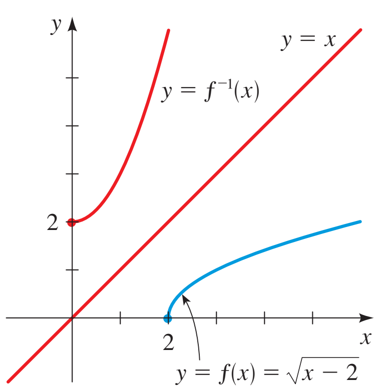


FIGURE 12

In Example 10 note how f^{-1} reverses the effect of f. The function f is the rule "Subtract 2, then take the square root," whereas f^{-1} is the rule "Square, then add 2."

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **67** ■

2.7 EXERCISES

CONCEPTS

1. A function f is one-to-one if different inputs produce ________ outputs. You can tell from the graph that a function is one-to-one by using the ______________ Test.

2. (a) For a function to have an inverse, it must be ________. So which one of the following functions has an inverse?

$$f(x) = x^2 \qquad g(x) = x^3$$

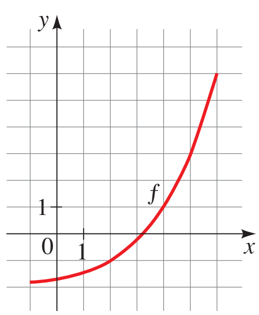
(b) What is the inverse of the function that you chose in part (a)?

3. A function f has the following verbal description: "Multiply by 3, add 5, and then take the third power of the result."

(a) Write a verbal description for f^{-1}.

(b) Find algebraic formulas that express f and f^{-1} in terms of the input x.

4. A graph of a function f is given. Does f have an inverse? If so, find $f^{-1}(1) =$ ________ and $f^{-1}(3) =$ ________.



5. If the point (3, 4) is on the graph of the function f, then the point (____, ____) is on the graph of f^{-1}.

6. *True or false?*

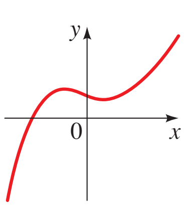
(a) If f has an inverse, then $f^{-1}(x)$ is the same as $\dfrac{1}{f(x)}$.

(b) If f has an inverse, then $f^{-1}(f(x)) = x$.

SKILLS

7–12 ■ A graph of a function f is given. Determine whether f is one-to-one.

7.



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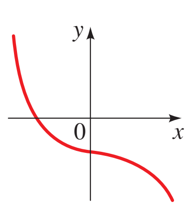

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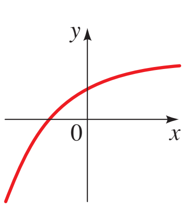

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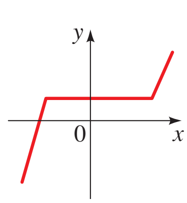

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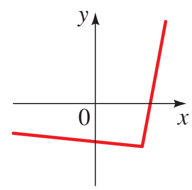
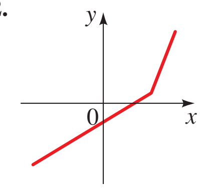


12.



13–22 ■ Determine whether the function is one-to-one.

13. $f(x) = -2x + 4$

14. $f(x) = 3x - 2$

15. $g(x) = \sqrt{x}$

16. $g(x) = |x|$

17. $h(x) = x^2 - 2x$

18. $h(x) = x^3 + 8$

19. $f(x) = x^4 + 5$

20. $f(x) = x^4 + 5, \quad 0 \le x \le 2$

21. $f(x) = \dfrac{1}{x^2}$

22. $f(x) = \dfrac{1}{x}$

23–26 ■ Assume that f is a one-to-one function.

23. (a) If $f(2) = 7$, find $f^{-1}(7)$.
(b) If $f^{-1}(3) = -1$, find $f(-1)$.

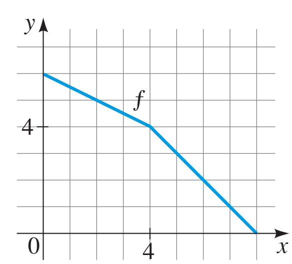
24. (a) If $f(5) = 18$, find $f^{-1}(18)$.
(b) If $f^{-1}(4) = 2$, find $f(2)$.

25. If $f(x) = 5 - 2x$, find $f^{-1}(3)$.

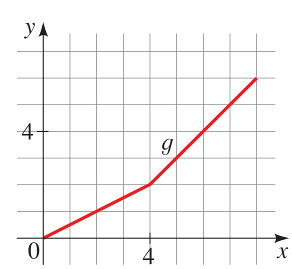
26. If $g(x) = x^2 + 4x$ with $x \ge -2$, find $g^{-1}(5)$.

27–28 ■ A graph of a function is given. Use the graph to find the indicated values.

27. (a) $f^{-1}(2)$ **(b)** $f^{-1}(5)$ **(c)** $f^{-1}(6)$



28. (a) $g^{-1}(2)$ **(b)** $g^{-1}(5)$ **(c)** $g^{-1}(6)$



29–40 ■ Use the Inverse Function Property to show that f and g are inverses of each other.

29. $f(x) = x - 6; \quad g(x) = x + 6$

30. $f(x) = 3x; \quad g(x) = \dfrac{x}{3}$

31. $f(x) = 2x - 5$; $g(x) = \dfrac{x + 5}{2}$

32. $f(x) = \dfrac{3 - x}{4}$; $g(x) = 3 - 4x$

33. $f(x) = \dfrac{1}{x}$; $g(x) = \dfrac{1}{x}$

34. $f(x) = x^5$; $g(x) = \sqrt[5]{x}$

35. $f(x) = x^2 - 4$, $x \geq 0$; $g(x) = \sqrt{x + 4}$, $x \geq -4$

36. $f(x) = x^3 + 1$; $g(x) = (x - 1)^{1/3}$

37. $f(x) = \dfrac{1}{x - 1}$; $g(x) = \dfrac{1}{x} + 1$

38. $f(x) = \sqrt{4 - x^2}$, $0 \leq x \leq 2$;
$g(x) = \sqrt{4 - x^2}$, $0 \leq x \leq 2$

39. $f(x) = \dfrac{x + 2}{x - 2}$; $g(x) = \dfrac{2x + 2}{x - 1}$

40. $f(x) = \dfrac{x - 5}{3x + 4}$; $g(x) = \dfrac{5 + 4x}{1 - 3x}$

41–64 ■ Find the inverse function of f.

41. $f(x) = 2x + 1$

42. $f(x) = 6 - x$

43. $f(x) = 4x + 7$

44. $f(x) = 3 - 5x$

45. $f(x) = 5 - 4x^3$

46. $f(x) = \dfrac{1}{x^2}$, $x > 0$

47. $f(x) = \dfrac{1}{x + 2}$

48. $f(x) = \dfrac{x - 2}{x + 2}$

49. $f(x) = \dfrac{x}{x + 4}$

50. $f(x) = \dfrac{3x}{x - 2}$

51. $f(x) = \dfrac{2x + 5}{x - 7}$

52. $f(x) = \dfrac{4x - 2}{3x + 1}$

53. $f(x) = \dfrac{1 + 3x}{5 - 2x}$

54. $f(x) = \dfrac{2x - 1}{x - 3}$

55. $f(x) = \sqrt{2 + 5x}$

56. $f(x) = x^2 + x$, $x \geq -\frac{1}{2}$

57. $f(x) = 4 - x^2$, $x \geq 0$

58. $f(x) = \sqrt{2x - 1}$

59. $f(x) = 4 + \sqrt[3]{x}$

60. $f(x) = (2 - x^3)^5$

61. $f(x) = 1 + \sqrt{1 + x}$

62. $f(x) = \sqrt{9 - x^2}$, $0 \leq x \leq 3$

63. $f(x) = x^4$, $x \geq 0$

64. $f(x) = 1 - x^3$

65–68 ■ A function f is given. **(a)** Sketch the graph of f. **(b)** Use the graph of f to sketch the graph of f^{-1}. **(c)** Find f^{-1}.

65. $f(x) = 3x - 6$

66. $f(x) = 16 - x^2$, $x \geq 0$

67. $f(x) = \sqrt{x + 1}$

68. $f(x) = x^3 - 1$

69–74 ■ Draw the graph of f and use it to determine whether the function is one-to-one.

69. $f(x) = x^3 - x$

70. $f(x) = x^3 + x$

71. $f(x) = \dfrac{x + 12}{x - 6}$

72. $f(x) = \sqrt{x^3 - 4x + 1}$

73. $f(x) = |x| - |x - 6|$

74. $f(x) = x \cdot |x|$

75–78 ■ A one-to-one function is given. **(a)** Find the inverse of the function. **(b)** Graph both the function and its inverse on the same screen to verify that the graphs are reflections of each other in the line $y = x$.

75. $f(x) = 2 + x$

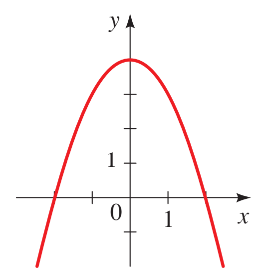
76. $f(x) = 2 - \frac{1}{2}x$

77. $g(x) = \sqrt{x + 3}$

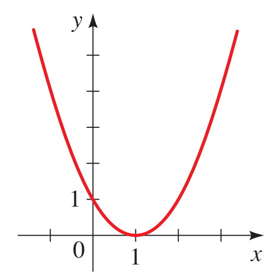
78. $g(x) = x^2 + 1$, $x \geq 0$

79–82 ■ The given function is not one-to-one. Restrict its domain so that the resulting function *is* one-to-one. Find the inverse of the function with the restricted domain. (There is more than one correct answer.)

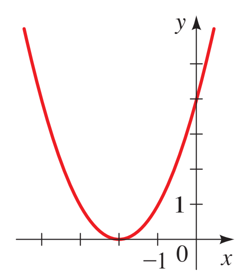
79. $f(x) = 4 - x^2$



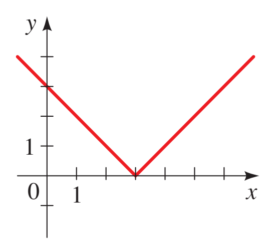
80. $g(x) = (x - 1)^2$



81. $h(x) = (x + 2)^2$

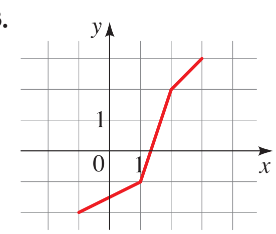


82. $k(x) = |x - 3|$



83–84 ■ Use the graph of f to sketch the graph of f^{-1}.

83.



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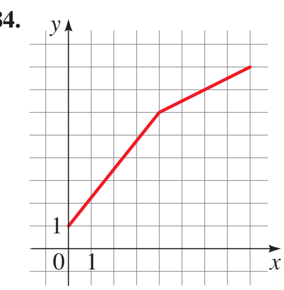


APPLICATIONS

85. Fee for Service For his services, a private investigator requires a \$500 retention fee plus \$80 per hour. Let x represent the number of hours the investigator spends working on a case.

(a) Find a function f that models the investigator's fee as a function of x.

(b) Find f^{-1}. What does f^{-1} represent?

(c) Find $f^{-1}(1220)$. What does your answer represent?

86. Toricelli's Law A tank holds 100 gallons of water, which drains from a leak at the bottom, causing the tank to empty in 40 minutes. Toricelli's Law gives the volume of water remaining in the tank after t minutes as

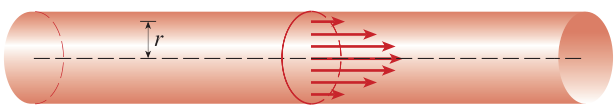
$$V(t) = 100\left(1 - \frac{t}{40}\right)^2$$

(a) Find V^{-1}. What does V^{-1} represent?
(b) Find $V^{-1}(15)$. What does your answer represent?

87. Blood Flow As blood moves through a vein or artery, its velocity v is greatest along the central axis and decreases as the distance r from the central axis increases (see the figure below). For an artery with radius 0.5 cm, v (in cm/s) is given as a function of r (in cm) by

$$v(r) = 18{,}500(0.25 - r^2)$$

(a) Find v^{-1}. What does v^{-1} represent?
(b) Find $v^{-1}(30)$. What does your answer represent?



88. Demand Function The amount of a commodity that is sold is called the *demand* for the commodity. The demand D for a certain commodity is a function of the price given by

$$D(p) = -3p + 150$$

(a) Find D^{-1}. What does D^{-1} represent?
(b) Find $D^{-1}(30)$. What does your answer represent?

89. Temperature Scales The relationship between the Fahrenheit (F) and Celsius (C) scales is given by

$$F(C) = \tfrac{9}{5}C + 32$$

(a) Find F^{-1}. What does F^{-1} represent?
(b) Find $F^{-1}(86)$. What does your answer represent?

90. Exchange Rates The relative value of currencies fluctuates every day. When this problem was written, one Canadian dollar was worth 1.0573 U.S. dollars.

(a) Find a function f that gives the U.S. dollar value $f(x)$ of x Canadian dollars.
(b) Find f^{-1}. What does f^{-1} represent?
(c) How much Canadian money would \$12,250 in U.S. currency be worth?

91. Income Tax In a certain country, the tax on incomes less than or equal to €20,000 is 10%. For incomes that are more than €20,000, the tax is €2000 plus 20% of the amount over €20,000.

(a) Find a function f that gives the income tax on an income x. Express f as a piecewise defined function.
(b) Find f^{-1}. What does f^{-1} represent?
(c) How much income would require paying a tax of €10,000?

92. Multiple Discounts A car dealership advertises a 15% discount on all its new cars. In addition, the manufacturer offers a \$1000 rebate on the purchase of a new car. Let x represent the sticker price of the car.

(a) Suppose that only the 15% discount applies. Find a function f that models the purchase price of the car as a function of the sticker price x.
(b) Suppose that only the \$1000 rebate applies. Find a function g that models the purchase price of the car as a function of the sticker price x.
(c) Find a formula for $H = f \circ g$.
(d) Find H^{-1}. What does H^{-1} represent?
(e) Find $H^{-1}(13{,}000)$. What does your answer represent?

93. Pizza Cost Marcello's Pizza charges a base price of \$7 for a large pizza plus \$2 for each topping. Thus if you order a large pizza with x toppings, the price of your pizza is given by the function $f(x) = 7 + 2x$. Find f^{-1}. What does the function f^{-1} represent?

DISCOVERY ▪ DISCUSSION ▪ WRITING

94. Determining When a Linear Function Has an Inverse For the linear function $f(x) = mx + b$ to be one-to-one, what must be true about its slope? If it is one-to-one, find its inverse. Is the inverse linear? If so, what is its slope?

95. Finding an Inverse "in Your Head" In the margin notes in this section we pointed out that the inverse of a function can be found by simply reversing the operations that make up the function. For instance, in Example 7 we saw that the inverse of

$$f(x) = 3x - 2 \qquad \text{is} \qquad f^{-1}(x) = \frac{x+2}{3}$$

because the "reverse" of "Multiply by 3 and subtract 2" is "Add 2 and divide by 3." Use the same procedure to find the inverse of the following functions.

(a) $f(x) = \dfrac{2x+1}{5}$ **(b)** $f(x) = 3 - \dfrac{1}{x}$
(c) $f(x) = \sqrt{x^3 + 2}$ **(d)** $f(x) = (2x - 5)^3$

Now consider another function:

$$f(x) = x^3 + 2x + 6$$

Is it possible to use the same sort of simple reversal of operations to find the inverse of this function? If so, do it. If not, explain what is different about this function that makes this task difficult.

96. The Identity Function The function $I(x) = x$ is called the **identity function**. Show that for any function f we have $f \circ I = f$, $I \circ f = f$, and $f \circ f^{-1} = f^{-1} \circ f = I$. (This means that the identity function I behaves for functions and composition just the way the number 1 behaves for real numbers and multiplication.)

97. Solving an Equation for an Unknown Function In Exercise 73 of Section 2.6 you were asked to solve equations in which the unknowns were functions. Now that we know about inverses and the identity function (see Exercise 96), we can use algebra to solve such equations. For instance, to solve $f \circ g = h$ for the unknown function f, we perform the following steps:

$f \circ g = h$	Problem: Solve for f
$f \circ g \circ g^{-1} = h \circ g^{-1}$	Compose with g^{-1} on the right
$f \circ I = h \circ g^{-1}$	Because $g \circ g^{-1} = I$
$f = h \circ g^{-1}$	Because $f \circ I = f$

So the solution is $f = h \circ g^{-1}$. Use this technique to solve the equation $f \circ g = h$ for the indicated unknown function.

(a) Solve for f, where $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$.

(b) Solve for g, where $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$.

CHAPTER 2 | REVIEW

PROPERTIES AND FORMULAS

Function Notation (p. 175)

If a function is given by the formula $y = f(x)$, then x is the independent variable and denotes the **input**; y is the dependent variable and denotes the **output**; the **domain** is the set of all possible inputs x; the **range** is the set of all possible outputs y.

Net Change (p. 177)

The **net change** in the value of the function f between $x = a$ and $x = b$ is

$$\text{net change} = f(b) - f(a)$$

The Graph of a Function (p. 185)

The graph of a function f is the graph of the equation $y = f(x)$ that defines f.

The Vertical Line Test (p. 189)

A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the graph more than once.

Increasing and Decreasing Functions (p. 196)

A function f is **increasing** on an interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in the interval.

A function f is **decreasing** on an interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in the interval.

Local Maximum and Minimum Values (p. 198)

The function value $f(a)$ is a **local maximum value** of the function f if $f(a) \geq f(x)$ for all x near a. In this case we also say that f has a **local maximum** at $x = a$.

The function value $f(b)$ is a **local minimum value** of the function f if $f(b) \leq f(x)$ for all x near b. In this case we also say that f has a **local minimum** at $x = b$.

Average Rate of Change (p. 205)

The **average rate of change** of the function f between $x = a$ and $x = b$ is the slope of the **secant** line between $(a, f(a))$ and $(b, f(b))$:

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

Vertical and Horizontal Shifts of Graphs (pp. 211–213)

Let c be a positive constant.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ **upward** by c units.

To graph $y = f(x) - c$, shift the graph of $y = f(x)$ **downward** by c units.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ **to the right** by c units.

To graph $y = f(x + c)$, shift the graph of $y = f(x)$ **to the left** by c units.

Reflecting Graphs (p. 214)

To graph $y = -f(x)$, **reflect** the graph of $y = f(x)$ in the **x-axis**.

To graph $y = f(-x)$, **reflect** the graph of $y = f(x)$ in the **y-axis**.

Vertical and Horizontal Stretching and Shrinking of Graphs (pp. 215, 216)

If $c > 1$, then to graph $y = cf(x)$, **stretch** the graph of $y = f(x)$ **vertically** by a factor of c.

If $0 < c < 1$, then to graph $y = cf(x)$, **shrink** the graph of $y = f(x)$ **vertically** by a factor of c.

If $c > 1$, then to graph $y = f(cx)$, **shrink** the graph of $y = f(x)$ **horizontally** by a factor of $1/c$.

If $0 < c < 1$, then to graph $y = f(cx)$, **stretch** the graph of $y = f(x)$ **horizontally** by a factor of $1/c$.

Even and Odd Functions (p. 217)

A function f is

even if $f(-x) = f(x)$

odd if $f(-x) = -f(x)$

for every x in the domain of f.

Composition of Functions (p. 225)

Given two functions f and g, the **composition** of f and g is the function $f \circ g$ defined by

$$(f \circ g)(x) = f(g(x))$$

The **domain** of $f \circ g$ is the set of all x for which both $g(x)$ and $f(g(x))$ are defined.

One-to-One Functions (p. 231)

A function f is **one-to-one** if $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 are *different* elements of the domain of f.

Horizontal Line test (p. 231)

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Inverse of a Function (p. 233)

Let f be a one-to-one function with domain A and range B.

The **inverse** of f is the function f^{-1} defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

The inverse function f^{-1} has domain B and range A.

The functions f and f^{-1} satisfy the following **cancellation properties**:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

LEARNING OBJECTIVES SUMMARY

Section	After completing this chapter, you should be able to . . .	Review Exercises
2.1	▪ Recognize functions in the real world	7–8
	▪ Work with function notation	9–10
	▪ Evaluate functions	5–10
	▪ Find net change	7–8, 11–12, 14, 57–60
	▪ Find domains of functions	15–24
	▪ Represent functions verbally, algebraically, graphically, and numerically	1–6, 25–42
2.2	▪ Graph functions by plotting points	25–42
	▪ Graph functions with a graphing calculator	47–54
	▪ Graph piecewise defined functions	39–42
	▪ Use the Vertical Line Test	13
	▪ Determine whether an equation defines a function	43–46
2.3	▪ Find function values from a graph	14
	▪ Find domain and range from a graph	14, 49–54
	▪ Find where a function is increasing or decreasing from a graph	14, 55–56
	▪ Find local maxima and minima from a graph	14, 69–74
2.4	▪ Find average rates of change	57–60
	▪ Interpret average rate of change in real-world situations	61–62
	▪ Recognize that a function with constant average rate of change is linear	63–64
2.5	▪ Shift graphs vertically	65–66
	▪ Shift graphs horizontally	65–66
	▪ Stretch or shrink graphs vertically	65–66
	▪ Stretch or shrink graphs horizontally	65–66
	▪ Reflect graphs	65–66
	▪ Determine whether a function is even or odd	67–68
2.6	▪ Find sums, differences, products, and quotients of functions	77
	▪ Add functions graphically	75–76
	▪ Find the composition of two functions	77–80
	▪ Express a given function as a composite function	81–82
2.7	▪ Determine whether a function is one-to-one	13–14, 83–88, 93–94
	▪ Find the inverse of a one-to-one function	89–92, 93–96
	▪ Draw the graph of the inverse of a function	93–96

EXERCISES

1–2 ■ A verbal description of a function f is given. Find a formula that expresses f in function notation.

1. "Square, then subtract 5."
2. "Divide by 2, then add 9."

3–4 ■ A formula for a function f is given. Give a verbal description of the function.

3. $f(x) = 3(x + 10)$
4. $f(x) = \sqrt{6x - 10}$

5–6 ■ Complete the table of values for the given function.

5. $g(x) = x^2 - 4x$

x	$g(x)$
-1	
0	
1	
2	
3	

6. $h(x) = 3x^2 + 2x - 5$

x	$h(x)$
-2	
-1	
0	
1	
2	

7. A publisher estimates that the cost $C(x)$ of printing a run of x copies of a certain mathematics textbook is given by the function $C(x) = 5000 + 30x - 0.001x^2$.
   (a) Find $C(1000)$ and $C(10{,}000)$.
   (b) What do your answers in part (a) represent?
   (c) Find $C(0)$. What does this number represent?
   (d) Find the net change in the cost C as x changes from 1000 to 10,000.

8. Reynalda works as a salesperson in the electronics division of a department store. She earns a base weekly salary plus a commission based on the retail price of the goods she has sold. If she sells x dollars worth of goods in a week, her earnings for that week are given by the function $E(x) = 400 + 0.03x$.
   (a) Find $E(2000)$ and $E(15{,}000)$.
   (b) What do your answers in part (a) represent?
   (c) Find $E(0)$. What does this number represent?
   (d) Find the net change in earnings E as x changes from 2000 to 15,000.
   (e) From the formula for E, determine what percentage Reynalda earns on the goods that she sells.

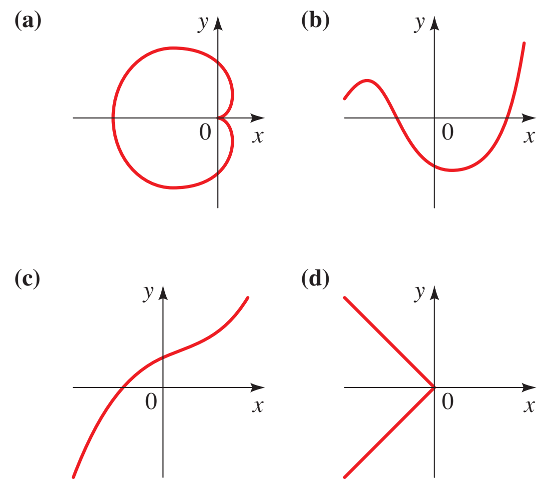
9. If $f(x) = x^2 - 4x + 6$, find $f(0)$, $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(x + 1)$, and $f(2x)$.

10. If $f(x) = 4 - \sqrt{3x - 6}$, find $f(5)$, $f(9)$, $f(a + 2)$, $f(-x)$, and $f(x^2)$.

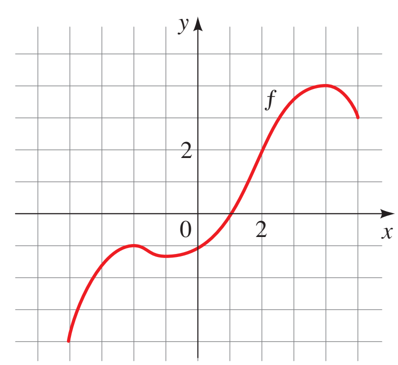
11–12 ■ Find the net change in the value of the function between the given inputs.

11. $f(x) = x^4 - 3x^2$; from -2 to 1
12. $f(x) = x^4 - 3x^2$; from 0 to 1

13. Which of the following figures are graphs of functions? Which of the functions are one-to-one?



14. A graph of a function f is given.
   (a) Find $f(-2)$ and $f(2)$.
   (b) Find the net change in the value of f from $x = -2$ to $x = 2$.
   (c) Find the domain and range of f.
   (d) On what intervals is f increasing? On what intervals is f decreasing?
   (e) What are the local maximum values of f?
   (f) Is f one-to-one?



15–16 ■ Find the domain and range of the function.

15. $f(x) = \sqrt{x + 3}$
16. $F(t) = t^2 + 2t + 5$

17–24 ■ Find the domain of the function.

17. $f(x) = 7x + 15$
18. $f(x) = \dfrac{2x + 1}{2x - 1}$
19. $f(x) = \sqrt{x + 4}$
20. $f(x) = 3x - \dfrac{2}{\sqrt{x + 1}}$
21. $f(x) = \dfrac{1}{x} + \dfrac{1}{x + 1} + \dfrac{1}{x + 2}$
22. $g(x) = \dfrac{2x^2 + 5x + 3}{2x^2 - 5x - 3}$
23. $h(x) = \sqrt{4 - x} + \sqrt{x^2 - 1}$
24. $f(x) = \dfrac{\sqrt[3]{2x + 1}}{\sqrt[3]{2x + 2}}$

25–42 ■ Sketch the graph of the function.

25. $f(x) = 1 - 2x$

26. $f(x) = \frac{1}{3}(x - 5),\ 2 \le x \le 8$

27. $f(t) = 1 - \frac{1}{2}t^2$

28. $g(t) = t^2 - 2t$

29. $f(x) = x^2 - 6x + 6$

30. $f(x) = 3 - 8x - 2x^2$

31. $g(x) = 1 - \sqrt{x}$

32. $g(x) = -|x|$

33. $h(x) = \frac{1}{2}x^3$

34. $h(x) = \sqrt{x + 3}$

35. $h(x) = \sqrt[3]{x}$

36. $H(x) = x^3 - 3x^2$

37. $g(x) = \dfrac{1}{x^2}$

38. $G(x) = \dfrac{1}{(x - 3)^2}$

39. $f(x) = \begin{cases} 1 - x & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$

40. $f(x) = \begin{cases} 1 - 2x & \text{if } x \le 0 \\ 2x - 1 & \text{if } x > 0 \end{cases}$

41. $f(x) = \begin{cases} x + 6 & \text{if } x < -2 \\ x^2 & \text{if } x \ge -2 \end{cases}$

42. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$

43–46 ■ Determine whether the equation defines y as a function of x.

43. $x + y^2 = 14$

44. $3x - \sqrt{y} = 8$

45. $x^3 - y^3 = 27$

46. $2x = y^4 - 16$

47. Determine which viewing rectangle produces the most appropriate graph of the function

$$f(x) = 6x^3 - 15x^2 + 4x - 1$$

(i) $[-2, 2]$ by $[-2, 2]$ (ii) $[-8, 8]$ by $[-8, 8]$
(iii) $[-4, 4]$ by $[-12, 12]$ (iv) $[-100, 100]$ by $[-100, 100]$

48. Determine which viewing rectangle produces the most appropriate graph of the function $f(x) = \sqrt{100 - x^3}$.
(i) $[-4, 4]$ by $[-4, 4]$
(ii) $[-10, 10]$ by $[-10, 10]$
(iii) $[-10, 10]$ by $[-10, 40]$
(iv) $[-100, 100]$ by $[-100, 100]$

49–54 ■ A function f is given. **(a)** Use a graphing calculator to draw the graph of f. **(b)** Find the domain and range of f from the graph.

49. $f(x) = \sqrt{9 - x^2}$

50. $f(x) = -\sqrt{x^2 - 3}$

51. $f(x) = -2,\ -2 \le x \le 3$

52. $f(x) = \sqrt{x + 3}$

53. $f(x) = \sqrt{x^3 - 4x + 1}$

54. $f(x) = x^4 - x^3 + x^2 + 3x - 6$

55–56 ■ Draw a graph of the function f, and determine the intervals on which f is increasing and on which f is decreasing.

55. $f(x) = x^3 - 4x^2$

56. $f(x) = |x^4 - 16|$

57–60 ■ Find the net change and the average rate of change of the function between the given points.

57. $f(x) = x^2 + 3x;\quad x = 0, x = 2$

58. $f(x) = \dfrac{1}{x - 2};\quad x = 4, x = 8$

59. $f(x) = \dfrac{1}{x};\quad x = 3, x = 3 + h$

60. $f(x) = (x + 1)^2;\quad x = a, x = a + h$

61. The population of a planned seaside community in Florida is given by the function $P(t) = 3000 + 200t + 0.1t^2$, where t represents the number of years since the community was incorporated in 1985.
(a) Find $P(10)$ and $P(20)$. What do these values represent?
(b) Find the average rate of change of P between $t = 10$ and $t = 20$. What does this number represent?

62. Ella is saving for her retirement by making regular deposits into a 401(k) plan. As her salary rises, she finds that she can deposit increasing amounts each year. Between 1995 and 2008, the annual amount (in dollars) that she deposited was given by the function $D(t) = 3500 + 15t^2$, where t represents the year of the deposit measured from the start of the plan (so 1995 corresponds to $t = 0$ and 1996 corresponds to $t = 1$, and so on).
(a) Find $D(0)$ and $D(15)$. What do these values represent?
(b) Assuming that her deposits continue to be modeled by the function D, in what year will she deposit $17,000?
(c) Find the average rate of change of D between $t = 0$ and $t = 15$. What does this number represent?

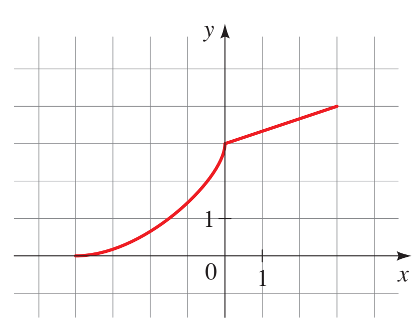
63–64 ■ A function f is given. **(a)** Find the average rate of change of f between $x = 0$ and $x = 2$, and the average rate of change of f between $x = 15$ and $x = 50$. **(b)** Were the two average rates of change that you found in part (a) the same? Explain why or why not.

63. $f(x) = \frac{1}{2}x - 6$

64. $f(x) = 8 - 3x$

65. Suppose the graph of f is given. Describe how the graphs of the following functions can be obtained from the graph of f.
(a) $y = f(x) + 8$ **(b)** $y = f(x + 8)$
(c) $y = 1 + 2f(x)$ **(d)** $y = f(x - 2) - 2$
(e) $y = f(-x)$ **(f)** $y = -f(-x)$
(g) $y = -f(x)$ **(h)** $y = f^{-1}(x)$

66. The graph of f is given. Draw the graphs of the following functions.
(a) $y = f(x - 2)$ **(b)** $y = -f(x)$
(c) $y = 3 - f(x)$ **(d)** $y = \frac{1}{2}f(x) - 1$
(e) $y = f^{-1}(x)$ **(f)** $y = f(-x)$



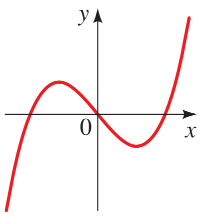
67. Determine whether f is even, odd, or neither.

(a) $f(x) = 2x^5 - 3x^2 + 2$ **(b)** $f(x) = x^3 - x^7$

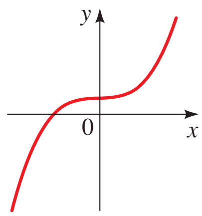
(c) $f(x) = \dfrac{1 - x^2}{1 + x^2}$ **(d)** $f(x) = \dfrac{1}{x + 2}$

68. Determine whether the function in the figure is even, odd, or neither.

(a)



(b)



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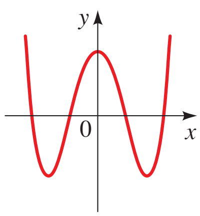

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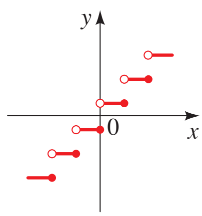


69. Find the minimum value of the function $g(x) = 2x^2 + 4x - 5$.

70. Find the maximum value of the function $f(x) = 1 - x - x^2$.

71. A stone is thrown upward from the top of a building. Its height (in feet) above the ground after t seconds is given by

$$h(t) = -16t^2 + 48t + 32$$

What maximum height does it reach?

72. The profit P (in dollars) generated by selling x units of a certain commodity is given by

$$P(x) = -1500 + 12x - 0.0004x^2$$

What is the maximum profit, and how many units must be sold to generate it?

73–74 ■ Find the local maximum and minimum values of the function and the values of x at which they occur. State each answer rounded to two decimal places.

73. $f(x) = 3.3 + 1.6x - 2.5x^3$

74. $f(x) = x^{2/3}(6 - x)^{1/3}$

75–76 ■ Two functions, f and g, are given. Draw graphs of f, g, and $f + g$ on the same graphing calculator screen to illustrate the concept of graphical addition.

75. $f(x) = x + 2, \quad g(x) = x^2$

76. $f(x) = x^2 + 1, \quad g(x) = 3 - x^2$

77. If $f(x) = x^2 - 3x + 2$ and $g(x) = 4 - 3x$, find the following functions.

(a) $f + g$ **(b)** $f - g$ **(c)** fg

(d) f/g **(e)** $f \circ g$ **(f)** $g \circ f$

78. If $f(x) = 1 + x^2$ and $g(x) = \sqrt{x - 1}$, find the following.

(a) $f \circ g$ **(b)** $g \circ f$ **(c)** $(f \circ g)(2)$

(d) $(f \circ f)(2)$ **(e)** $f \circ g \circ f$ **(f)** $g \circ f \circ g$

79–80 ■ Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

79. $f(x) = 3x - 1, \quad g(x) = 2x - x^2$

80. $f(x) = \sqrt{x}, \quad g(x) = \dfrac{2}{x - 4}$

81. Find $f \circ g \circ h$, where $f(x) = \sqrt{1 - x}$, $g(x) = 1 - x^2$, and $h(x) = 1 + \sqrt{x}$.

82. If $T(x) = \dfrac{1}{\sqrt{1 + \sqrt{x}}}$, find functions f, g, and h such that $f \circ g \circ h = T$.

83–88 ■ Determine whether the function is one-to-one.

83. $f(x) = 3 + x^3$

84. $g(x) = 2 - 2x + x^2$

85. $h(x) = \dfrac{1}{x^4}$

86. $r(x) = 2 + \sqrt{x + 3}$

87. $p(x) = 3.3 + 1.6x - 2.5x^3$

88. $q(x) = 3.3 + 1.6x + 2.5x^3$

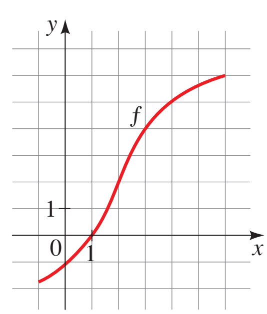
89–92 ■ Find the inverse of the function.

89. $f(x) = 3x - 2$ **90.** $f(x) = \dfrac{2x + 1}{3}$

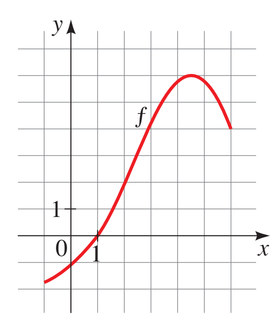
91. $f(x) = (x + 1)^3$ **92.** $f(x) = 1 + \sqrt[5]{x - 2}$

93–94 ■ A graph of a function f is given. Does f have an inverse? If so, find $f^{-1}(0)$ and $f^{-1}(4)$.

93.



94.



95. (a) Sketch the graph of the function

$$f(x) = x^2 - 4 \qquad x \geq 0$$

(b) Use part (a) to sketch the graph of f^{-1}.

(c) Find an equation for f^{-1}.

96. (a) Show that the function $f(x) = 1 + \sqrt[4]{x}$ is one-to-one.

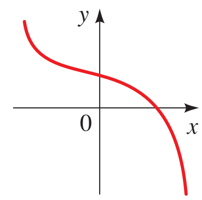
(b) Sketch the graph of f.

(c) Use part (b) to sketch the graph of f^{-1}.

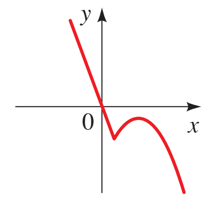
(d) Find an equation for f^{-1}.

1. Which of the following are graphs of functions? If the graph is that of a function, is it one-to-one?

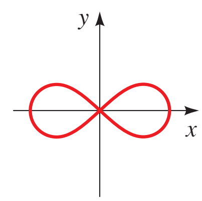
(a)



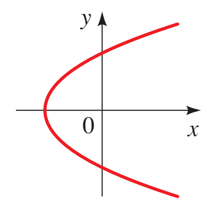
(b)



(c)



(d)



2. Let $f(x) = \dfrac{\sqrt{x + 1}}{x}$.

 (a) Evaluate $f(3)$, $f(5)$, and $f(a - 1)$.

 (b) Find the domain of f.

3. A function f has the following verbal description: "Subtract 2, then cube the result."

 (a) Find a formula that expresses f algebraically.

 (b) Make a table of values of f, for the inputs -1, 0, 1, 2, 3, and 4.

 (c) Sketch a graph of f, using the table of values from part (b) to help you.

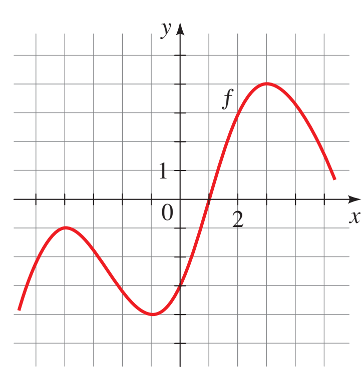
 (d) How do we know that f has an inverse? Give a verbal description for f^{-1}.

 (e) Find a formula that expresses f^{-1} algebraically.

4. A graph of a function f is given.

 (a) Find the local minimum and maximum values of f and the values of x at which they occur.

 (b) Find the intervals on which f is increasing and on which f is decreasing.



5. A school fund-raising group sells chocolate bars to help finance a swimming pool for their physical education program. The group finds that when they set their price at x dollars per bar (where $0 < x \le 5$), their total sales revenue (in dollars) is given by the function $R(x) = -500x^2 + 3000x$.

 (a) Evaluate $R(2)$ and $R(4)$. What do these values represent?

 (b) Use a graphing calculator to draw a graph of R. What does the graph tell us about what happens to revenue as the price increases from 0 to 5 dollars?

 (c) What is the maximum revenue, and at what price is it achieved?

6. Determine the net change and the average rate of change for the function $f(t) = t^2 - 2t$ between $t = 2$ and $t = 5$.

7. **(a)** Sketch the graph of the function $f(x) = x^3$.

(b) Use part (a) to graph the function $g(x) = (x - 1)^3 - 2$.

8. **(a)** How is the graph of $y = f(x - 3) + 2$ obtained from the graph of f?

(b) How is the graph of $y = f(-x)$ obtained from the graph of f?

9. Let $f(x) = \begin{cases} 1 - x & \text{if } x \le 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$

(a) Evaluate $f(-2)$ and $f(1)$.

(b) Sketch the graph of f.

10. If $f(x) = x^2 + x + 1$ and $g(x) = x - 3$, find the following.

(a) $f + g$

(b) $f - g$

(c) $f \circ g$

(d) $g \circ f$

(e) $f(g(2))$

(f) $g(f(2))$

(g) $g \circ g \circ g$

11. **(a)** If $f(x) = \sqrt{3 - x}$, find the inverse function f^{-1}.

(b) Sketch the graphs of f and f^{-1} on the same coordinate axes.

12–17 ■ A graph of a function f is given.

12. Find the domain and range of f.

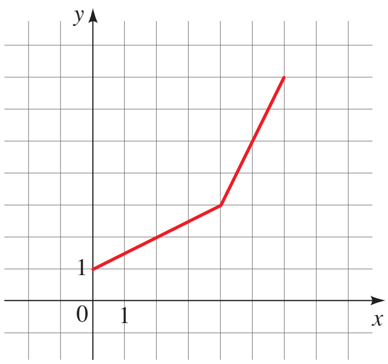
13. Find $f(0)$ and $f(4)$.

14. Graph $f(x - 2)$ and $f(x) + 2$.

15. Find the net change and the average rate of change of f between $x = 2$ and $x = 6$.

16. Find $f^{-1}(1)$ and $f^{-1}(3)$.

17. Sketch the graph of f^{-1}.



18. Let $f(x) = 3x^4 - 14x^2 + 5x - 3$.

(a) Draw the graph of f in an appropriate viewing rectangle.

(b) Is f one-to-one?

(c) Find the local maximum and minimum values of f and the values of x at which they occur. State each answer correct to two decimal places.

(d) Use the graph to determine the range of f.

(e) Find the intervals on which f is increasing and on which f is decreasing.

Modeling with Functions

Many of the processes that are studied in the physical and social sciences involve understanding how one quantity varies with respect to another. Finding a function that describes the dependence of one quantity on another is called *modeling*. For example, a biologist observes that the number of bacteria in a certain culture increases with time. He tries to model this phenomenon by finding the precise function (or rule) that relates the bacteria population to the elapsed time.

In this *Focus on Modeling* we will learn how to find models that can be constructed using geometric or algebraic properties of the object under study. Once the model is found, we use it to analyze and predict properties of the object or process being studied.

▼ Modeling with Functions

We begin with a simple real-life situation that illustrates the modeling process.

EXAMPLE 1 | Modeling the Volume of a Box

A breakfast cereal company manufactures boxes to package their product. For aesthetic reasons, the box must have the following proportions: Its width is 3 times its depth, and its height is 5 times its depth.

(a) Find a function that models the volume of the box in terms of its depth.

(b) Find the volume of the box if the depth is 1.5 in.

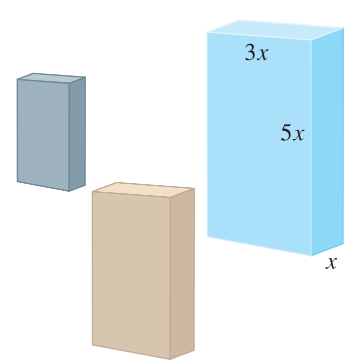
(c) For what depth is the volume 90 in^3?

(d) For what depth is the volume greater than 60 in^3?

THINKING ABOUT THE PROBLEM

Let's experiment with the problem. If the depth is 1 in., then the width is 3 in. and the height is 5 in. So in this case, the volume is $V = 1 \times 3 \times 5 = 15 \text{ in}^3$. The table gives other values. Notice that all the boxes have the same shape, and the greater the depth, the greater the volume.

Depth	Volume
1	$1 \times 3 \times 5 = 15$
2	$2 \times 6 \times 10 = 120$
3	$3 \times 9 \times 15 = 405$
4	$4 \times 12 \times 20 = 96$



SOLUTION

(a) To find the function that models the volume of the box, we use the following steps.

▶ Express the Model in Words

We know that the volume of a rectangular box is

$$\boxed{\text{volume}} = \boxed{\text{depth}} \times \boxed{\text{width}} \times \boxed{\text{height}}$$

▶ Choose the Variable

There are three varying quantities: width, depth, and height. Because the function we want depends on the depth, we let

$$x = \text{depth of the box}$$

Then we express the other dimensions of the box in terms of x:

In Words	In Algebra
Depth	x
Width	$3x$
Height	$5x$

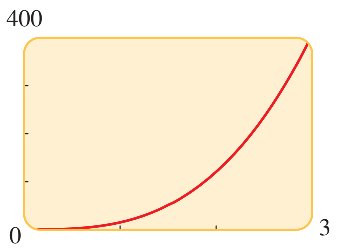


FIGURE 1

▶ Set Up the Model

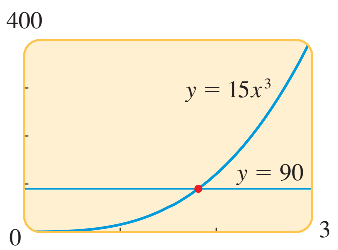
The model is the function V that gives the volume of the box in terms of the depth x.

$$\boxed{\text{volume}} = \boxed{\text{depth}} \times \boxed{\text{width}} \times \boxed{\text{height}}$$

$$V(x) = x \cdot 3x \cdot 5x$$

$$V(x) = 15x^3$$

The volume of the box is modeled by the function $V(x) = 15x^3$. The function V is graphed in Figure 1.



$15x^3 = 90$

FIGURE 2

▶ Use the Model

We use the model to answer the questions in parts (b), (c), and (d).

(b) If the depth is 1.5 in., the volume is $V(1.5) = 15(1.5)^3 = 50.625\text{ in}^3$.

(c) We need to solve the equation $V(x) = 90$ or

$$15x^3 = 90$$

$$x^3 = 6$$

$$x = \sqrt[3]{6} \approx 1.82\text{ in.}$$

The volume is 90 in^3 when the depth is about 1.82 in. (We can also solve this equation graphically, as shown in Figure 2.)

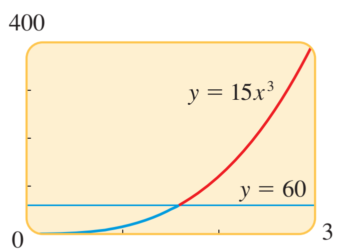
(d) We need to solve the inequality $V(x) \geq 60$ or

$$15x^3 \geq 60$$

$$x^3 \geq 4$$

$$x \geq \sqrt[3]{4} \approx 1.59$$

The volume will be greater than 60 in^3 if the depth is greater than 1.59 in. (We can also solve this inequality graphically, as shown in Figure 3.) ■



$15x^3 \geq 60$

FIGURE 3

The steps in Example 1 are typical of how we model with functions. They are summarized in the following box.

GUIDELINES FOR MODELING WITH FUNCTIONS

1. **Express the Model in Words.** Identify the quantity you want to model, and express it, in words, as a function of the other quantities in the problem.
2. **Choose the Variable.** Identify all the variables that are used to express the function in Step 1. Assign a symbol, such as x, to one variable, and express the other variables in terms of this symbol.
3. **Set up the Model.** Express the function in the language of algebra by writing it as a function of the single variable chosen in Step 2.
4. **Use the Model.** Use the function to answer the questions posed in the problem. (To find a maximum or a minimum, use the methods described in Section 2.3.)

EXAMPLE 2 | Fencing a Garden

A gardener has 140 feet of fencing to fence in a rectangular vegetable garden.

(a) Find a function that models the area of the garden she can fence.

(b) For what range of widths is the area greater than 825 ft^2?

(c) Can she fence a garden with area 1250 ft^2?

(d) Find the dimensions of the largest area she can fence.

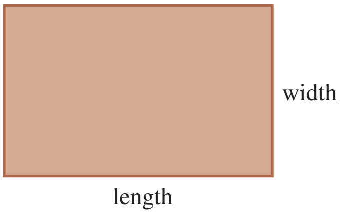
THINKING ABOUT THE PROBLEM

If the gardener fences a plot with width 10 ft, then the length must be 60 ft, because $10 + 10 + 60 + 60 = 140$. So the area is

$$A = \text{width} \times \text{length} = 10 \cdot 60 = 600 \text{ ft}^2$$

The table shows various choices for fencing the garden. We see that as the width increases, the fenced area increases, then decreases.

Width	Length	Area
10	60	600
20	50	1000
30	40	1200
40	30	1200
50	20	1000
60	10	600



SOLUTION

(a) The model that we want is a function that gives the area she can fence.

▶ **Express the Model in Words**

We know that the area of a rectangular garden is

$$\boxed{\text{area}} = \boxed{\text{width}} \times \boxed{\text{length}}$$

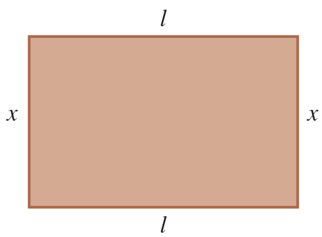


FIGURE 4

▶ Choose the Variable

There are two varying quantities: width and length. Because the function we want depends on only one variable, we let

$$x = \text{width of the garden}$$

Then we must express the length in terms of x. The perimeter is fixed at 140 ft, so the length is determined once we choose the width. If we let the length be l, as in Figure 4, then $2x + 2l = 140$, so $l = 70 - x$. We summarize these facts:

In Words	In Algebra
Width	x
Length	$70 - x$

▶ Set Up the Model

The model is the function A that gives the area of the garden for any width x.

$$\text{area} = \text{width} \times \text{length}$$

$$A(x) = x(70 - x)$$

$$A(x) = 70x - x^2$$

The area that she can fence is modeled by the function $A(x) = 70x - x^2$.

▶ Use the Model

We use the model to answer the questions in parts (b)–(d).

(b) We need to solve the inequality $A(x) \geq 825$. To solve graphically, we graph $y = 70x - x^2$ and $y = 825$ in the same viewing rectangle (see Figure 5). We see that $15 \leq x \leq 55$.

(c) From Figure 6 we see that the graph of $A(x)$ always lies below the line $y = 1250$, so an area of 1250 ft^2 is never attained.

Maximum values of functions are discussed on page 198.

(d) We need to find where the maximum value of the function $A(x) = 70x - x^2$ occurs. The function is graphed in Figure 7. Using the $\boxed{\text{TRACE}}$ feature on a graphing calculator, we find that the function achieves its maximum value at $x = 35$. So the maximum area that she can fence is that when the garden's width is 35 ft and its length is $70 - 35 = 35$ ft. The maximum area then is $35 \times 35 = 1225$ ft^2.

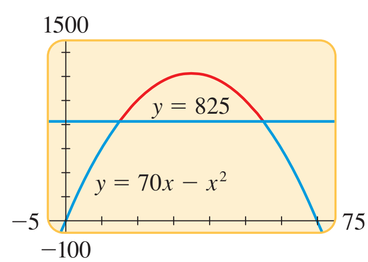


FIGURE 5

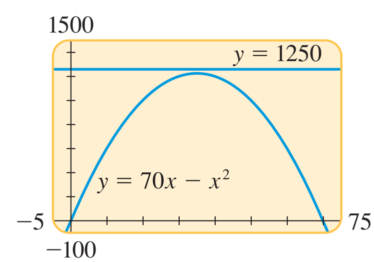


FIGURE 6

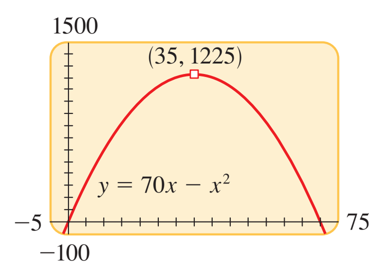


FIGURE 7

■

EXAMPLE 3 | Minimizing the Metal in a Can

A manufacturer makes a metal can that holds 1 L (liter) of oil. What radius minimizes the amount of metal in the can?

THINKING ABOUT THE PROBLEM

To use the least amount of metal, we must minimize the surface area of the can, that is, the area of the top, bottom, and the sides. The area of the top and bottom is $2\pi r^2$ and the area of the sides is $2\pi rh$ (see Figure 8), so the surface area of the can is

$$S = 2\pi r^2 + 2\pi rh$$

The radius and height of the can must be chosen so that the volume is exactly 1 L, or 1000 cm^3. If we want a small radius, say $r = 3$, then the height must be just tall enough to make the total volume 1000 cm^3. In other words, we must have

$$\pi(3)^2 h = 1000 \qquad \text{Volume of the can is } \pi r^2 h$$

$$h = \frac{1000}{9\pi} \approx 35.4 \text{ cm} \qquad \text{Solve for } h$$

Now that we know the radius and height, we can find the surface area of the can:

$$\text{surface area} = 2\pi(3)^2 + 2\pi(3)(35.4) \approx 723.8 \text{ cm}^3$$

If we want a different radius, we can find the corresponding height and surface area in a similar fashion.

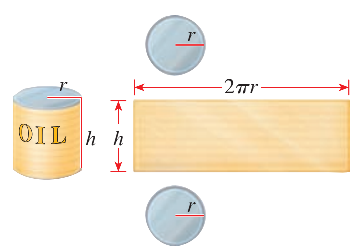


FIGURE 8

SOLUTION The model that we want is a function that gives the surface area of the can.

▶ Express the Model in Words

We know that for a cylindrical can

surface area	=	area of top and bottom	+	area of sides

▶ Choose the Variable

There are two varying quantities: radius and height. Because the function we want depends on the radius, we let

$$r = \text{radius of can}$$

Next, we must express the height in terms of the radius r. Because the volume of a cylindrical can is $V = \pi r^2 h$ and the volume must be 1000 cm^3, we have

$$\pi r^2 h = 1000 \qquad \text{Volume of can is 1000 cm}^3$$

$$h = \frac{1000}{\pi r^2} \qquad \text{Solve for } h$$

We can now express the areas of the top, bottom, and sides in terms of r only:

In Words	In Algebra
Radius of can	r
Height of can	$\dfrac{1000}{\pi r^2}$
Area of top and bottom	$2\pi r^2$
Area of sides ($2\pi rh$)	$2\pi r\left(\dfrac{1000}{\pi r^2}\right)$

▶ Set Up the Model

The model is the function S that gives the surface area of the can as a function of the radius r.

$$\boxed{\text{surface area}} = \boxed{\text{area of top and bottom}} + \boxed{\text{area of sides}}$$

$$S(r) = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$S(r) = 2\pi r^2 + \frac{2000}{r}$$

▶ Use the Model

We use the model to find the minimum surface area of the can. We graph S in Figure 9 and zoom in on the minimum point to find that the minimum value of S is about 554 cm^2 and occurs when the radius is about 5.4 cm.

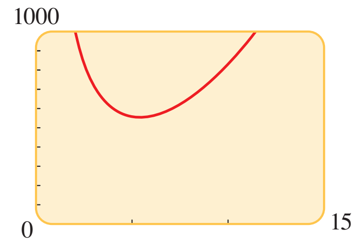


FIGURE 9 $S(r) = 2\pi r^2 + \dfrac{2000}{r}$

PROBLEMS

1–18 ■ In these problems you are asked to find a function that models a real-life situation. Use the principles of modeling described in this *Focus* to help you.

1. **Area** A rectangular building lot is three times as long as it is wide. Find a function that models its area A in terms of its width w.

2. **Area** A poster is 10 inches longer than it is wide. Find a function that models its area A in terms of its width w.

3. **Volume** A rectangular box has a square base. Its height is half the width of the base. Find a function that models its volume V in terms of its width w.

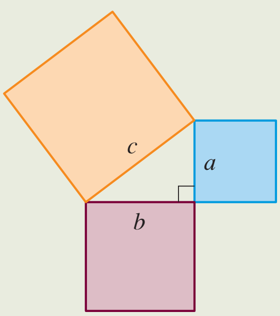
4. **Volume** The height of a cylinder is four times its radius. Find a function that models the volume V of the cylinder in terms of its radius r.

5. **Area** A rectangle has a perimeter of 20 ft. Find a function that models its area A in terms of the length x of one of its sides.

6. **Perimeter** A rectangle has an area of 16 m^2. Find a function that models its perimeter P in terms of the length x of one of its sides.

PYTHAGORAS (circa 580–500 B.C.) founded a school in Croton in southern Italy, devoted to the study of arithmetic, geometry, music, and astronomy. The Pythagoreans, as they were called, were a secret society with peculiar rules and initiation rites. They wrote nothing down and were not to reveal to anyone what they had learned from the Master. Although women were barred by law from attending public meetings, Pythagoras allowed women in his school, and his most famous student was Theano (whom he later married).

According to Aristotle, the Pythagoreans were convinced that "the principles of mathematics are the principles of all things." Their motto was "Everything is Number," by which they meant *whole* numbers. The outstanding contribution of Pythagoras is the theorem that bears his name: In a right triangle the area of the square on the hypotenuse is equal to the sum of the areas of the square on the other two sides.



$$c^2 = a^2 + b^2$$

The converse of Pythagoras's Theorem is also true; that is, a triangle whose sides a, b, and c satisfy $a^2 + b^2 = c^2$ is a right triangle.

7. Area Find a function that models the area A of an equilateral triangle in terms of the length x of one of its sides.

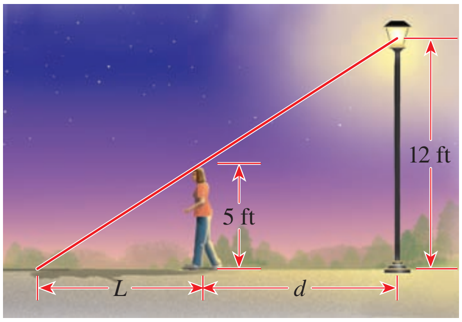
8. Area Find a function that models the surface area S of a cube in terms of its volume V.

9. Radius Find a function that models the radius r of a circle in terms of its area A.

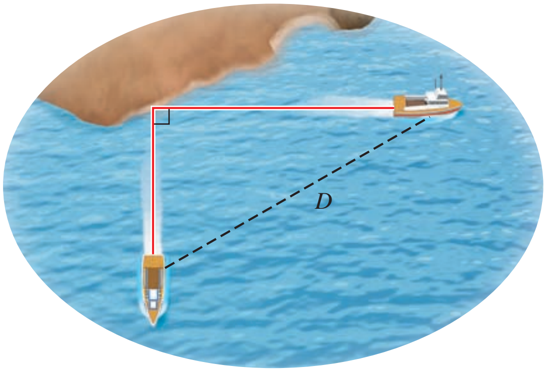
10. Area Find a function that models the area A of a circle in terms of its circumference C.

11. Area A rectangular box with a volume of 60 ft^3 has a square base. Find a function that models its surface area S in terms of the length x of one side of its base.

12. Length A woman 5 ft tall is standing near a street lamp that is 12 ft tall, as shown in the figure. Find a function that models the length L of her shadow in terms of her distance d from the base of the lamp.



13. Distance Two ships leave port at the same time. One sails south at 15 mi/h, and the other sails east at 20 mi/h. Find a function that models the distance D between the ships in terms of the time t (in hours) elapsed since their departure.

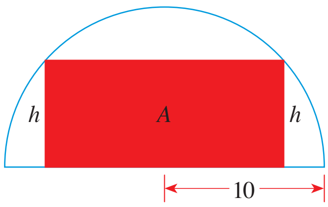


14. Product The sum of two positive numbers is 60. Find a function that models their product P in terms of x, one of the numbers.

15. Area An isosceles triangle has a perimeter of 8 cm. Find a function that models its area A in terms of the length of its base b.

16. Perimeter A right triangle has one leg twice as long as the other. Find a function that models its perimeter P in terms of the length x of the shorter leg.

17. Area A rectangle is inscribed in a semicircle of radius 10, as shown in the figure. Find a function that models the area A of the rectangle in terms of its height h.



18. Height The volume of a cone is 100 in^3. Find a function that models the height h of the cone in terms of its radius r.

19–32 ■ In these problems you are asked to find a function that models a real-life situation, and then use the model to answer questions about the situation. Use the guidelines on page 249 to help you.

19. Maximizing a Product Consider the following problem: Find two numbers whose sum is 19 and whose product is as large as possible.

(a) Experiment with the problem by making a table like the one following, showing the product of different pairs of numbers that add up to 19. On the basis of the evidence in your table, estimate the answer to the problem.

(b) Find a function that models the product in terms of one of the two numbers.

(c) Use your model to solve the problem, and compare with your answer to part (a).

20. Minimizing a Sum Find two positive numbers whose sum is 100 and the sum of whose squares is a minimum.

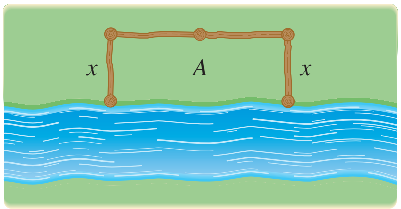
21. Fencing a Field Consider the following problem: A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He does not need a fence along the river (see the figure). What are the dimensions of the field of largest area that he can fence?

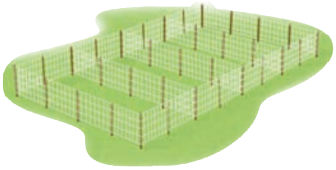
(a) Experiment with the problem by drawing several diagrams illustrating the situation. Calculate the area of each configuration, and use your results to estimate the dimensions of the largest possible field.

First number	Second number	Product
1	18	18
2	17	34
3	16	48
⋮	⋮	⋮

(b) Find a function that models the area of the field in terms of one of its sides.

(c) Use your model to solve the problem, and compare with your answer to part (a).





22. Dividing a Pen A rancher with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle (see the figure).

(a) Find a function that models the total area of the four pens.

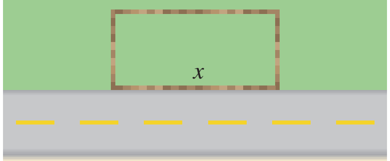
(b) Find the largest possible total area of the four pens.

23. Fencing a Garden Plot A property owner wants to fence a garden plot adjacent to a road, as shown in the figure. The fencing next to the road must be sturdier and costs \$5 per foot, but the other fencing costs just \$3 per foot. The garden is to have an area of 1200 ft^2.

(a) Find a function that models the cost of fencing the garden.

(b) Find the garden dimensions that minimize the cost of fencing.

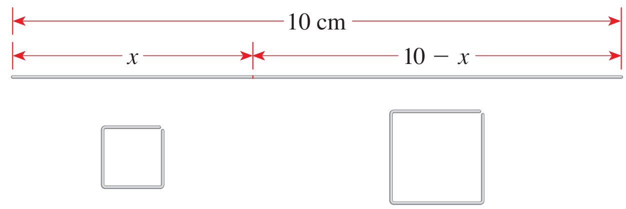
(c) If the owner has at most \$600 to spend on fencing, find the range of lengths he can fence along the road.



24. Maximizing Area A wire 10 cm long is cut into two pieces, one of length x and the other of length $10 - x$, as shown in the figure. Each piece is bent into the shape of a square.

(a) Find a function that models the total area enclosed by the two squares.

(b) Find the value of x that minimizes the total area of the two squares.



25. Light from a Window A Norman window has the shape of a rectangle surmounted by a semicircle, as shown in the figure to the left. A Norman window with perimeter 30 ft is to be constructed.

(a) Find a function that models the area of the window.

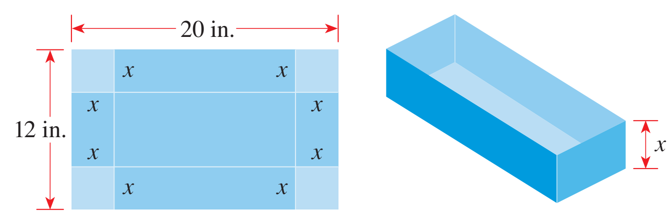
(b) Find the dimensions of the window that admits the greatest amount of light.

26. Volume of a Box A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides (see the figure).

(a) Find a function that models the volume of the box.

(b) Find the values of x for which the volume is greater than 200 in^3.

(c) Find the largest volume that such a box can have.

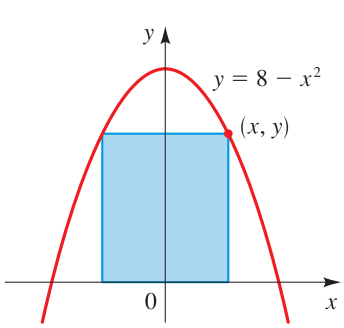


27. Area of a Box An open box with a square base is to have a volume of 12 ft^3.

(a) Find a function that models the surface area of the box.

(b) Find the box dimensions that minimize the amount of material used.

28. Inscribed Rectangle Find the dimensions that give the largest area for the rectangle shown in the figure. Its base is on the x-axis and its other two vertices are above the x-axis, lying on the parabola $y = 8 - x^2$.



29. Minimizing Costs A rancher wants to build a rectangular pen with an area of 100 m^2.

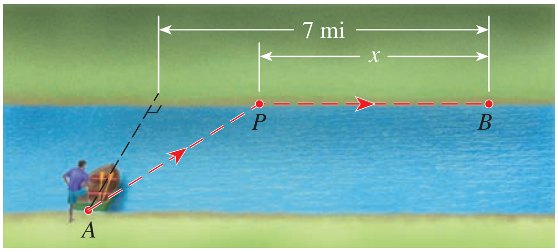
(a) Find a function that models the length of fencing required.

(b) Find the pen dimensions that require the minimum amount of fencing.

30. Minimizing Time A man stands at a point A on the bank of a straight river, 2 mi wide. To reach point B, 7 mi downstream on the opposite bank, he first rows his boat to point P on the opposite bank and then walks the remaining distance x to B, as shown in the figure. He can row at a speed of 2 mi/h and walk at a speed of 5 mi/h.

(a) Find a function that models the time needed for the trip.

(b) Where should he land so that he reaches B as soon as possible?



31. Bird Flight A bird is released from point A on an island, 5 mi from the nearest point B on a straight shoreline. The bird flies to a point C on the shoreline and then flies along the shoreline to its nesting area D (see the figure). Suppose the bird requires 10 kcal/mi of energy to fly over land and 14 kcal/mi to fly over water.

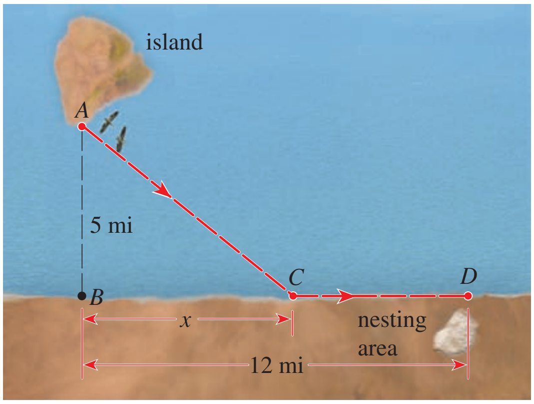
(a) Use the fact that

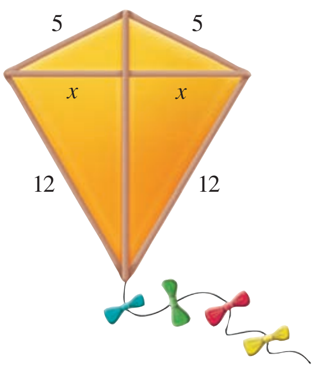
$$\text{energy used} = \text{energy per mile} \times \text{miles flown}$$

to show that the total energy used by the bird is modeled by the function

$$E(x) = 14\sqrt{x^2 + 25} + 10(12 - x)$$

(b) If the bird instinctively chooses a path that minimizes its energy expenditure, to what point does it fly?





32. Area of a Kite A kite frame is to be made from six pieces of wood. The four pieces that form its border have been cut to the lengths indicated in the figure. Let x be as shown in the figure.

(a) Show that the area of the kite is given by the function

$$A(x) = x\left(\sqrt{25 - x^2} + \sqrt{144 - x^2}\right)$$

(b) How long should each of the two crosspieces be to maximize the area of the kite?

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CHAPTER 3

Polynomial and Rational Functions

Modeling Variability Functions defined by polynomial expressions are called *polynomial functions.* The graphs of polynomial functions can have many peaks and valleys. This property makes them suitable models for many real-world situations. For example, a factory owner notices that if she increases the number of workers, productivity increases, but if there are too many workers, productivity begins to decrease. This situation is modeled by a polynomial function of degree 2 (a quadratic function). The growth of many animal species follows a predictable pattern, beginning with a period of rapid growth, followed by a period of slow growth and then a final growth spurt. This variability in growth is modeled by a polynomial of degree 3. Polynomials of higher degree are used for modeling situations with even more variability. We also study how to model real-world situations where direct or inverse variation is involved. Inverse variation is modeled by rational functions, which are functions defined by rational expressions.

In the *Focus on Modeling* at the end of this chapter we explore different ways of using polynomial functions to model real-world situations.

3.1 Quadratic Functions and Models

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Express quadratic functions in standard form ▶ Graph quadratic functions using the standard form ▶ Find maximum or minimum values of quadratic functions ▶ Model with quadratic functions

GET READY **Prepare for this section by reviewing Section 1.6 on how to solve quadratic equations.**

A polynomial function is a function that is defined by a polynomial expression. So a **polynomial function of degree *n*** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Polynomial expressions are defined in Section P.5.

We have already studied polynomial functions of degree 0 and 1. These are functions of the form $P(x) = a_0$ and $P(x) = a_1 x + a_0$, respectively, whose graphs are lines. In this section we study polynomial functions of degree 2. These are called quadratic functions.

QUADRATIC FUNCTIONS

A **quadratic function** is a polynomial function of degree 2. So a quadratic function is a function of the form

$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

We see in this section how quadratic functions model many real-world phenomena. We begin by analyzing the graphs of quadratic functions.

▼ Graphing Quadratic Functions Using the Standard Form

For a geometric definition of parabolas, see Section 7.1.

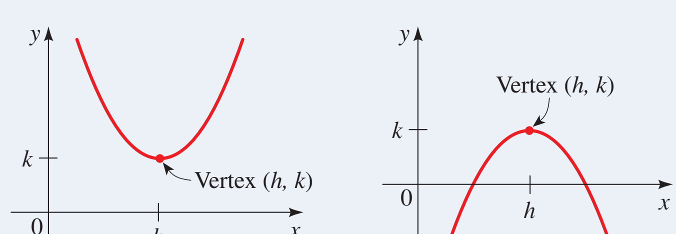
If we take $a = 1$ and $b = c = 0$ in the quadratic function $f(x) = ax^2 + bx + c$, we get the quadratic function $f(x) = x^2$, whose graph is the parabola graphed in Example 1 of Section 2.2. In fact, the graph of any quadratic function is a **parabola**; it can be obtained from the graph of $f(x) = x^2$ by the transformations given in Section 2.5.

STANDARD FORM OF A QUADRATIC FUNCTION

A quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **standard form**

$$f(x) = a(x - h)^2 + k$$

by completing the square. The graph of f is a parabola with **vertex** (h, k); the parabola opens upward if $a > 0$ or downward if $a < 0$.



$f(x) = a(x - h)^2 + k, \ a > 0$ $\qquad$ $f(x) = a(x - h)^2 + k, \ a < 0$

EXAMPLE 1 | Standard Form of a Quadratic Function

Let $f(x) = 2x^2 - 12x + 23$.

(a) Express f in standard form. **(b)** Sketch the graph of f.

SOLUTION

Completing the square is discussed in Section 1.6.

(a) Since the coefficient of x^2 is not 1, we must factor this coefficient from the terms involving x before we complete the square:

$$\begin{aligned} f(x) &= 2x^2 - 12x + 23 \\ &= 2(x^2 - 6x) + 23 && \text{Factor 2 from the } x\text{-terms} \\ &= 2(x^2 - 6x + 9) + 23 - 2 \cdot 9 && \text{Complete the square: Add 9 inside parentheses, subtract } 2 \cdot 9 \text{ outside} \\ &= 2(x - 3)^2 + 5 && \text{Factor and simplify} \end{aligned}$$

The standard form is $f(x) = 2(x - 3)^2 + 5$.

(b) The standard form tells us that we get the graph of f by taking the parabola $y = x^2$, shifting it to the right 3 units, stretching it by a factor of 2, and moving it upward 5 units. The vertex of the parabola is at $(3, 5)$, and the parabola opens upward. We sketch the graph in Figure 1 after noting that the y-intercept is $f(0) = 23$.

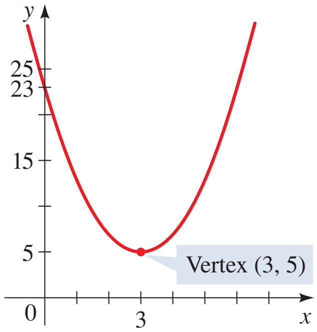


FIGURE 1 The graph of $f(x) = 2(x - 3)^2 + 5$, with vertex at $(3, 5)$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17**

Maximum and Minimum Values of Quadratic Functions

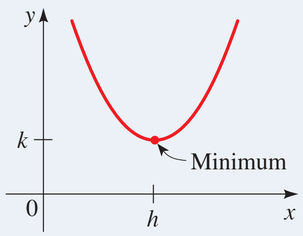
If a quadratic function has vertex (h, k), then the function has a minimum value at the vertex if its graph opens upward and a maximum value at the vertex if its graph opens downward. For example, the function graphed in Figure 1 has minimum value 5 when $x = 3$, since the vertex $(3, 5)$ is the lowest point on the graph.

MAXIMUM OR MINIMUM VALUE OF A QUADRATIC FUNCTION

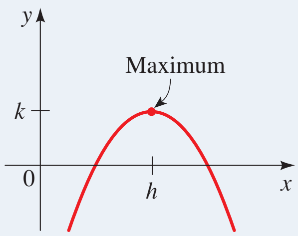
Let f be a quadratic function with standard form $f(x) = a(x - h)^2 + k$. The maximum or minimum value of f occurs at $x = h$.

If $a > 0$, then the **minimum value** of f is $f(h) = k$.

If $a < 0$, then the **maximum value** of f is $f(h) = k$.



$f(x) = a(x - h)^2 + k,\ a > 0$



$f(x) = a(x - h)^2 + k,\ a < 0$

EXAMPLE 2 | Minimum Value of a Quadratic Function

Consider the quadratic function $f(x) = 5x^2 - 30x + 49$.

(a) Express f in standard form.

(b) Sketch the graph of f.

(c) Find the minimum value of f.

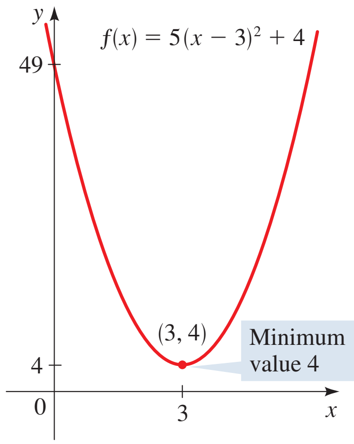


FIGURE 2

SOLUTION

(a) To express this quadratic function in standard form, we complete the square:

$$\begin{aligned} f(x) &= 5x^2 - 30x + 49 \\ &= 5(x^2 - 6x) + 49 && \text{Factor 5 from the } x\text{-terms} \\ &= 5(x^2 - 6x + 9) + 49 - 5 \cdot 9 && \text{Complete the square: Add 9 inside parentheses, subtract } 5 \cdot 9 \text{ outside} \\ &= 5(x - 3)^2 + 4 && \text{Factor and simplify} \end{aligned}$$

(b) The graph is a parabola that has its vertex at $(3, 4)$ and opens upward, as sketched in Figure 2.

(c) Since the coefficient of x^2 is positive, f has a minimum value. The minimum value is $f(3) = 4$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **29**

EXAMPLE 3 | Maximum Value of a Quadratic Function

Consider the quadratic function $f(x) = -x^2 + x + 2$.

(a) Express f in standard form.

(b) Sketch the graph of f.

(c) Find the maximum value of f.

SOLUTION

(a) To express this quadratic function in standard form, we complete the square:

$$\begin{aligned} y &= -x^2 + x + 2 \\ &= -(x^2 - x) + 2 && \text{Factor } -1 \text{ from the } x\text{-terms} \\ &= -\left(x^2 - x + \tfrac{1}{4}\right) + 2 - (-1)\tfrac{1}{4} && \text{Complete the square: Add } \tfrac{1}{4} \text{ inside parentheses, subtract } (-1)\tfrac{1}{4} \text{ outside} \\ &= -\left(x - \tfrac{1}{2}\right)^2 + \tfrac{9}{4} && \text{Factor and simplify} \end{aligned}$$

(b) From the standard form we see that the graph is a parabola that opens downward and has vertex $\left(\frac{1}{2}, \frac{9}{4}\right)$. As an aid to sketching the graph, we find the intercepts. The y-intercept is $f(0) = 2$. To find the x-intercepts, we set $f(x) = 0$ and solve the resulting quadratic equation. We can solve a quadratic equation by any of the methods we studied in Section 1.6. In this case we solve the equation by factoring.

$$\begin{aligned} -x^2 + x + 2 &= 0 && \text{Set } y = 0 \\ x^2 - x - 2 &= 0 && \text{Multiply by } -1 \\ (x - 2)(x + 1) &= 0 && \text{Factor} \end{aligned}$$

Thus, the x-intercepts are $x = 2$ and $x = -1$. The graph of f is sketched in Figure 3.

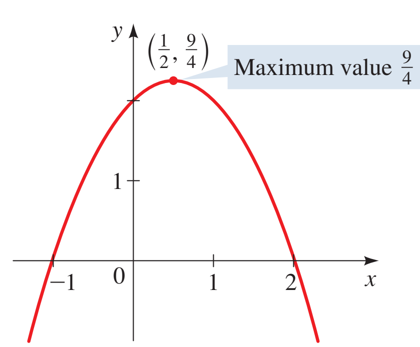
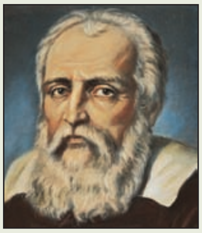


FIGURE 3 Graph of $f(x) = -x^2 + x + 2$

(c) Since the coefficient of x^2 is negative, f has a maximum value, which is $f(\frac{1}{2}) = \frac{9}{4}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **31** ■



The Granger Collection, New York

GALILEO GALILEI (1564–1642) was born in Pisa, Italy. He studied medicine but later abandoned this in favor of science and mathematics. At the age of 25, by dropping cannonballs of various sizes from the Leaning Tower of Pisa, he demonstrated that light objects fall at the same rate as heavier ones. This contradicted the then-accepted view of Aristotle that heavier objects fall more quickly. He also showed that the distance an object falls is proportional to the square of the time it has been falling, and from this he was able to prove that the path of a projectile is a parabola.

Galileo constructed the first telescope and, using it, discovered the moons of Jupiter. His advocacy of the Copernican view that the earth revolves around the sun (rather than being stationary) led to his being called before the Inquisition. By then an old man, he was forced to recant his views, but he is said to have muttered under his breath, "Nevertheless, it does move." Galileo revolutionized science by expressing scientific principles in the language of mathematics. He said, "The great book of nature is written in mathematical symbols."

Expressing a quadratic function in standard form helps us to sketch its graph as well as to find its maximum or minimum value. If we are interested only in finding the maximum or minimum value, then a formula is available for doing so. This formula is obtained by completing the square for the general quadratic function as follows:

$$f(x) = ax^2 + bx + c$$

$$= a\left(x^2 + \frac{b}{a}x\right) + c \qquad \text{Factor } a \text{ from the } x\text{-terms}$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) \qquad \text{Complete the square: Add } \frac{b^2}{4a^2} \text{ inside parentheses, subtract } a\left(\frac{b^2}{4a^2}\right) \text{ outside}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \qquad \text{Factor}$$

This equation is in standard form with $h = -b/(2a)$ and $k = c - b^2/(4a)$. Since the maximum or minimum value occurs at $x = h$, we have the following result.

MAXIMUM OR MINIMUM VALUE OF A QUADRATIC FUNCTION

The maximum or minimum value of a quadratic function $f(x) = ax^2 + bx + c$ occurs at

$$x = -\frac{b}{2a}$$

If $a > 0$, then the **minimum value** is $f\left(-\frac{b}{2a}\right)$.

If $a < 0$, then the **maximum value** is $f\left(-\frac{b}{2a}\right)$.

EXAMPLE 4 | Finding Maximum and Minimum Values of Quadratic Functions

Find the maximum or minimum value of each quadratic function.

(a) $f(x) = x^2 + 4x$

(b) $g(x) = -2x^2 + 4x - 5$

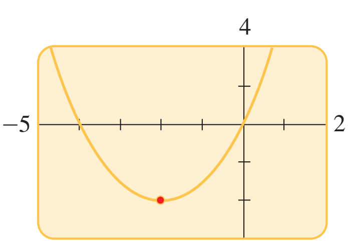
SOLUTION

(a) This is a quadratic function with $a = 1$ and $b = 4$. Thus the maximum or minimum value occurs at

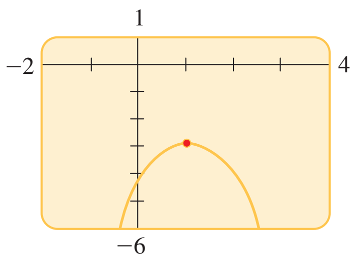
$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot 1} = -2$$

Since $a > 0$, the function has the *minimum* value

$$f(-2) = (-2)^2 + 4(-2) = -4$$



The minimum value occurs at $x = -2$.



The maximum value occurs at $x = 1$.

(b) This is a quadratic function with $a = -2$ and $b = 4$. Thus the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2\cdot(-2)} = 1$$

Since $a < 0$, the function has the *maximum* value

$$f(1) = -2(1)^2 + 4(1) - 5 = -3$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **37** AND **39** ■

Modeling with Quadratic Functions

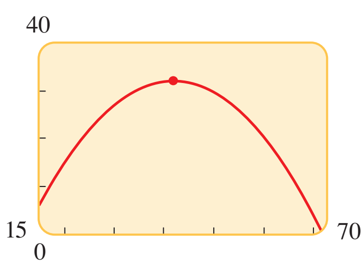
We study some examples of real-world phenomena that are modeled by quadratic functions. These examples and the *Applications* exercises for this section show some of the variety of situations that are naturally modeled by quadratic functions.

EXAMPLE 5 | Maximum Gas Mileage for a Car

Most cars get their best gas mileage when traveling at a relatively modest speed. The gas mileage M for a certain new car is modeled by the function

$$M(s) = -\frac{1}{28}s^2 + 3s - 31, \qquad 15 \le s \le 70$$

where s is the speed in mi/h and M is measured in mi/gal. What is the car's best gas mileage, and at what speed is it attained?



The maximum gas mileage occurs at 42 mi/h.

SOLUTION The function M is a quadratic function with $a = -\frac{1}{28}$ and $b = 3$. Thus its maximum value occurs when

$$s = -\frac{b}{2a} = -\frac{3}{2\left(-\frac{1}{28}\right)} = 42$$

The maximum is $M(42) = -\frac{1}{28}(42)^2 + 3(42) - 31 = 32$. So the car's best gas mileage is 32 mi/gal, when it is traveling at 42 mi/h.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **59** ■

EXAMPLE 6 | Maximizing Revenue from Ticket Sales

A hockey team plays in an arena that has a seating capacity of 15,000 spectators. With the ticket price set at \$14, average attendance at recent games has been 9500. A market survey indicates that for each dollar the ticket price is lowered, the average attendance increases by 1000.

(a) Find a function that models the revenue in terms of ticket price.

(b) Find the price that maximizes revenue from ticket sales.

(c) What ticket price is so high that no one attends and so no revenue is generated?

SOLUTION

(a) Express the model in words. The model that we want is a function that gives the revenue for any ticket price:

$$\text{revenue} = \text{ticket price} \times \text{attendance}$$

Choose the variable. There are two varying quantities: ticket price and attendance. Since the function we want depends on price, we let

$$x = \text{ticket price}$$

Next, we express attendance in terms of x.

In Words	In Algebra
Ticket price	x
Amount ticket price is lowered	$14 - x$
Increase in attendance	$1000(14 - x)$
Attendance	$9500 + 1000(14 - x)$

Set up the model. The model that we want is the function R that gives the revenue for a given ticket price x:

$$\text{revenue} = \text{ticket price} \times \text{attendance}$$

$$R(x) = x \times [9500 + 1000(14 - x)]$$

$$R(x) = x(23{,}500 - 1000x)$$

$$R(x) = 23{,}500x - 1000x^2$$

(b) Use the model. Since R is a quadratic function with $a = -1000$ and $b = 23{,}500$, the maximum occurs at

$$x = -\frac{b}{2a} = -\frac{23{,}500}{2(-1000)} = 11.75$$

So a ticket price of \$11.75 gives the maximum revenue.

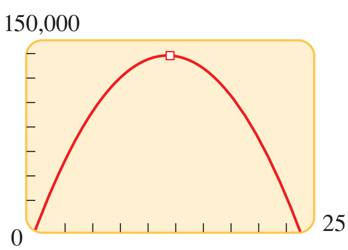
(c) Use the model. We want to find the ticket price for which $R(x) = 0$.

$$23{,}500x - 1000x^2 = 0 \qquad \text{Set } R(x) = 0$$

$$23.5x - x^2 = 0 \qquad \text{Divide by 1000}$$

$$x(23.5 - x) = 0 \qquad \text{Factor}$$

$$x = 0 \quad \text{or} \quad x = 23.5 \qquad \text{Solve for } x$$



Maximum attendance occurs when ticket price is \$11.75.

So according to this model, a ticket price of \$23.50 is just too high; at that price, no one attends to watch this team play. (Of course, revenue is also zero if the ticket price is zero.)

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **69** ■

3.1 EXERCISES

CONCEPTS

1. To put the quadratic function $f(x) = ax^2 + bx + c$ in standard form, we complete the ________.

2. The quadratic function $f(x) = a(x - h)^2 + k$ is in standard form.

(a) The graph of f is a parabola with vertex (____, ____).

(b) If $a > 0$, the graph of f opens ________. In this case $f(h) = k$ is the ________ value of f.

(c) If $a < 0$, the graph of f opens ________. In this case $f(h) = k$ is the ________ value of f.

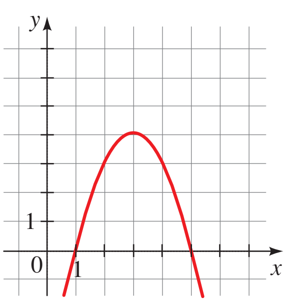
3. The graph of $f(x) = 2(x - 3)^2 + 5$ is a parabola that opens ________, with its vertex at (____, ____), and $f(3) =$ ________ is the (minimum/maximum) ________ value of f.

4. The graph of $f(x) = -2(x - 3)^2 + 5$ is a parabola that opens ________, with its vertex at (____, ____), and $f(3) =$ ________ is the (minimum/maximum) ________ value of f.

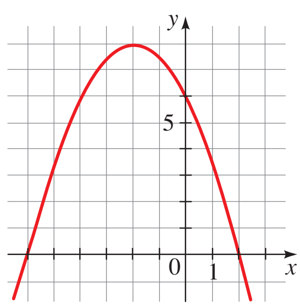
SKILLS

5–8 ■ The graph of a quadratic function f is given. **(a)** Find the coordinates of the vertex. **(b)** Find the maximum or minimum value of f. **(c)** Find the domain and range of f.

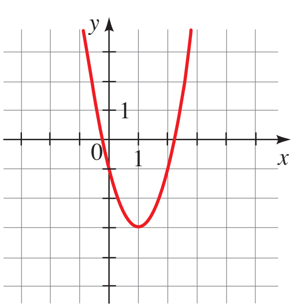
5. $f(x) = -x^2 + 6x - 5$



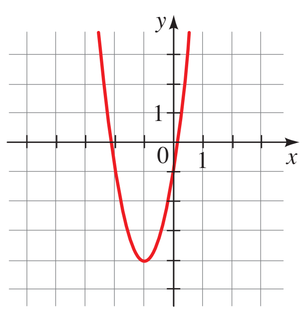
6. $f(x) = -\frac{1}{2}x^2 - 2x + 6$



7. $f(x) = 2x^2 - 4x - 1$



8. $f(x) = 3x^2 + 6x - 1$



9–26 ■ A quadratic function is given. **(a)** Express the quadratic function in standard form. **(b)** Find its vertex and its x- and y-intercept(s). **(c)** Sketch its graph.

9. $f(x) = x^2 - 2x + 3$
10. $f(x) = x^2 + 4x - 1$
11. $f(x) = x^2 - 6x$
12. $f(x) = x^2 + 8x$
13. $f(x) = 3x^2 + 6x$
14. $f(x) = 2x^2 - 16x$
15. $f(x) = -x^2 + 6x$
16. $f(x) = -x^2 + 10x$
17. $f(x) = x^2 + 4x + 3$
18. $f(x) = x^2 - 2x + 2$
19. $f(x) = -x^2 + 6x + 4$
20. $f(x) = -x^2 - 4x + 4$
21. $f(x) = 2x^2 + 4x + 3$
22. $f(x) = -3x^2 + 6x - 2$
23. $f(x) = 2x^2 - 20x + 57$
24. $f(x) = 2x^2 + x - 6$
25. $f(x) = -4x^2 - 12x + 1$
26. $f(x) = 3x^2 + 2x - 2$

27–36 ■ A quadratic function is given. **(a)** Express the quadratic function in standard form. **(b)** Sketch its graph. **(c)** Find its maximum or minimum value.

27. $f(x) = x^2 + 2x - 1$
28. $f(x) = x^2 - 8x + 8$
29. $f(x) = 3x^2 - 6x + 1$
30. $f(x) = 5x^2 + 30x + 4$
31. $f(x) = -x^2 - 3x + 3$
32. $f(x) = 1 - 6x - x^2$
33. $g(x) = 3x^2 - 12x + 13$
34. $g(x) = 2x^2 + 8x + 11$
35. $h(x) = 1 - x - x^2$
36. $h(x) = 3 - 4x - 4x^2$

37–46 ■ Find the maximum or minimum value of the function.

37. $f(x) = x^2 + x + 1$
38. $f(x) = 1 + 3x - x^2$
39. $f(t) = 100 - 49t - 7t^2$
40. $f(t) = 10t^2 + 40t + 113$
41. $f(s) = s^2 - 1.2s + 16$
42. $g(x) = 100x^2 - 1500x$
43. $h(x) = \frac{1}{2}x^2 + 2x - 6$
44. $f(x) = -\dfrac{x^2}{3} + 2x + 7$
45. $f(x) = 3 - x - \frac{1}{2}x^2$
46. $g(x) = 2x(x - 4) + 7$

47. Find a function whose graph is a parabola with vertex $(1, -2)$ and that passes through the point $(4, 16)$.

48. Find a function whose graph is a parabola with vertex $(3, 4)$ and that passes through the point $(1, -8)$.

49–52 ■ Find the domain and range of the function.

49. $f(x) = -x^2 + 4x - 3$
50. $f(x) = x^2 - 2x - 3$
51. $f(x) = 2x^2 + 6x - 7$
52. $f(x) = -3x^2 + 6x + 4$

53–54 ■ A quadratic function is given. **(a)** Use a graphing device to find the maximum or minimum value of the quadratic function f, rounded to two decimal places. **(b)** Find the exact maximum or minimum value of f, and compare it with your answer to part (a).

53. $f(x) = x^2 + 1.79x - 3.21$

54. $f(x) = 1 + x - \sqrt{2}x^2$

APPLICATIONS

55. Height of a Ball If a ball is thrown directly upward with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. What is the maximum height attained by the ball?

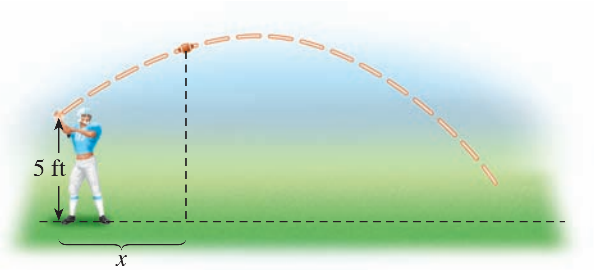
56. Path of a Ball A ball is thrown across a playing field from a height of 5 ft above the ground at an angle of 45° to the horizontal at a speed of 20 ft/s. It can be deduced from physical principles that the path of the ball is modeled by the function

$$y = -\frac{32}{(20)^2}x^2 + x + 5$$

where x is the distance in feet that the ball has traveled horizontally.

(a) Find the maximum height attained by the ball.

(b) Find the horizontal distance the ball has traveled when it hits the ground.



57. Revenue A manufacturer finds that the revenue generated by selling x units of a certain commodity is given by the function $R(x) = 80x - 0.4x^2$, where the revenue $R(x)$ is measured in dollars. What is the maximum revenue, and how many units should be manufactured to obtain this maximum?

58. Sales A soft-drink vendor at a popular beach analyzes his sales records and finds that if he sells x cans of soda pop in one day, his profit (in dollars) is given by

$$P(x) = -0.001x^2 + 3x - 1800$$

What is his maximum profit per day, and how many cans must he sell for maximum profit?

59. Advertising The effectiveness of a television commercial depends on how many times a viewer watches it. After some experiments an advertising agency found that if the effectiveness E is measured on a scale of 0 to 10, then

$$E(n) = \tfrac{2}{3}n - \tfrac{1}{90}n^2$$

where n is the number of times a viewer watches a given commercial. For a commercial to have maximum effectiveness, how many times should a viewer watch it?

60. Pharmaceuticals When a certain drug is taken orally, the concentration of the drug in the patient's bloodstream after t minutes is given by $C(t) = 0.06t - 0.0002t^2$, where $0 \le t \le 240$ and the concentration is measured in mg/L. When is the maximum serum concentration reached, and what is that maximum concentration?

61. Agriculture The number of apples produced by each tree in an apple orchard depends on how densely the trees are planted. If n trees are planted on an acre of land, then each tree produces $900 - 9n$ apples. So the number of apples produced per acre is

$$A(n) = n(900 - 9n)$$

How many trees should be planted per acre to obtain the maximum yield of apples?

62. Agriculture At a certain vineyard it is found that each grape vine produces about 10 pounds of grapes in a season when about 700 vines are planted per acre. For each additional vine that is planted, the production of each vine decreases by about 1 percent. So the number of pounds of grapes produced per acre is modeled by

$$A(n) = (700 + n)(10 - 0.01n)$$

where n is the number of additional vines planted. Find the number of vines that should be planted to maximize grape production.

63–66 ■ Use the formulas of this section to give an alternative solution to the indicated problem in *Focus on Modeling: Modeling with Functions* on pages 254–255.

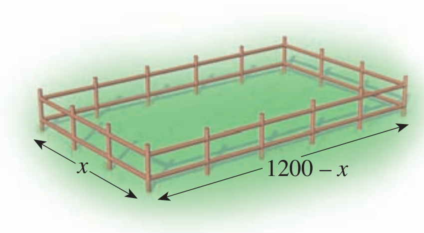
63. Problem 21 **64.** Problem 22

65. Problem 25 **66.** Problem 24

67. Fencing a Horse Corral Carol has 2400 ft of fencing to fence in a rectangular horse corral.

(a) Find a function that models the area of the corral in terms of the width x of the corral.

(b) Find the dimensions of the rectangle that maximize the area of the corral.

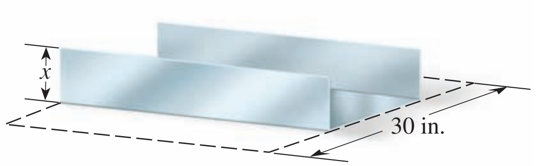


68. Making a Rain Gutter A rain gutter is formed by bending up the sides of a 30-inch-wide rectangular metal sheet as shown in the figure.

(a) Find a function that models the cross-sectional area of the gutter in terms of x.

(b) Find the value of x that maximizes the cross-sectional area of the gutter.

(c) What is the maximum cross-sectional area for the gutter?



69. Stadium Revenue A baseball team plays in a stadium that holds 55,000 spectators. With the ticket price at \$10, the average attendance at recent games has been 27,000. A market survey indicates that for every dollar the ticket price is lowered, attendance increases by 3000.

(a) Find a function that models the revenue in terms of ticket price.

(b) Find the price that maximizes revenue from ticket sales.

(c) What ticket price is so high that no revenue is generated?

70. Maximizing Profit A community bird-watching society makes and sells simple bird feeders to raise money for its conservation activities. The materials for each feeder cost \$6, and the society sells an average of 20 per week at a price of \$10 each. The society has been considering raising the price, so it conducts a survey and finds that for every dollar increase, it loses 2 sales per week.

(a) Find a function that models weekly profit in terms of price per feeder.

(b) What price should the society charge for each feeder to maximize profits? What is the maximum weekly profit?

DISCOVERY ■ DISCUSSION ■ WRITING

71. Vertex and x-Intercepts We know that the graph of the quadratic function $f(x) = (x - m)(x - n)$ is a parabola. Sketch a rough graph of what such a parabola would look like. What are the x-intercepts of the graph of f? Can you tell from your graph the x-coordinate of the vertex in terms of m and n? (Use the symmetry of the parabola.) Confirm your answer by expanding and using the formulas of this section.

72. Maximum of a Fourth-Degree Polynomial Find the maximum value of the function

$$f(x) = 3 + 4x^2 - x^4$$

[*Hint:* Let $t = x^2$.]

3.2 POLYNOMIAL FUNCTIONS AND THEIR GRAPHS

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Graph basic polynomial functions ▶ Describe the end behavior of a polynomial function ▶ Graph a polynomial function using its zeroes ▶ Use multiplicity to help graph a polynomial function ▶ Find local maxima and minima of polynomial functions

GET READY **Prepare for this section by reviewing Section P.6 on factoring.**

In this section we study polynomial functions of any degree. But before we work with polynomial functions, we must agree on some terminology.

POLYNOMIAL FUNCTIONS

A **polynomial function of degree n** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n \neq 0$.

The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial.

The number a_0 is the **constant coefficient** or **constant term**.

The number a_n, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.

We often refer to polynomial functions simply as *polynomials*. The following polynomial has degree 5, leading coefficient 3, and constant term -6.

Leading coefficient 3

Degree 5

Constant term -6

$$3x^5 + 6x^4 - 2x^3 + x^2 + 7x - 6$$

Leading term $3x^5$

Coefficients 3, 6, -2, 1, 7, and -6

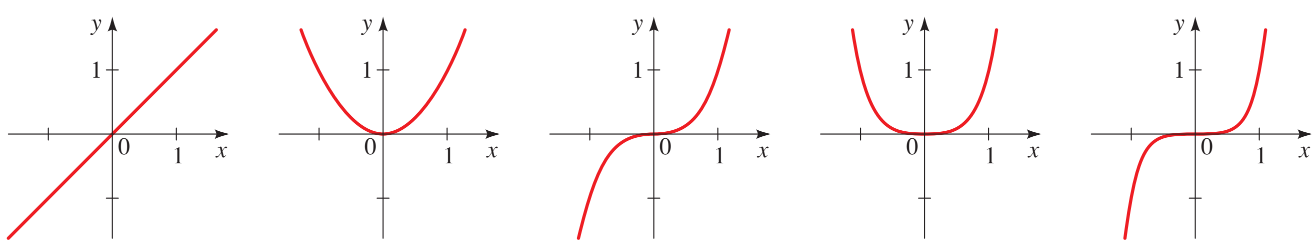
Here are some more examples of polynomials:

Polynomial	Degree	Leading term	Constant term
$P(x) = 4x - 7$	1	$4x$	-7
$P(x) = x^2 + x$	2	x^2	0
$P(x) = 2x^3 - 6x^2 + 10$	3	$2x^3$	10
$P(x) = -5x^4 + x - 2$	4	$-5x^4$	-2

If a polynomial consists of just a single term, then it is called a **monomial**. For example, $P(x) = x^3$ and $Q(x) = -6x^5$ are monomials.

Graphing Basic Polynomial Functions

The simplest polynomial functions are the monomials $P(x) = x^n$, whose graphs are shown in Figure 1. As the figure suggests, the graph of $P(x) = x^n$ has the same general shape as the graph of $y = x^2$ when n is even and the same general shape as the graph of $y = x^3$ when n is odd. However, as the degree n becomes larger, the graphs become flatter around the origin and steeper elsewhere.



(a) $y = x$ (b) $y = x^2$ (c) $y = x^3$ (d) $y = x^4$ (e) $y = x^5$

FIGURE 1 Graphs of monomials

EXAMPLE 1 | Transformations of Monomials

Sketch the graphs of the following functions.

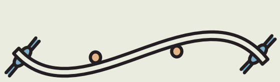
(a) $P(x) = -x^3$ **(b)** $Q(x) = (x - 2)^4$

(c) $R(x) = -2x^5 + 4$

MATHEMATICS IN THE MODERN WORLD

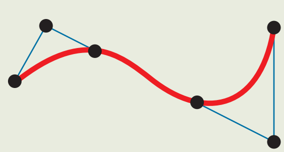
Splines

A spline is a long strip of wood that is curved while held fixed at certain points. In the old days shipbuilders used splines to create the curved shape of a boat's hull. Splines are also used to make the curves of a piano, a violin, or the spout of a teapot.



Mathematicians discovered that the shapes of splines can be obtained by piecing together parts of polynomials. For example, the graph of a cubic polynomial can be made to fit specified points by adjusting the coefficients of the polynomial (see Example 10, page 277).

Curves obtained in this way are called cubic splines. In modern computer design programs, such as Adobe Illustrator or Microsoft Paint, a curve can be drawn by fixing two points, then using the mouse to drag one or more anchor points. Moving the anchor points amounts to adjusting the coefficients of a cubic polynomial.



SOLUTION We use the graphs in Figure 1 and transform them using the techniques of Section 2.5.

(a) The graph of $P(x) = -x^3$ is the reflection of the graph of $y = x^3$ in the x-axis, as shown in Figure 2(a) below.

(b) The graph of $Q(x) = (x - 2)^4$ is the graph of $y = x^4$ shifted to the right 2 units, as shown in Figure 2(b).

(c) We begin with the graph of $y = x^5$. The graph of $y = -2x^5$ is obtained by stretching the graph vertically and reflecting it in the x-axis (see the dashed blue graph in Figure 2(c)). Finally, the graph of $R(x) = -2x^5 + 4$ is obtained by shifting upward 4 units (see the red graph in Figure 2(c)).

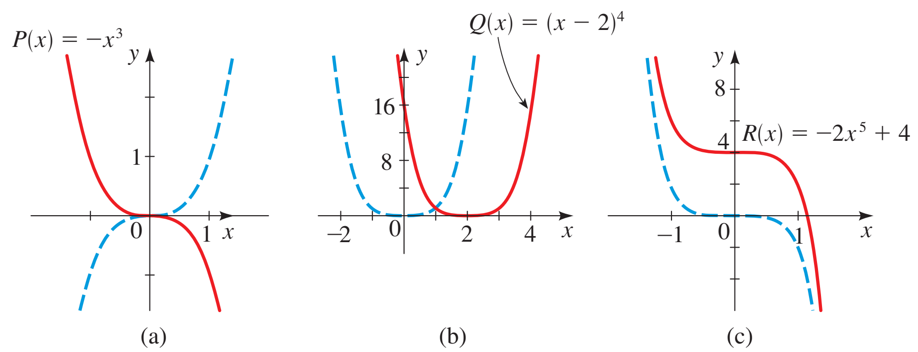


FIGURE 2

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5**

Graphs of Polynomial Functions: End Behavior

The graphs of polynomials of degree 0 or 1 are lines (Section 1.3), and the graphs of polynomials of degree 2 are parabolas (Section 3.1). The greater the degree of a polynomial, the more complicated its graph can be. However, the graph of a polynomial function is **continuous**. This means that the graph has no breaks or holes (see Figure 3). Moreover, the graph of a polynomial function is a smooth curve; that is, it has no corners or sharp points (cusps) as shown in Figure 3.

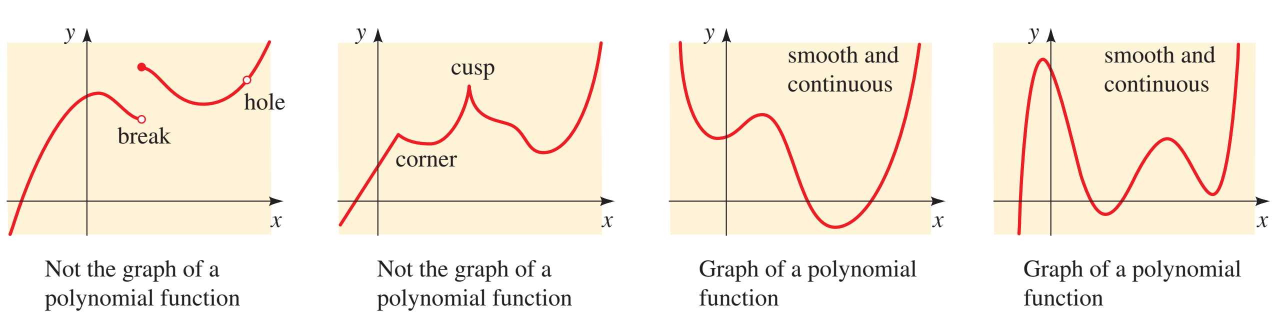


FIGURE 3

The domain of a polynomial function is the set of all real numbers, so we can sketch only a small portion of the graph. However, for values of x outside the portion of the graph we have drawn, we can describe the behavior of the graph.

The **end behavior** of a polynomial is a description of what happens as x becomes large in the positive or negative direction. To describe end behavior, we use the following notation:

$x \to \infty$	means	"x becomes large in the positive direction"

$x \to -\infty$	means	"x becomes large in the negative direction"

For example, the monomial $y = x^2$ in Figure 1(b) has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to -\infty$$

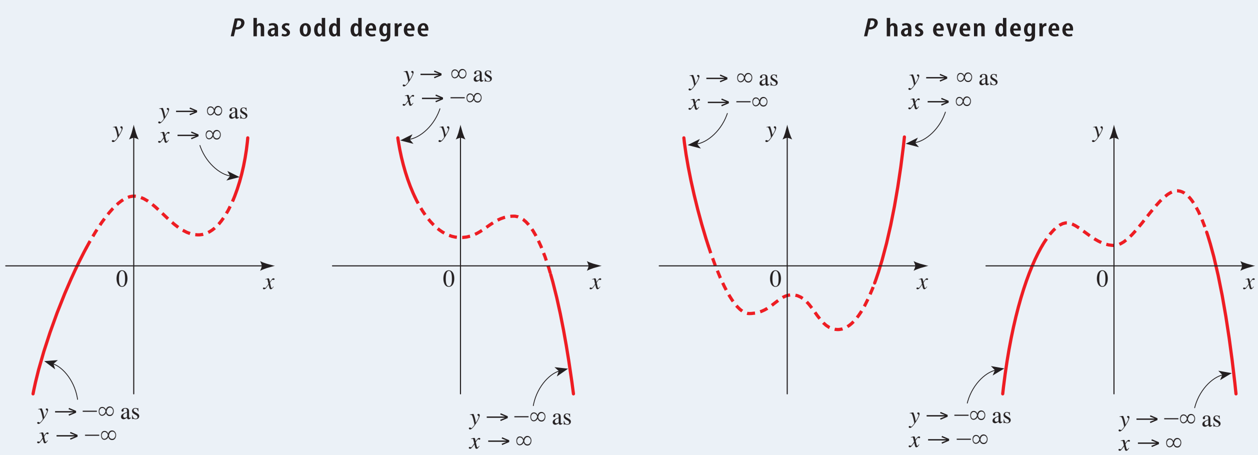
The monomial $y = x^3$ in Figure 1(c) has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

For any polynomial *the end behavior is determined by the term that contains the highest power of* x, because when x is large, the other terms are relatively insignificant in size. The following box shows the four possible types of end behavior, based on the highest power and the sign of its coefficient.

END BEHAVIOR OF POLYNOMIALS

The end behavior of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is determined by the degree n and the sign of the leading coefficient a_n, as indicated in the following graphs.



Leading coefficient positive | Leading coefficient negative | Leading coefficient positive | Leading coefficient negative

EXAMPLE 2 | End Behavior of a Polynomial

Determine the end behavior of the polynomial

$$P(x) = -2x^4 + 5x^3 + 4x - 7$$

SOLUTION The polynomial P has degree 4 and leading coefficient -2. Thus P has *even* degree and *negative* leading coefficient, so it has the following end behavior:

$$y \to -\infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

The graph in Figure 4 illustrates the end behavior of P.

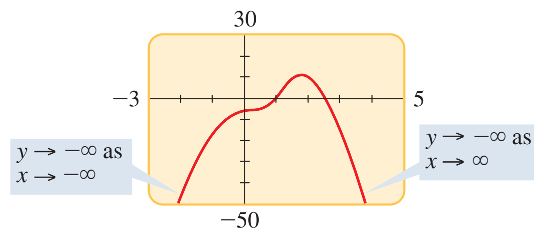


FIGURE 4
$P(x) = -2x^4 + 5x^3 + 4x - 7$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **11**

EXAMPLE 3 | End Behavior of a Polynomial

(a) Determine the end behavior of the polynomial $P(x) = 3x^5 - 5x^3 + 2x$.

(b) Confirm that P and its leading term $Q(x) = 3x^5$ have the same end behavior by graphing them together.

SOLUTION

(a) Since P has odd degree and positive leading coefficient, it has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

(b) Figure 5 shows the graphs of P and Q in progressively larger viewing rectangles. The larger the viewing rectangle, the more the graphs look alike. This confirms that they have the same end behavior.

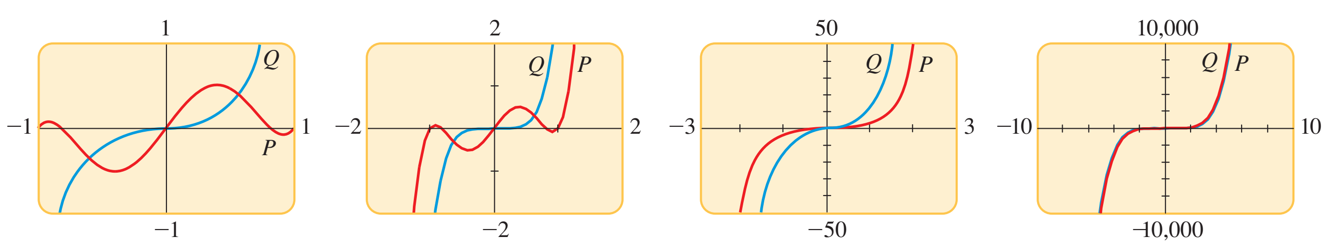


FIGURE 5
$P(x) = 3x^5 - 5x^3 + 2x$
$Q(x) = 3x^5$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **43**

To see algebraically why P and Q in Example 3 have the same end behavior, factor P as follows and compare with Q.

$$P(x) = 3x^5\left(1 - \frac{5}{3x^2} + \frac{2}{3x^4}\right) \qquad Q(x) = 3x^5$$

When x is large, the terms $5/(3x^2)$ and $2/(3x^4)$ are close to 0 (see Exercise 95 on page 17). So for large x, we have

$$P(x) \approx 3x^5(1 - 0 - 0) = 3x^5 = Q(x)$$

So when x is large, P and Q have approximately the same values. We can also see this numerically by making a table like the one shown below.

x	$P(x)$	$Q(x)$
15	2,261,280	2,278,125
30	72,765,060	72,900,000
50	936,875,100	937,500,000

By the same reasoning we can show that the end behavior of *any* polynomial is determined by its leading term.

Using Zeros to Graph Polynomials

If P is a polynomial function, then c is called a **zero** of P if $P(c) = 0$. In other words, the zeros of P are the solutions of the polynomial equation $P(x) = 0$. Note that if $P(c) = 0$, then the graph of P has an x-intercept at $x = c$, so the x-intercepts of the graph are the zeros of the function.

REAL ZEROS OF POLYNOMIALS

If P is a polynomial and c is a real number, then the following are equivalent:

1. c is a zero of P.
2. $x = c$ is a solution of the equation $P(x) = 0$.
3. $x - c$ is a factor of $P(x)$.
4. c is an x-intercept of the graph of P.

To find the zeros of a polynomial P, we factor and then use the Zero-Product Property (see page 121). For example, to find the zeros of $P(x) = x^2 + x - 6$, we factor P to get

$$P(x) = (x - 2)(x + 3)$$

From this factored form we easily see that

1. 2 is a zero of P.
2. $x = 2$ is a solution of the equation $x^2 + x - 6 = 0$.
3. $x - 2$ is a factor of $x^2 + x - 6$.
4. 2 is an x-intercept of the graph of P.

The same facts are true for the other zero, -3.

The following theorem has many important consequences. (See, for instance, the *Discovery Project* referenced on page 297.) Here we use it to help us graph polynomial functions.

INTERMEDIATE VALUE THEOREM FOR POLYNOMIALS

If P is a polynomial function and $P(a)$ and $P(b)$ have opposite signs, then there exists at least one value c between a and b for which $P(c) = 0$.

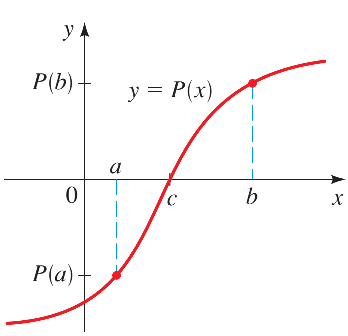


FIGURE 6

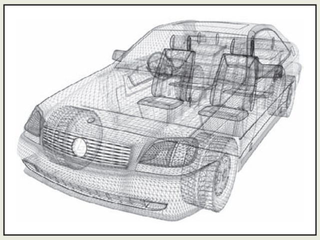
We will not prove this theorem, but Figure 6 shows why it is intuitively plausible.

One important consequence of this theorem is that between any two successive zeros the values of a polynomial are either all positive or all negative. That is, between two successive zeros the graph of a polynomial lies *entirely above* or *entirely below* the x-axis. To see why, suppose c_1 and c_2 are successive zeros of P. If P has both positive and negative values between c_1 and c_2, then by the Intermediate Value Theorem P must have another zero between c_1 and c_2. But that's not possible because c_1 and c_2 are successive zeros. This observation allows us to use the following guidelines to graph polynomial functions.

GUIDELINES FOR GRAPHING POLYNOMIAL FUNCTIONS

1. **Zeros.** Factor the polynomial to find all its real zeros; these are the x-intercepts of the graph.
2. **Test Points.** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the x-axis on the intervals determined by the zeros. Include the y-intercept in the table.
3. **End Behavior.** Determine the end behavior of the polynomial.
4. **Graph.** Plot the intercepts and other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

MATHEMATICS IN THE MODERN WORLD



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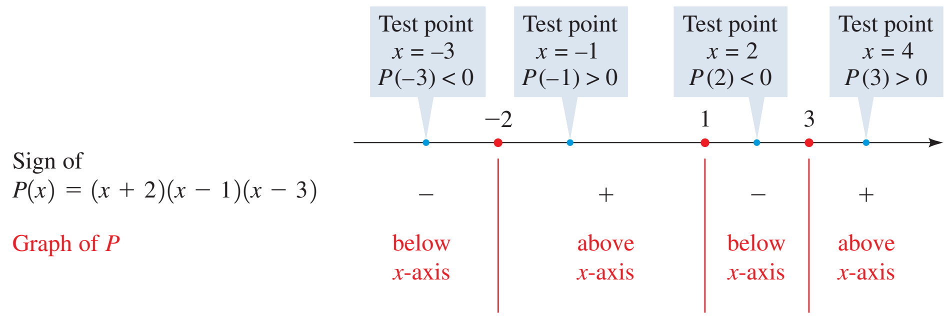
Automotive Design

Computer-aided design (CAD) has completely changed the way in which car companies design and manufacture cars. Before the 1980s automotive engineers would build a full-scale "nuts and bolts" model of a proposed new car; this was really the only way to tell whether the design was feasible. Today automotive engineers build a mathematical model, one that exists only in the memory of a computer. The model incorporates all the main design features of the car. Certain polynomial curves, called *splines* (see page 267), are used in shaping the body of the car. The resulting "mathematical car" can be tested for structural stability, handling, aerodynamics, suspension response, and more. All this testing is done before a prototype is built. As you can imagine, CAD saves car manufacturers millions of dollars each year. More importantly, CAD gives automotive engineers far more flexibility in design; desired changes can be created and tested within seconds. With the help of computer graphics, designers can see how good the "mathematical car" looks before they build the real one. Moreover, the mathematical car can be viewed from any perspective; it can be moved, rotated, or seen from the inside. These manipulations of the car on the computer monitor translate mathematically into solving large systems of linear equations.

EXAMPLE 4 | Using Zeros to Graph a Polynomial Function

Sketch the graph of the polynomial function $P(x) = (x + 2)(x - 1)(x - 3)$.

SOLUTION The zeros are $x = -2$, 1, and 3. These determine the intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$, and $(3, \infty)$. Using test points in these intervals, we get the information in the following sign diagram (see Section 1.8).



Plotting a few additional points and connecting them with a smooth curve helps us to complete the graph in Figure 7.

	x	$P(x)$
Test point →	-3	-24
	-2	0
Test point →	-1	8
	0	6
	1	0
Test point →	2	-4
	3	0
Test point →	4	18

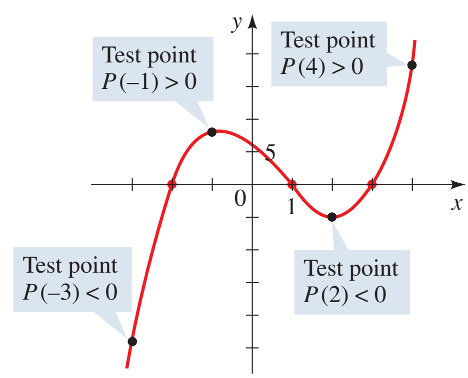


FIGURE 7 $P(x) = (x + 2)(x - 1)(x - 3)$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17**

EXAMPLE 5 | Finding Zeros and Graphing a Polynomial Function

Let $P(x) = x^3 - 2x^2 - 3x$.

(a) Find the zeros of P.

(b) Sketch a graph of P.

SOLUTION

(a) To find the zeros, we factor completely:

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 3x \\ &= x(x^2 - 2x - 3) && \text{Factor } x \\ &= x(x - 3)(x + 1) && \text{Factor quadratic} \end{aligned}$$

Thus the zeros are $x = 0$, $x = 3$, and $x = -1$.

(b) The x-intercepts are $x = 0$, $x = 3$, and $x = -1$. The y-intercept is $P(0) = 0$. We make a table of values of $P(x)$, making sure that we choose test points between (and to the right and left of) successive zeros.

Since P is of odd degree and its leading coefficient is positive, it has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

We plot the points in the table and connect them by a smooth curve to complete the graph, as shown in Figure 8.

	x	$P(x)$
Test point →	-2	-10
	-1	0
Test point →	$-\frac{1}{2}$	$\frac{7}{8}$
	0	0
Test point →	1	-4
	2	-6
	3	0
Test point →	4	20

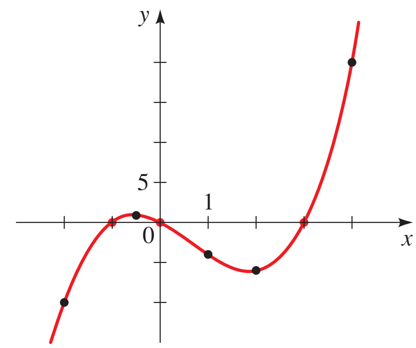


FIGURE 8 $P(x) = x^3 - 2x^2 - 3x$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **29**

EXAMPLE 6 | Finding Zeros and Graphing a Polynomial Function

Let $P(x) = -2x^4 - x^3 + 3x^2$.

(a) Find the zeros of P. **(b)** Sketch a graph of P.

SOLUTION

(a) To find the zeros, we factor completely:

$$\begin{aligned} P(x) &= -2x^4 - x^3 + 3x^2 \\ &= -x^2(2x^2 + x - 3) && \text{Factor } -x^2 \\ &= -x^2(2x + 3)(x - 1) && \text{Factor quadratic} \end{aligned}$$

Thus the zeros are $x = 0$, $x = -\frac{3}{2}$, and $x = 1$.

(b) The x-intercepts are $x = 0$, $x = -\frac{3}{2}$, and $x = 1$. The y-intercept is $P(0) = 0$. We make a table of values of $P(x)$, making sure that we choose test points between (and to the right and left of) successive zeros.

Since P is of even degree and its leading coefficient is negative, it has the following end behavior:

$$y \to -\infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

We plot the points from the table and connect the points by a smooth curve to complete the graph in Figure 9.

A table of values is most easily calculated by using a programmable calculator or a graphing calculator.

x	$P(x)$
-2	-12
-1.5	0
-1	2
-0.5	0.75
0	0
0.5	0.5
1	0
1.5	-6.75

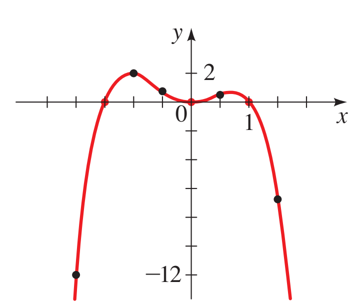


FIGURE 9 $P(x) = -2x^4 - x^3 + 3x^2$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **33**

EXAMPLE 7 | Finding Zeros and Graphing a Polynomial Function

Let $P(x) = x^3 - 2x^2 - 4x + 8$.

(a) Find the zeros of P.

(b) Sketch a graph of P.

SOLUTION

(a) To find the zeros, we factor completely:

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 4x + 8 \\ &= x^2(x - 2) - 4(x - 2) && \text{Group and factor} \\ &= (x^2 - 4)(x - 2) && \text{Factor } x - 2 \\ &= (x + 2)(x - 2)(x - 2) && \text{Difference of squares} \\ &= (x + 2)(x - 2)^2 && \text{Simplify} \end{aligned}$$

Thus the zeros are $x = -2$ and $x = 2$.

(b) The x-intercepts are $x = -2$ and $x = 2$. The y-intercept is $P(0) = 8$. The table gives additional values of $P(x)$.

Since P is of odd degree and its leading coefficient is positive, it has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

We connect the points by a smooth curve to complete the graph in Figure 10.

x	$P(x)$
-3	-25
-2	0
-1	9
0	8
1	3
2	0
3	5

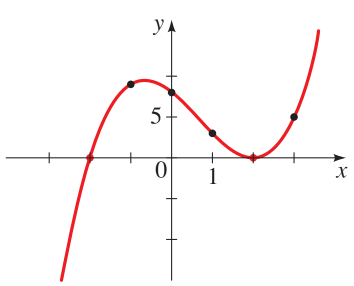


FIGURE 10 $P(x) = x^3 - 2x^2 - 4x + 8$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **35**

Shape of the Graph Near a Zero

Although $x = 2$ is a zero of the polynomial in Example 7, the graph does not cross the x-axis at the x-intercept 2. This is because the factor $(x - 2)^2$ corresponding to that zero is raised to an even power, so it doesn't change sign as we test points on either side of 2. In the same way the graph does not cross the x-axis at $x = 0$ in Example 6.

In general, if c is a zero of P and the corresponding factor $x - c$ occurs exactly m times in the factorization of P, then we say that c is a **zero of multiplicity m**. By considering test points on either side of the x-intercept c, we conclude that the graph crosses the x-axis at c if the multiplicity m is odd and does not cross the x-axis if m is even. Moreover, it can be shown by using calculus that near $x = c$ the graph has the same general shape as the graph of $y = A(x - c)^m$.

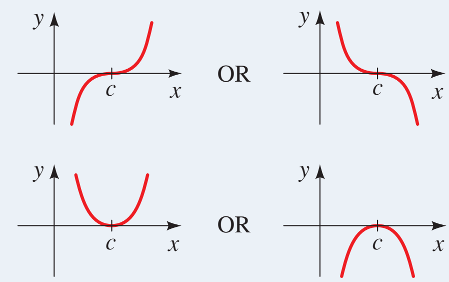
SHAPE OF THE GRAPH NEAR A ZERO OF MULTIPLICITY m

If c is a zero of P of multiplicity m, then the shape of the graph of P near c is as follows.

Multiplicity of c — **Shape of the graph of P near the x-intercept c**

m odd, $m > 1$

m even, $m > 1$



EXAMPLE 8 | Graphing a Polynomial Function Using Its Zeros

Graph the polynomial $P(x) = x^4(x - 2)^3(x + 1)^2$.

SOLUTION The zeros of P are -1, 0, and 2 with multiplicities 2, 4, and 3, respectively:

0 is a zero of multiplicity 4

2 is a zero of multiplicity 3

–1 is a zero of multiplicity 2

$$P(x) = x^4(x - 2)^3(x + 1)^2$$

The zero 2 has *odd* multiplicity, so the graph crosses the x-axis at the x-intercept 2. But the zeros 0 and -1 have *even* multiplicity, so the graph does not cross the x-axis at the x-intercepts 0 and -1.

Since P is a polynomial of degree 9 and has positive leading coefficient, it has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

With this information and a table of values we sketch the graph in Figure 11.

x	$P(x)$
-1.3	-9.2
-1	0
-0.5	-3.9
0	0
1	-4
2	0
2.3	8.2

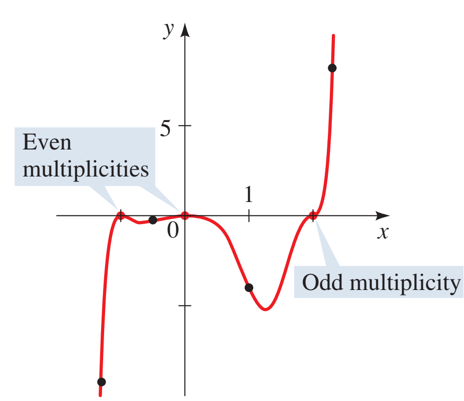


FIGURE 11 $P(x) = x^4(x - 2)^3(x + 1)^2$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **27**

Local Maxima and Minima of Polynomials

Recall from Section 2.3 that if the point $(a, f(a))$ is the highest point on the graph of f within some viewing rectangle, then $f(a)$ is a local maximum value of f, and if $(b, f(b))$

is the lowest point on the graph of f within a viewing rectangle, then $f(b)$ is a local minimum value (see Figure 12). We say that such a point $(a, f(a))$ is a **local maximum point** on the graph and that $(b, f(b))$ is a **local minimum point**. The local maximum and minimum points on the graph of a function are called its **local extrema**.

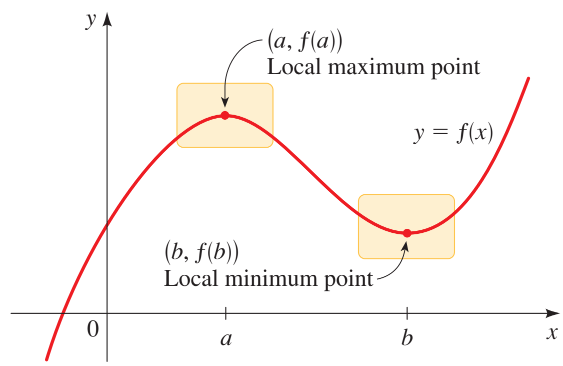


FIGURE 12

For a polynomial function the number of local extrema must be less than the degree, as the following principle indicates. (A proof of this principle requires calculus.)

LOCAL EXTREMA OF POLYNOMIALS

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then the graph of P has at most $n - 1$ local extrema.

A polynomial of degree n may in fact have less than $n - 1$ local extrema. For example, $P(x) = x^5$ (graphed in Figure 1) has *no* local extrema, even though it is of degree 5. The preceding principle tells us only that a polynomial of degree n can have no more than $n - 1$ local extrema.

EXAMPLE 9 | The Number of Local Extrema

Graph the polynomial and determine how many local extrema it has.

(a) $P_1(x) = x^4 + x^3 - 16x^2 - 4x + 48$

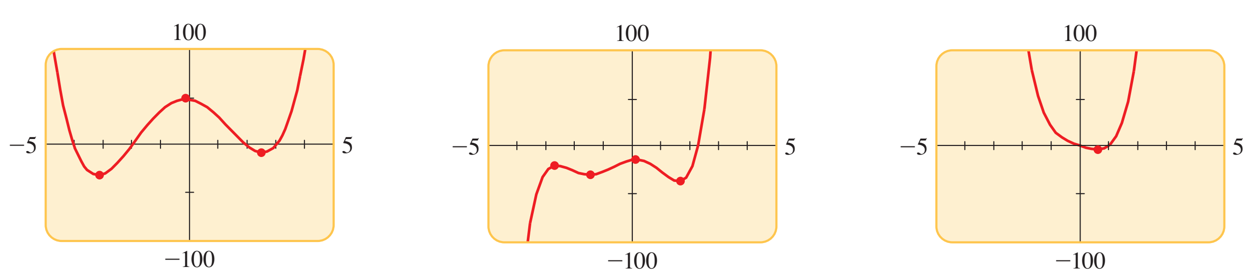
(b) $P_2(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15$ **(c)** $P_3(x) = 7x^4 + 3x^2 - 10x$

SOLUTION The graphs are shown in Figure 13.

(a) P_1 has two local minimum points and one local maximum point, for a total of three local extrema.

(b) P_2 has two local minimum points and two local maximum points, for a total of four local extrema.

(c) P_3 has just one local extremum, a local minimum.



(a) $P_1(x) = x^4 + x^3 - 16x^2 - 4x + 48$

(b) $P_2(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15$

(c) $P_3(x) = 7x^4 + 3x^2 - 10x$

FIGURE 13

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **63** AND **65**

With a graphing calculator we can quickly draw the graphs of many functions at once, on the same viewing screen. This allows us to see how changing a value in the definition of the functions affects the shape of its graph. In the next example we apply this principle to a family of third-degree polynomials.

EXAMPLE 10 | A Family of Polynomials

Sketch the family of polynomials $P(x) = x^3 - cx^2$ for $c = 0, 1, 2,$ and 3. How does changing the value of c affect the graph?

SOLUTION The polynomials

$$P_0(x) = x^3 \qquad P_1(x) = x^3 - x^2$$
$$P_2(x) = x^3 - 2x^2 \qquad P_3(x) = x^3 - 3x^2$$

are graphed in Figure 14. We see that increasing the value of c causes the graph to develop an increasingly deep "valley" to the right of the y-axis, creating a local maximum at the origin and a local minimum at a point in Quadrant IV. This local minimum moves lower and farther to the right as c increases. To see why this happens, factor $P(x) = x^2(x - c)$. The polynomial P has zeros at 0 and c, and the larger c gets, the farther to the right the minimum between 0 and c will be.

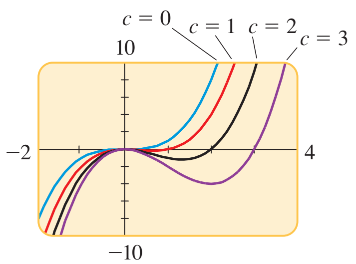


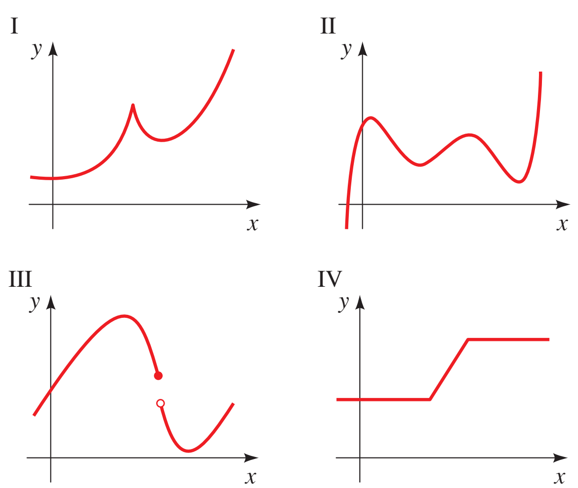
FIGURE 14 A family of polynomials $P(x) = x^3 - cx^2$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **73** ■

3.2 EXERCISES

CONCEPTS

1. Only one of the following graphs could be the graph of a polynomial function. Which one? Why are the others not graphs of polynomials?



2. Describe the end behavior of each polynomial.

(a) $y = x^3 - 8x^2 + 2x - 15$

End behavior: $y \to$ ________ as $x \to \infty$

$y \to$ ________ as $x \to -\infty$

(b) $y = -2x^4 + 12x + 100$

End behavior: $y \to$ ________ as $x \to \infty$

$y \to$ ________ as $x \to -\infty$

3. If c is a zero of the polynomial P, then

(a) $P(c) =$ ________.

(b) $x - c$ is a ________ of $P(x)$.

(c) c is a(n) ______ -intercept of the graph of P.

4. Which of the following statements couldn't possibly be true about the polynomial function P?
(a) P has degree 3, two local maxima, and two local minima.
(b) P has degree 3 and no local maxima or minima.
(c) P has degree 4, one local maximum, and no local minima.

SKILLS

5–8 ■ Sketch the graph of each function by transforming the graph of an appropriate function of the form $y = x^n$ from Figure 1. Indicate all x- and y-intercepts on each graph.

5. **(a)** $P(x) = x^2 - 4$ **(b)** $Q(x) = (x - 4)^2$
(c) $R(x) = 2x^2 - 2$ **(d)** $S(x) = 2(x - 2)^2$

6. **(a)** $P(x) = x^4 - 16$ **(b)** $Q(x) = (x + 2)^4$
(c) $R(x) = (x + 2)^4 - 16$ **(d)** $S(x) = -2(x + 2)^4$

7. **(a)** $P(x) = x^3 - 8$ **(b)** $Q(x) = -x^3 + 27$
(c) $R(x) = -(x + 2)^3$ **(d)** $S(x) = \frac{1}{2}(x - 1)^3 + 4$

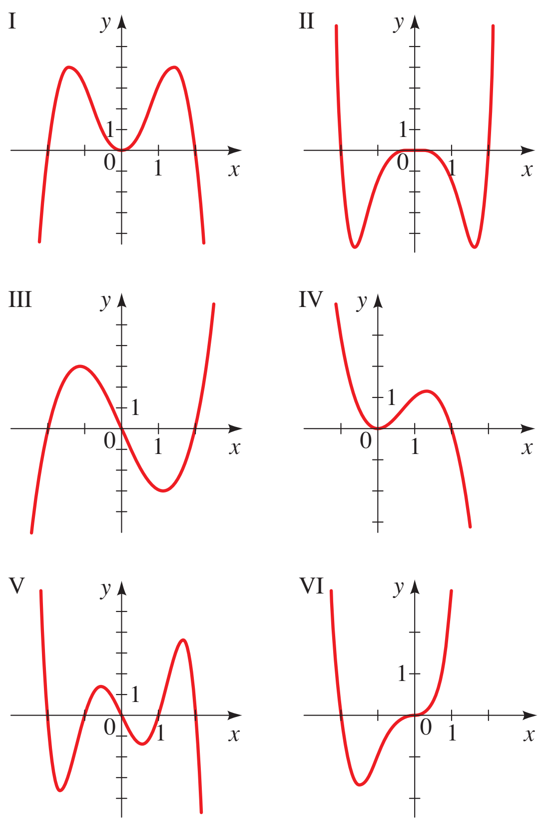
8. **(a)** $P(x) = (x + 3)^5$ **(b)** $Q(x) = 2(x + 3)^5 - 64$
(c) $R(x) = -\frac{1}{2}(x - 2)^5$ **(d)** $S(x) = -\frac{1}{2}(x - 2)^5 + 16$

9–14 ■ A polynomial function is given. **(a)** Describe the end behavior of the polynomial function. **(b)** Match the polynomial function with one of the graphs I–VI.

9. $P(x) = x(x^2 - 4)$ **10.** $Q(x) = -x^2(x^2 - 4)$

11. $R(x) = -x^5 + 5x^3 - 4x$ **12.** $S(x) = \frac{1}{2}x^6 - 2x^4$

13. $T(x) = x^4 + 2x^3$ **14.** $U(x) = -x^3 + 2x^2$



15–28 ■ Sketch the graph of the polynomial function. Make sure your graph shows all intercepts and exhibits the proper end behavior.

15. $P(x) = (x - 1)(x + 2)$

16. $P(x) = (2 - x)(x + 5)$

17. $P(x) = x(x - 3)(x + 2)$

18. $P(x) = -x(x - 3)(x + 2)$

19. $P(x) = -(2x - 1)(x + 1)(x + 3)$

20. $P(x) = (x - 3)(x + 2)(3x - 2)$

21. $P(x) = -2x(x - 2)^2$ **22.** $P(x) = \frac{1}{5}x(x - 5)^2$

23. $P(x) = -(x + 4)(x + 3)(x - 5)^2$

24. $P(x) = \frac{1}{4}(x + 1)^3(x - 3)$

25. $P(x) = \frac{1}{12}(x + 2)^2(x - 3)^2$ **26.** $P(x) = (x - 1)^2(x + 2)^3$

27. $P(x) = x^3(x + 2)(x - 3)^2$ **28.** $P(x) = (x - 3)^2(x + 1)^2$

29–42 ■ Factor the polynomial and use the factored form to find the zeros. Then sketch the graph.

29. $P(x) = x^3 - x^2 - 6x$ **30.** $P(x) = x^3 + 2x^2 - 8x$

31. $P(x) = -x^3 + x^2 + 12x$ **32.** $P(x) = -2x^3 - x^2 + x$

33. $P(x) = x^4 - 3x^3 + 2x^2$ **34.** $P(x) = x^5 - 9x^3$

35. $P(x) = x^3 + x^2 - x - 1$ **36.** $P(x) = x^3 + 3x^2 - 4x - 12$

37. $P(x) = 2x^3 - x^2 - 18x + 9$

38. $P(x) = \frac{1}{8}(2x^4 + 3x^3 - 16x - 24)^2$

39. $P(x) = x^4 - 2x^3 - 8x + 16$

40. $P(x) = x^4 - 2x^3 + 8x - 16$

41. $P(x) = x^4 - 3x^2 - 4$ **42.** $P(x) = x^6 - 2x^3 + 1$

43–48 ■ Determine the end behavior of P. Compare the graphs of P and Q in large and small viewing rectangles, as in Example 3(b).

43. $P(x) = 3x^3 - x^2 + 5x + 1$; $Q(x) = 3x^3$

44. $P(x) = -\frac{1}{8}x^3 + \frac{1}{4}x^2 + 12x$; $Q(x) = -\frac{1}{8}x^3$

45. $P(x) = x^4 - 7x^2 + 5x + 5$; $Q(x) = x^4$

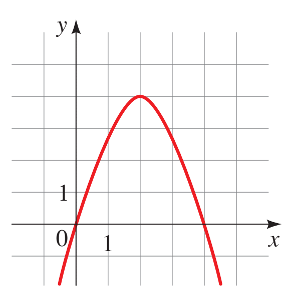
46. $P(x) = -x^5 + 2x^2 + x$; $Q(x) = -x^5$

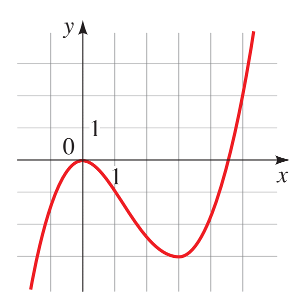
47. $P(x) = x^{11} - 9x^9$; $Q(x) = x^{11}$

48. $P(x) = 2x^2 - x^{12}$; $Q(x) = -x^{12}$

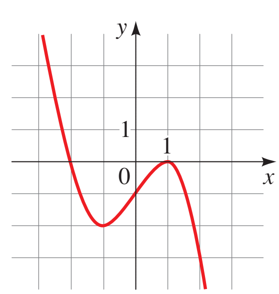
49–52 ■ The graph of a polynomial function is given. From the graph, find **(a)** the x- and y-intercepts, and **(b)** the coordinates of all local extrema.

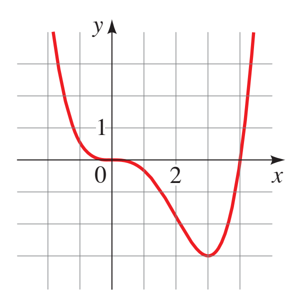
49. $P(x) = -x^2 + 4x$ **50.** $P(x) = \frac{2}{9}x^3 - x^2$





51. $P(x) = -\frac{1}{2}x^3 + \frac{3}{2}x - 1$ **52.** $P(x) = \frac{1}{9}x^4 - \frac{4}{9}x^3$





53–60 ■ Graph the polynomial in the given viewing rectangle. Find the coordinates of all local extrema. State each answer rounded to two decimal places.

53. $y = -x^2 + 8x$, $[-4, 12]$ by $[-50, 30]$

54. $y = x^3 - 3x^2$, $[-2, 5]$ by $[-10, 10]$

55. $y = x^3 - 12x + 9$, $[-5, 5]$ by $[-30, 30]$

56. $y = 2x^3 - 3x^2 - 12x - 32$, $[-5, 5]$ by $[-60, 30]$

57. $y = x^4 + 4x^3$, $[-5, 5]$ by $[-30, 30]$

58. $y = x^4 - 18x^2 + 32$, $[-5, 5]$ by $[-100, 100]$

59. $y = 3x^5 - 5x^3 + 3$, $[-3, 3]$ by $[-5, 10]$

60. $y = x^5 - 5x^2 + 6$, $[-3, 3]$ by $[-5, 10]$

61–70 ■ Graph the polynomial and determine how many local maxima and minima it has.

61. $y = -2x^2 + 3x + 5$

62. $y = x^3 + 12x$

63. $y = x^3 - x^2 - x$

64. $y = 6x^3 + 3x + 1$

65. $y = x^4 - 5x^2 + 4$

66. $y = 1.2x^5 + 3.75x^4 - 7x^3 - 15x^2 + 18x$

67. $y = (x - 2)^5 + 32$

68. $y = (x^2 - 2)^3$

69. $y = x^8 - 3x^4 + x$

70. $y = \frac{1}{3}x^7 - 17x^2 + 7$

71–76 ■ Graph the family of polynomials in the same viewing rectangle, using the given values of c. Explain how changing the value of c affects the graph.

71. $P(x) = cx^3; \quad c = 1, 2, 5, \frac{1}{2}$

72. $P(x) = (x - c)^4; \quad c = -1, 0, 1, 2$

73. $P(x) = x^4 + c; \quad c = -1, 0, 1, 2$

74. $P(x) = x^3 + cx; \quad c = 2, 0, -2, -4$

75. $P(x) = x^4 - cx; \quad c = 0, 1, 8, 27$

76. $P(x) = x^c; \quad c = 1, 3, 5, 7$

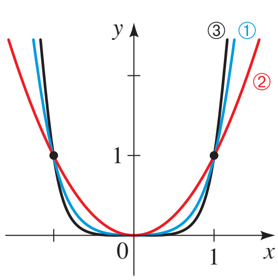
77. **(a)** On the same coordinate axes, sketch graphs (as accurately as possible) of the functions

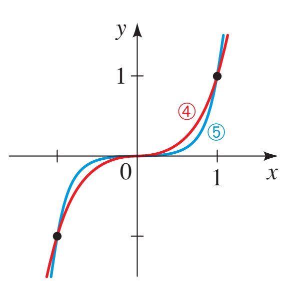
$$y = x^3 - 2x^2 - x + 2 \qquad \text{and} \qquad y = -x^2 + 5x + 2$$

(b) On the basis of your sketch in part (a), at how many points do the two graphs appear to intersect?

(c) Find the coordinates of all intersection points.

78. Portions of the graphs of $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$, and $y = x^6$ are plotted in the figures. Determine which function belongs to each graph.





79. Recall that a function f is *odd* if $f(-x) = -f(x)$ or *even* if $f(-x) = f(x)$ for all real x.

(a) Show that a polynomial $P(x)$ that contains only odd powers of x is an odd function.

(b) Show that a polynomial $P(x)$ that contains only even powers of x is an even function.

(c) Show that if a polynomial $P(x)$ contains both odd and even powers of x, then it is neither an odd nor an even function.

(d) Express the function

$$P(x) = x^5 + 6x^3 - x^2 - 2x + 5$$

as the sum of an odd function and an even function.

80. **(a)** Graph the function $P(x) = (x - 1)(x - 3)(x - 4)$ and find all local extrema, correct to the nearest tenth.

(b) Graph the function

$$Q(x) = (x - 1)(x - 3)(x - 4) + 5$$

and use your answers to part (a) to find all local extrema, correct to the nearest tenth.

81. **(a)** Graph the function $P(x) = (x - 2)(x - 4)(x - 5)$ and determine how many local extrema it has.

(b) If $a < b < c$, explain why the function

$$P(x) = (x - a)(x - b)(x - c)$$

must have two local extrema.

82. **(a)** How many x-intercepts and how many local extrema does the polynomial $P(x) = x^3 - 4x$ have?

(b) How many x-intercepts and how many local extrema does the polynomial $Q(x) = x^3 + 4x$ have?

(c) If $a > 0$, how many x-intercepts and how many local extrema does each of the polynomials $P(x) = x^3 - ax$ and $Q(x) = x^3 + ax$ have? Explain your answer.

APPLICATIONS

83. Market Research A market analyst working for a small-appliance manufacturer finds that if the firm produces and sells x blenders annually, the total profit (in dollars) is

$$P(x) = 8x + 0.3x^2 - 0.0013x^3 - 372$$

Graph the function P in an appropriate viewing rectangle and use the graph to answer the following questions.

(a) When just a few blenders are manufactured, the firm loses money (profit is negative). (For example, $P(10) = -263.3$, so the firm loses \$263.30 if it produces and sells only 10 blenders.) How many blenders must the firm produce to break even?

(b) Does profit increase indefinitely as more blenders are produced and sold? If not, what is the largest possible profit the firm could have?

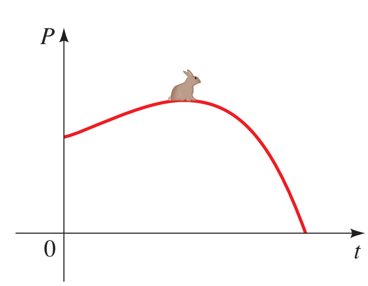
84. Population Change The rabbit population on a small island is observed to be given by the function

$$P(t) = 120t - 0.4t^4 + 1000$$

where t is the time (in months) since observations of the island began.

(a) When is the maximum population attained, and what is that maximum population?

(b) When does the rabbit population disappear from the island?

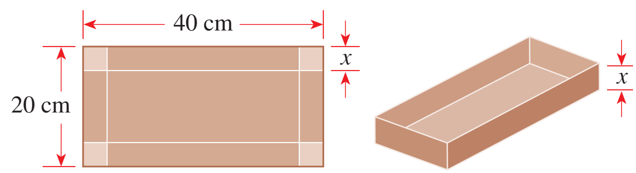


85. Volume of a Box An open box is to be constructed from a piece of cardboard 20 cm by 40 cm by cutting squares of side length x from each corner and folding up the sides, as shown in the figure.

(a) Express the volume V of the box as a function of x.

(b) What is the domain of V? (Use the fact that length and volume must be positive.)

(c) Draw a graph of the function V, and use it to estimate the maximum volume for such a box.

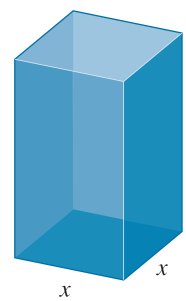


86. Volume of a Box A cardboard box has a square base, with each edge of the base having length x inches, as shown in the figure. The total length of all 12 edges of the box is 144 in.

(a) Show that the volume of the box is given by the function $V(x) = 2x^2(18 - x)$.

(b) What is the domain of V? (Use the fact that length and volume must be positive.)

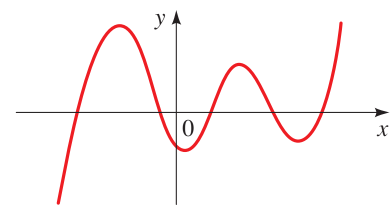
(c) Draw a graph of the function V and use it to estimate the maximum volume for such a box.



DISCOVERY ■ DISCUSSION ■ WRITING

87. Graphs of Large Powers Graph the functions $y = x^2$, $y = x^3$, $y = x^4$, and $y = x^5$, for $-1 \le x \le 1$, on the same coordinate axes. What do you think the graph of $y = x^{100}$ would look like on this same interval? What about $y = x^{101}$? Make a table of values to confirm your answers.

88. Maximum Number of Local Extrema What is the smallest possible degree that the polynomial whose graph is shown can have? Explain.



89. Possible Number of Local Extrema Is it possible for a third-degree polynomial to have exactly one local extremum? Can a fourth-degree polynomial have exactly two local extrema? How many local extrema can polynomials of third, fourth, fifth, and sixth degree have? (Think about the end behavior of such polynomials.) Now give an example of a polynomial that has six local extrema.

90. Impossible Situation? Is it possible for a polynomial to have two local maxima and no local minimum? Explain.

3.3 Dividing Polynomials

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Use long division to divide polynomials ▶ Use synthetic division to divide polynomials ▶ Use the Remainder Theorem to find values of a polynomial ▶ Use the Factor Theorem to factor a polynomial ▶ Find a polynomial with specified zeros

So far in this chapter we have been studying polynomial functions *graphically*. In this section we begin to study polynomials *algebraically*. Most of our work will be concerned with factoring polynomials, and to factor, we need to know how to divide polynomials.

▼ Long Division of Polynomials

Dividing polynomials is much like the familiar process of dividing numbers. When we divide 38 by 7, the quotient is 5 and the remainder is 3. We write

Dividend

Remainder

$$\frac{38}{7} = 5 + \frac{3}{7}$$

Divisor

Quotient

To divide polynomials, we use long division, as follows.

DIVISION ALGORITHM

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree less than the degree of $D(x)$, such that

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \qquad \text{or} \qquad \underbrace{P(x)}_{\text{Dividend}} = \underbrace{D(x)}_{\text{Divisor}} \cdot \underbrace{Q(x)}_{\text{Quotient}} + \overbrace{R(x)}^{\text{Remainder}}$$

The polynomials $P(x)$ and $D(x)$ are called the **dividend** and **divisor**, respectively, $Q(x)$ is the **quotient**, and $R(x)$ is the **remainder**.

EXAMPLE 1 | Long Division of Polynomials

Divide $6x^2 - 26x + 12$ by $x - 4$. Express the result in each of the two forms shown in the above box.

SOLUTION The *dividend* is $6x^2 - 26x + 12$, and the *divisor* is $x - 4$. We begin by arranging them as follows:

$$x - 4 \overline{\smash{)}\,6x^2 - 26x + 12}$$

Next we divide the leading term in the dividend by the leading term in the divisor to get the first term of the quotient: $6x^2/x = 6x$. Then we multiply the divisor by $6x$ and subtract the result from the dividend:

$$\begin{array}{rl} 6x & \text{Divide leading terms: } \dfrac{6x^2}{x} = 6x \\ x - 4 \overline{\smash{)}\,6x^2 - 26x + 12} & \\ \underline{6x^2 - 24x} & \text{Multiply: } 6x(x - 4) = 6x^2 - 24x \\ -2x + 12 & \text{Subtract and "bring down" 12} \end{array}$$

We repeat the process using the last line $-2x + 12$ as the dividend.

$$\begin{array}{rl} 6x - 2 & \text{Divide leading terms: } \dfrac{-2x}{x} = -2 \\ x - 4 \overline{\smash{)}\,6x^2 - 26x + 12} & \\ \underline{6x^2 - 24x} & \\ -2x + 12 & \\ \underline{-2x + 8} & \text{Multiply: } -2(x - 4) = -2x + 8 \\ 4 & \text{Subtract} \end{array}$$

The division process ends when the last line is of lesser degree than the divisor. The last line then contains the *remainder*, and the top line contains the *quotient*. The result of the division can be interpreted in either of two ways:

$$\frac{\overbrace{6x^2 - 26x + 12}^{\text{Dividend}}}{\underbrace{x - 4}_{\text{Divisor}}} = \overbrace{6x - 2}^{\text{Quotient}} + \frac{\overbrace{4}^{\text{Remainder}}}{x - 4} \qquad \text{or} \qquad \underbrace{6x^2 - 26x + 12}_{\text{Dividend}} = \underbrace{(x - 4)}_{\text{Divisor}}\underbrace{(6x - 2)}_{\text{Quotient}} + \overbrace{4}^{\text{Remainder}}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **3** AND **9**

EXAMPLE 2 | Long Division of Polynomials

Let $P(x) = 8x^4 + 6x^2 - 3x + 1$ and $D(x) = 2x^2 - x + 2$. Find polynomials $Q(x)$ and $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$.

SOLUTION We use long division after first inserting the term $0x^3$ into the dividend to ensure that the columns line up correctly.

$$
\begin{array}{rl}
 & 4x^2 + 2x \\
2x^2 - x + 2 \overline{)} & 8x^4 + 0x^3 + 6x^2 - 3x + 1 \\
 & \underline{8x^4 - 4x^3 + 8x^2} \qquad \text{Multiply divisor by } 4x^2 \\
 & 4x^3 - 2x^2 - 3x \qquad \text{Subtract} \\
 & \underline{4x^3 - 2x^2 + 4x} \qquad \text{Multiply divisor by } 2x \\
 & -7x + 1 \qquad \text{Subtract}
\end{array}
$$

The process is complete at this point because $-7x + 1$ is of lesser degree than the divisor $2x^2 - x + 2$. From the above long division we see that $Q(x) = 4x^2 + 2x$ and $R(x) = -7x + 1$, so

$$8x^4 + 6x^2 - 3x + 1 = (2x^2 - x + 2)(4x^2 + 2x) + (-7x + 1)$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **19**

Synthetic Division

Synthetic division is a quick method of dividing polynomials; it can be used when the divisor is of the form $x - c$. In synthetic division we write only the essential parts of the long division. Compare the following long and synthetic divisions, in which we divide $2x^3 - 7x^2 + 5$ by $x - 3$. (We'll explain how to perform the synthetic division in Example 3.)

Long Division

$$
\begin{array}{rl}
 & \underbrace{2x^2 - x - 3}_{\text{Quotient}} \\
x - 3 \overline{)} & 2x^3 - 7x^2 + 0x + 5 \\
 & \underline{2x^3 - 6x^2} \\
 & -x^2 + 0x \\
 & \underline{-x^2 + 3x} \\
 & -3x + 5 \\
 & \underline{-3x + 9} \\
 & -4 \quad \text{Remainder}
\end{array}
$$

Synthetic Division

$$
\begin{array}{r|rrrr}
3 & 2 & -7 & 0 & 5 \\
 & & 6 & -3 & -9 \\
\hline
 & \underbrace{2 \quad -1 \quad -3}_{\text{Quotient}} & & & \underbrace{-4}_{\text{Remainder}}
\end{array}
$$

Note that in synthetic division we abbreviate $2x^3 - 7x^2 + 5$ by writing only the coefficients: 2 -7 0 5, and instead of $x - 3$, we simply write 3. (Writing 3 instead of -3 allows us to add instead of subtract, but this changes the sign of all the numbers that appear in the gold boxes.)

The next example shows how synthetic division is performed.

EXAMPLE 3 | Synthetic Division

Use synthetic division to divide $2x^3 - 7x^2 + 5$ by $x - 3$.

SOLUTION We begin by writing the appropriate coefficients to represent the divisor and the dividend:

$$
\begin{array}{lr|rrrrl}
\text{Divisor } x - 3 & 3 & 2 & -7 & 0 & 5 & \text{Dividend } 2x^3 - 7x^2 + 0x + 5 \\
\hline
\end{array}
$$

We bring down the 2, multiply $3 \cdot 2 = 6$, and write the result in the middle row. Then we add:

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ & & 6 & & \\ \hline & \boxed{2} & -1 & & \end{array}$$

Multiply: $3 \cdot 2 = 6$

Add: $-7 + 6 = -1$

We repeat this process of multiplying and then adding until the table is complete.

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ & & 6 & -3 & \\ \hline & 2 & \boxed{-1} & -3 & \end{array}$$

Multiply: $3(-1) = -3$

Add: $0 + (-3) = -3$

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ & & 6 & -3 & -9 \\ \hline & 2 & -1 & \boxed{-3} & -4 \end{array}$$

Multiply: $3(-3) = -9$

Add: $5 + (-9) = -4$

Quotient
$2x^2 - x - 3$

Remainder
-4

From the last line of the synthetic division we see that the quotient is $2x^2 - x - 3$ and the remainder is -4. Thus

$$2x^3 - 7x^2 + 5 = (x - 3)(2x^2 - x - 3) - 4$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **31**

The Remainder and Factor Theorems

The next theorem shows how synthetic division can be used to evaluate polynomials easily.

REMAINDER THEOREM

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

PROOF If the divisor in the Division Algorithm is of the form $x - c$ for some real number c, then the remainder must be a constant (since the degree of the remainder is less than the degree of the divisor). If we call this constant r, then

$$P(x) = (x - c) \cdot Q(x) + r$$

Replacing x by c in this equation, we get $P(c) = (c - c) \cdot Q(x) + r = 0 + r = r$, that is, $P(c)$ is the remainder r.

EXAMPLE 4 | Using the Remainder Theorem to Find the Value of a Polynomial

Let $P(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3$.

(a) Find the quotient and remainder when $P(x)$ is divided by $x + 2$.

(b) Use the Remainder Theorem to find $P(-2)$.

SOLUTION

(a) Since $x + 2 = x - (-2)$, the synthetic division for this problem takes the following form:

$$\begin{array}{r|rrrrrr} -2 & 3 & 5 & -4 & 0 & 7 & 3 \\ & & -6 & 2 & 4 & -8 & 2 \\ \hline & 3 & -1 & -2 & 4 & -1 & 5 \end{array}$$

Remainder is 5, so $P(-2) = 5$

The quotient is $3x^4 - x^3 - 2x^2 + 4x - 1$, and the remainder is 5.

(b) By the Remainder Theorem, $P(-2)$ is the remainder when $P(x)$ is divided by $x - (-2) = x + 2$. From part (a) the remainder is 5, so $P(-2) = 5$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **39**

The next theorem says that *zeros* of polynomials correspond to *factors*; we used this fact in Section 3.2 to graph polynomials.

FACTOR THEOREM

c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

PROOF If $P(x)$ factors as $P(x) = (x - c) \cdot Q(x)$, then

$$P(c) = (c - c) \cdot Q(c) = 0 \cdot Q(c) = 0$$

Conversely, if $P(c) = 0$, then by the Remainder Theorem

$$P(x) = (x - c) \cdot Q(x) + 0 = (x - c) \cdot Q(x)$$

so $x - c$ is a factor of $P(x)$.

EXAMPLE 5 | Factoring a Polynomial Using the Factor Theorem

Let $P(x) = x^3 - 7x + 6$. Show that $P(1) = 0$, and use this fact to factor $P(x)$ completely.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -7 & 6 \\ & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

$$\begin{array}{r} x^2 + x - 6 \\ x - 1 \overline{)\, x^3 + 0x^2 - 7x + 6} \\ \underline{x^3 - x^2} \qquad\qquad \\ x^2 - 7x \quad\;\; \\ \underline{x^2 - x} \quad\;\; \\ -6x + 6 \\ \underline{-6x + 6} \\ 0 \end{array}$$

SOLUTION Substituting, we see that $P(1) = 1^3 - 7 \cdot 1 + 6 = 0$. By the Factor Theorem this means that $x - 1$ is a factor of $P(x)$. Using synthetic or long division (shown in the margin), we see that

$$\begin{aligned} P(x) &= x^3 - 7x + 6 && \text{Given polynomial} \\ &= (x - 1)(x^2 + x - 6) && \text{See margin} \\ &= (x - 1)(x - 2)(x + 3) && \text{Factor quadratic } x^2 + x - 6 \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **53** AND **57**

EXAMPLE 6 | Finding a Polynomial with Specified Zeros

Find a polynomial of degree 4 that has zeros -3, 0, 1, and 5.

SOLUTION By the Factor Theorem, $x - (-3)$, $x - 0$, $x - 1$, and $x - 5$ must all be factors of the desired polynomial.

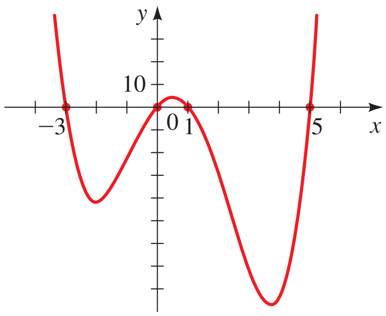


FIGURE 1
$P(x) = (x + 3)x(x - 1)(x - 5)$ has zeros -3, 0, 1, and 5.

Let

$$P(x) = (x + 3)(x - 0)(x - 1)(x - 5)$$
$$= x^4 - 3x^3 - 13x^2 + 15x$$

Since $P(x)$ is of degree 4, it is a solution of the problem. Any other solution of the problem must be a constant multiple of $P(x)$, since only multiplication by a constant does not change the degree.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **59**

The polynomial P of Example 6 is graphed in Figure 1. Note that the zeros of P correspond to the x-intercepts of the graph.

3.3 EXERCISES

CONCEPTS

1. If we divide the polynomial P by the factor $x - c$ and we obtain the equation $P(x) = (x - c)Q(x) + R(x)$, then we say that $x - c$ is the divisor, $Q(x)$ is the ________, and $R(x)$ is the ________.

2. (a) If we divide the polynomial $P(x)$ by the factor $x - c$ and we obtain a remainder of 0, then we know that c is a ________ of P.

(b) If we divide the polynomial $P(x)$ by the factor $x - c$ and we obtain a remainder of k, then we know that $P(c) =$ ________.

SKILLS

3–8 ■ Two polynomials P and D are given. Use either synthetic or long division to divide $P(x)$ by $D(x)$, and express the quotient $P(x)/D(x)$ in the form

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

3. $P(x) = x^2 + 4x - 8$, $D(x) = x + 3$

4. $P(x) = x^3 + 6x + 5$, $D(x) = x - 4$

5. $P(x) = 4x^2 - 3x - 7$, $D(x) = 2x - 1$

6. $P(x) = 6x^3 + x^2 - 12x + 5$, $D(x) = 3x - 4$

7. $P(x) = 2x^4 - x^3 + 9x^2$, $D(x) = x^2 + 4$

8. $P(x) = x^5 + x^4 - 2x^3 + x + 1$, $D(x) = x^2 + x - 1$

9–14 ■ Two polynomials P and D are given. Use either synthetic or long division to divide $P(x)$ by $D(x)$, and express P in the form $P(x) = D(x) \cdot Q(x) + R(x)$.

9. $P(x) = 3x^2 + 5x - 4$, $D(x) = x + 3$

10. $P(x) = x^3 + 4x^2 - 6x + 1$, $D(x) = x - 1$

11. $P(x) = 2x^3 - 3x^2 - 2x$, $D(x) = 2x - 3$

12. $P(x) = 4x^3 + 7x + 9$, $D(x) = 2x + 1$

13. $P(x) = x^4 - x^3 + 4x + 2$, $D(x) = x^2 + 3$

14. $P(x) = 2x^5 + 4x^4 - 4x^3 - x - 3$, $D(x) = x^2 - 2$

15–24 ■ Find the quotient and remainder using long division.

15. $\dfrac{x^2 - 6x - 8}{x - 4}$ **16.** $\dfrac{x^3 - x^2 - 2x + 6}{x - 2}$

17. $\dfrac{4x^3 + 2x^2 - 2x - 3}{2x + 1}$ **18.** $\dfrac{x^3 + 3x^2 + 4x + 3}{3x + 6}$

19. $\dfrac{x^3 + 6x + 3}{x^2 - 2x + 2}$ **20.** $\dfrac{3x^4 - 5x^3 - 20x - 5}{x^2 + x + 3}$

21. $\dfrac{6x^3 + 2x^2 + 22x}{2x^2 + 5}$ **22.** $\dfrac{9x^2 - x + 5}{3x^2 - 7x}$

23. $\dfrac{x^6 + x^4 + x^2 + 1}{x^2 + 1}$ **24.** $\dfrac{2x^5 - 7x^4 - 13}{4x^2 - 6x + 8}$

25–38 ■ Find the quotient and remainder using synthetic division.

25. $\dfrac{x^2 - 5x + 4}{x - 3}$ **26.** $\dfrac{x^2 - 5x + 4}{x - 1}$

27. $\dfrac{3x^2 + 5x}{x - 6}$ **28.** $\dfrac{4x^2 - 3}{x + 5}$

29. $\dfrac{x^3 + 2x^2 + 2x + 1}{x + 2}$ **30.** $\dfrac{3x^3 - 12x^2 - 9x + 1}{x - 5}$

31. $\dfrac{x^3 - 8x + 2}{x + 3}$ **32.** $\dfrac{x^4 - x^3 + x^2 - x + 2}{x - 2}$

33. $\dfrac{x^5 + 3x^3 - 6}{x - 1}$ **34.** $\dfrac{x^3 - 9x^2 + 27x - 27}{x - 3}$

35. $\dfrac{2x^3 + 3x^2 - 2x + 1}{x - \frac{1}{2}}$

36. $\dfrac{6x^4 + 10x^3 + 5x^2 + x + 1}{x + \frac{2}{3}}$

37. $\dfrac{x^3 - 27}{x - 3}$ **38.** $\dfrac{x^4 - 16}{x + 2}$

39–51 ■ Use synthetic division and the Remainder Theorem to evaluate $P(c)$.

39. $P(x) = 4x^2 + 12x + 5, \quad c = -1$

40. $P(x) = 2x^2 + 9x + 1, \quad c = \frac{1}{2}$

41. $P(x) = x^3 + 3x^2 - 7x + 6, \quad c = 2$

42. $P(x) = x^3 - x^2 + x + 5, \quad c = -1$

43. $P(x) = x^3 + 2x^2 - 7, \quad c = -2$

44. $P(x) = 2x^3 - 21x^2 + 9x - 200, \quad c = 11$

45. $P(x) = 5x^4 + 30x^3 - 40x^2 + 36x + 14, \quad c = -7$

46. $P(x) = 6x^5 + 10x^3 + x + 1, \quad c = -2$

47. $P(x) = x^7 - 3x^2 - 1, \quad c = 3$

48. $P(x) = -2x^6 + 7x^5 + 40x^4 - 7x^2 + 10x + 112, \quad c = -3$

49. $P(x) = 3x^3 + 4x^2 - 2x + 1, \quad c = \frac{2}{3}$

50. $P(x) = x^3 - x + 1, \quad c = \frac{1}{4}$

51. $P(x) = x^3 + 2x^2 - 3x - 8, \quad c = 0.1$

52. Let

$$P(x) = 6x^7 - 40x^6 + 16x^5 - 200x^4 - 60x^3 - 69x^2 + 13x - 139$$

Calculate $P(7)$ by **(a)** using synthetic division and **(b)** substituting $x = 7$ into the polynomial and evaluating directly.

53–56 ■ Use the Factor Theorem to show that $x - c$ is a factor of $P(x)$ for the given value(s) of c.

53. $P(x) = x^3 - 3x^2 + 3x - 1, \quad c = 1$

54. $P(x) = x^3 + 2x^2 - 3x - 10, \quad c = 2$

55. $P(x) = 2x^3 + 7x^2 + 6x - 5, \quad c = \frac{1}{2}$

56. $P(x) = x^4 + 3x^3 - 16x^2 - 27x + 63, \quad c = 3, -3$

57–58 ■ Show that the given value(s) of c are zeros of $P(x)$, and find all other zeros of $P(x)$.

57. $P(x) = x^3 - x^2 - 11x + 15, \quad c = 3$

58. $P(x) = 3x^4 - x^3 - 21x^2 - 11x + 6, \quad c = \frac{1}{3}, -2$

59–62 ■ Find a polynomial of the specified degree that has the given zeros.

59. Degree 3; zeros $-1, 1, 3$

60. Degree 4; zeros $-2, 0, 2, 4$

61. Degree 4; zeros $-1, 1, 3, 5$

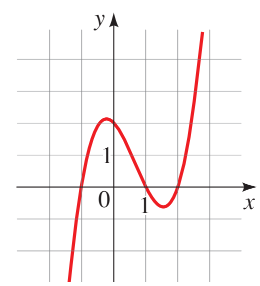
62. Degree 5; zeros $-2, -1, 0, 1, 2$

63. Find a polynomial of degree 3 that has zeros 1, -2, and 3 and in which the coefficient of x^2 is 3.

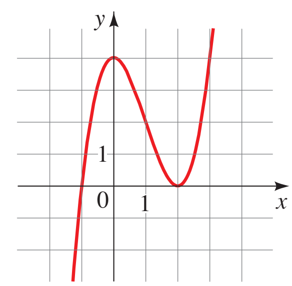
64. Find a polynomial of degree 4 that has integer coefficients and zeros 1, -1, 2, and $\frac{1}{2}$.

65–68 ■ Find the polynomial of the specified degree whose graph is shown.

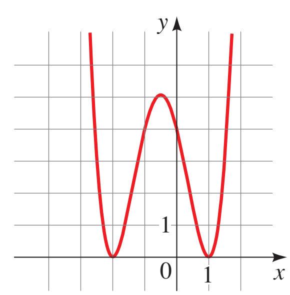
65. Degree 3



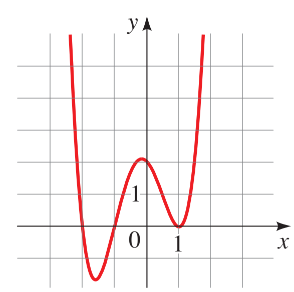
66. Degree 3



67. Degree 4



68. Degree 4



DISCOVERY ■ DISCUSSION ■ WRITING

69. Impossible Division? Suppose you were asked to solve the following two problems on a test:

A. Find the remainder when $6x^{1000} - 17x^{562} + 12x + 26$ is divided by $x + 1$.

B. Is $x - 1$ a factor of $x^{567} - 3x^{400} + x^9 + 2$?

Obviously, it's impossible to solve these problems by dividing, because the polynomials are of such large degree. Use one or more of the theorems in this section to solve these problems *without* actually dividing.

70. Nested Form of a Polynomial Expand Q to prove that the polynomials P and Q are the same.

$$P(x) = 3x^4 - 5x^3 + x^2 - 3x + 5$$

$$Q(x) = (((3x - 5)x + 1)x - 3)x + 5$$

Try to evaluate $P(2)$ and $Q(2)$ in your head, using the forms given. Which is easier? Now write the polynomial $R(x) = x^5 - 2x^4 + 3x^3 - 2x^2 + 3x + 4$ in "nested" form, like the polynomial Q. Use the nested form to find $R(3)$ in your head.

Do you see how calculating with the nested form follows the same arithmetic steps as calculating the value of a polynomial using synthetic division?

3.4 Real Zeros of Polynomials

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Use the Rational Zeros Theorem to find the rational zeros of polynomials ▶ Use Descartes' Rule of Signs to determine the possible number of positive and negative zeros of a polynomial ▶ Use the Upper and Lower Bounds Theorem to find upper and lower bounds for the zeros of a polynomial ▶ Use algebra and graphing devices to solve polynomial equations

The Factor Theorem tells us that finding the zeros of a polynomial is really the same thing as factoring it into linear factors. In this section we study some algebraic methods that help us to find the real zeros of a polynomial and thereby factor the polynomial. We begin with the *rational* zeros of a polynomial.

▼ Rational Zeros of Polynomials

To help us understand the next theorem, let's consider the polynomial

$$\begin{aligned} P(x) &= (x-2)(x-3)(x+4) && \text{Factored form} \\ &= x^3 - x^2 - 14x + 24 && \text{Expanded form} \end{aligned}$$

From the factored form we see that the zeros of P are 2, 3, and -4. When the polynomial is expanded, the constant 24 is obtained by multiplying $(-2) \times (-3) \times 4$. This means that the zeros of the polynomial are all factors of the constant term. The following generalizes this observation.

RATIONAL ZEROS THEOREM

If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients (where $a_n \neq 0$ and $a_0 \neq 0$), then every rational zero of P is of the form

$$\frac{p}{q}$$

where p an q are integers and

p is a factor of the constant coefficient a_0
q is a factor of the leading coefficient a_n

PROOF If p/q is a rational zero, in lowest terms, of the polynomial P, then we have

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} + a_0 q^n = 0 \qquad \text{Multiply by } q^n$$

$$p(a_n p^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1 q^{n-1}) = -a_0 q^n \qquad \text{Subtract } a_0q^n \text{ and factor LHS}$$

Now p is a factor of the left side, so it must be a factor of the right side as well. Since p/q is in lowest terms, p and q have no factor in common, so p must be a factor of a_0. A similar proof shows that q is a factor of a_n. ■

We see from the Rational Zeros Theorem that if the leading coefficient is 1 or -1, then the rational zeros must be factors of the constant term.

EVARISTE GALOIS (1811–1832) is one of the very few mathematicians to have an entire theory named in his honor. Not yet 21 when he died, he completely settled the central problem in the theory of equations by describing a criterion that reveals whether a polynomial equation can be solved by algebraic operations. Galois was one of the greatest mathematicians in the world at that time, although no one knew it but him. He repeatedly sent his work to the eminent mathematicians Cauchy and Poisson, who either lost his letters or did not understand his ideas. Galois wrote in a terse style and included few details, which probably played a role in his failure to pass the entrance exams at the Ecole Polytechnique in Paris. A political radical, Galois spent several months in prison for his revolutionary activities. His brief life came to a tragic end when he was killed in a duel over a love affair. The night before his duel, fearing that he would die, Galois wrote down the essence of his ideas and entrusted them to his friend Auguste Chevalier. He concluded by writing "there will, I hope, be people who will find it to their advantage to decipher all this mess." The mathematician Camille Jordan did just that, 14 years later.

EXAMPLE 1 | Using the Rational Zeros Theorem

Find the rational zeros of $P(x) = x^3 - 3x + 2$.

SOLUTION Since the leading coefficient is 1, any rational zero must be a divisor of the constant term 2. So the possible rational zeros are ± 1 and ± 2. We test each of these possibilities:

$$\begin{aligned} P(1) &= (1)^3 - 3(1) + 2 = 0 \\ P(-1) &= (-1)^3 - 3(-1) + 2 = 4 \\ P(2) &= (2)^3 - 3(2) + 2 = 4 \\ P(-2) &= (-2)^3 - 3(-2) + 2 = 0 \end{aligned}$$

The rational zeros of P are 1 and -2.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **15**

The following box explains how we use the Rational Zeros Theorem with synthetic division to factor a polynomial.

FINDING THE RATIONAL ZEROS OF A POLYNOMIAL

1. **List Possible Zeros.** List all possible rational zeros, using the Rational Zeros Theorem.
2. **Divide.** Use synthetic division to evaluate the polynomial at each of the candidates for the rational zeros that you found in Step 1. When the remainder is 0, note the quotient you have obtained.
3. **Repeat.** Repeat Steps 1 and 2 for the quotient. Stop when you reach a quotient that is quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

EXAMPLE 2 | Finding Rational Zeros

Write the polynomial $P(x) = 2x^3 + x^2 - 13x + 6$ in factored form, and find all its zeros.

SOLUTION By the Rational Zeros Theorem the rational zeros of P are of the form

$$\text{possible rational zero of } P = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}}$$

The constant term is 6 and the leading coefficient is 2, so

$$\text{possible rational zero of } P = \frac{\text{factor of 6}}{\text{factor of 2}}$$

The factors of 6 are $\pm 1, \pm 2, \pm 3, \pm 6$, and the factors of 2 are $\pm 1, \pm 2$. Thus the possible rational zeros of P are

$$\pm\frac{1}{1}, \quad \pm\frac{2}{1}, \quad \pm\frac{3}{1}, \quad \pm\frac{6}{1}, \quad \pm\frac{1}{2}, \quad \pm\frac{2}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{6}{2}$$

Simplifying the fractions and eliminating duplicates, we get the following list of possible rational zeros:

$$\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6, \quad \pm\frac{1}{2}, \quad \pm\frac{3}{2}$$

To check which of these *possible* zeros actually *are* zeros, we need to evaluate P at each of these numbers. An efficient way to do this is to use synthetic division.

Test whether 1 is a zero

$$\begin{array}{r|rrrr} 1 & 2 & 1 & -13 & 6 \\ & & 2 & 3 & -10 \\ \hline & 2 & 3 & -10 & -4 \end{array}$$

Remainder is *not* 0, so 1 is *not* a zero

Test whether 2 is a zero

$$\begin{array}{r|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

Remainder *is* 0, so 2 *is* a zero

From the last synthetic division we see that 2 is a zero of P and that P factors as

$$\begin{aligned} P(x) &= 2x^3 + x^2 - 13x + 6 && \text{Given polynomial} \\ &= (x-2)(2x^2 + 5x - 3) && \text{From synthetic division} \\ &= (x-2)(2x-1)(x+3) && \text{Factor } 2x^2 + 5x - 3 \end{aligned}$$

From the factored form we see that the zeros of P are 2, $\frac{1}{2}$, and –3.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **31**

EXAMPLE 3 | Using the Rational Zeros Theorem and the Quadratic Formula

Let $P(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$.

(a) Find the zeros of P. **(b)** Sketch the graph of P.

SOLUTION

$$\begin{array}{r|rrrrr} 1 & 1 & -5 & -5 & 23 & 10 \\ & & 1 & -4 & -9 & 14 \\ \hline & 1 & -4 & -9 & 14 & 24 \end{array}$$

$$\begin{array}{r|rrrrr} 2 & 1 & -5 & -5 & 23 & 10 \\ & & 2 & -6 & -22 & 2 \\ \hline & 1 & -3 & -11 & 1 & 12 \end{array}$$

$$\begin{array}{r|rrrrr} 5 & 1 & -5 & -5 & 23 & 10 \\ & & 5 & 0 & -25 & -10 \\ \hline & 1 & 0 & -5 & -2 & 0 \end{array}$$

(a) The leading coefficient of P is 1, so all the rational zeros are integers: They are divisors of the constant term 10. Thus the possible candidates are

$$\pm 1, \quad \pm 2, \quad \pm 5, \quad \pm 10$$

Using synthetic division (see the margin), we find that 1 and 2 are not zeros but that 5 is a zero and that P factors as

$$x^4 - 5x^3 - 5x^2 + 23x + 10 = (x-5)(x^3 - 5x - 2)$$

We now try to factor the quotient $x^3 - 5x - 2$. Its possible zeros are the divisors of -2, namely,

$$\pm 1, \quad \pm 2$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & 0 \end{array}$$

Since we already know that 1 and 2 are not zeros of the original polynomial P, we don't need to try them again. Checking the remaining candidates, -1 and -2, we see that -2 is a zero (see the margin), and P factors as

$$\begin{aligned} x^4 - 5x^3 - 5x^2 + 23x + 10 &= (x-5)(x^3 - 5x - 2) \\ &= (x-5)(x+2)(x^2 - 2x - 1) \end{aligned}$$

Now we use the Quadratic Formula to obtain the two remaining zeros of P:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2} = 1 \pm \sqrt{2}$$

The zeros of P are 5, -2, $1 + \sqrt{2}$, and $1 - \sqrt{2}$.

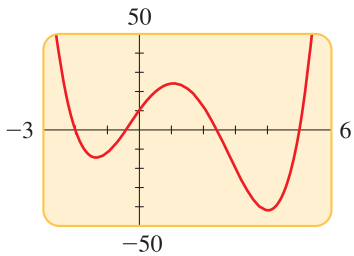


FIGURE 1
$P(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$

(b) Now that we know the zeros of P, we can use the methods of Section 3.2 to sketch the graph. If we want to use a graphing calculator instead, knowing the zeros allows us to choose an appropriate viewing rectangle—one that is wide enough to contain all the x-intercepts of P. Numerical approximations to the zeros of P are

$$5, \qquad -2, \qquad 2.4, \qquad \text{and} \qquad -0.4$$

So in this case we choose the rectangle $[-3, 6]$ by $[-50, 50]$ and draw the graph shown in Figure 1.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **49** AND **59**

Descartes' Rule of Signs and Upper and Lower Bounds for Roots

In some cases, the following rule—discovered by the French philosopher and mathematician René Descartes around 1637 (see page 213)—is helpful in eliminating candidates from lengthy lists of possible rational roots. To describe this rule, we need the concept of *variation in sign*. If $P(x)$ is a polynomial with real coefficients, written with descending powers of x (and omitting powers with coefficient 0), then a **variation in sign** occurs whenever adjacent coefficients have opposite signs. For example,

$$P(x) = 5x^7 - 3x^5 - x^4 + 2x^2 + x - 3$$

has three variations in sign.

Polynomial	Variations in sign
$x^2 + 4x + 1$	0
$2x^3 + x - 6$	1
$x^4 - 3x^2 - x + 4$	2

DESCARTES' RULE OF SIGNS

Let P be a polynomial with real coefficients.

1. The number of positive real zeros of $P(x)$ either is equal to the number of variations in sign in $P(x)$ or is less than that by an even whole number.
2. The number of negative real zeros of $P(x)$ either is equal to the number of variations in sign in $P(-x)$ or is less than that by an even whole number.

Multiplicity is discussed on page 274.

In Descartes' Rule of Signs a zero with multiplicity m is counted m times. For example, the polynomial $P(x) = x^2 - 2x + 1$ has two sign changes and has the positive zero $x = 1$. But this zero is counted twice because it has multiplicity 2.

EXAMPLE 4 | Using Descartes' Rule

Use Descartes' Rule of Signs to determine the possible number of positive and negative real zeros of the polynomial

$$P(x) = 3x^6 + 4x^5 + 3x^3 - x - 3$$

SOLUTION The polynomial has one variation in sign, so it has one positive zero. Now

$$\begin{aligned} P(-x) &= 3(-x)^6 + 4(-x)^5 + 3(-x)^3 - (-x) - 3 \\ &= 3x^6 - 4x^5 - 3x^3 + x - 3 \end{aligned}$$

So $P(-x)$ has three variations in sign. Thus $P(x)$ has either three or one negative zero(s), making a total of either two or four real zeros.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **69**

We say that a is a **lower bound** and b is an **upper bound** for the zeros of a polynomial if every real zero c of the polynomial satisfies $a \le c \le b$. The next theorem helps us to find such bounds for the zeros of a polynomial.

THE UPPER AND LOWER BOUNDS THEOREM

Let P be a polynomial with real coefficients.

1. If we divide $P(x)$ by $x - b$ (with $b > 0$) using synthetic division and if the row that contains the quotient and remainder has no negative entry, then b is an upper bound for the real zeros of P.
2. If we divide $P(x)$ by $x - a$ (with $a < 0$) using synthetic division and if the row that contains the quotient and remainder has entries that are alternately nonpositive and nonnegative, then a is a lower bound for the real zeros of P.

A proof of this theorem is suggested in Exercise 103. The phrase "alternately nonpositive and nonnegative" simply means that the signs of the numbers alternate, with 0 considered to be positive or negative as required.

EXAMPLE 5 | Upper and Lower Bounds for the Zeros of a Polynomial

Show that all the real zeros of the polynomial $P(x) = x^4 - 3x^2 + 2x - 5$ lie between -3 and 2.

SOLUTION We divide $P(x)$ by $x - 2$ and $x + 3$ using synthetic division:

2	1	0	−3	2	−5	
		2	4	2	8	
	1	2	1	4	3	All entries nonnegative

−3	1	0	−3	2	−5	
		−3	9	−18	48	
	1	−3	6	−16	43	Entries alternate in sign

By the Upper and Lower Bounds Theorem, -3 is a lower bound and 2 is an upper bound for the zeros. Since neither -3 nor 2 is a zero (the remainders are not 0 in the division table), all the real zeros lie between these numbers.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **73**

EXAMPLE 6 | A Lower Bound for the Zeros of a Polynomial

Show that all the real zeros of the polynomial $P(x) = x^4 + 4x^3 + 3x^2 + 7x - 5$ are greater than or equal to -4.

SOLUTION We divide $P(x)$ by $x + 4$ using synthetic division:

−4	1	4	3	7	−5	
		−4	0	−12	20	
	1	0	3	−5	15	Alternately nonnegative and nonpositive

Since 0 can be considered either nonnegative or nonpositive, the entries alternate in sign. So -4 is a lower bound for the real zeros of P.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **77**

EXAMPLE 7 | Factoring a Fifth-Degree Polynomial

Factor completely the polynomial

$$P(x) = 2x^5 + 5x^4 - 8x^3 - 14x^2 + 6x + 9$$

SOLUTION The possible rational zeros of P are $\pm\frac{1}{2}$, ± 1, $\pm\frac{3}{2}$, ± 3, $\pm\frac{9}{2}$, and ± 9. We check the positive candidates first, beginning with the smallest:

$$\begin{array}{r|rrrrrr} \frac{1}{2} & 2 & 5 & -8 & -14 & 6 & 9 \\ & & 1 & 3 & -\frac{5}{2} & -\frac{33}{4} & -\frac{9}{8} \\ \hline & 2 & 6 & -5 & -\frac{33}{2} & -\frac{9}{4} & \frac{63}{8} \end{array}$$

$\frac{1}{2}$ is not a zero

$$\begin{array}{r|rrrrrr} 1 & 2 & 5 & -8 & -14 & 6 & 9 \\ & & 2 & 7 & -1 & -15 & -9 \\ \hline & 2 & 7 & -1 & -15 & -9 & 0 \end{array}$$

$P(1) = 0$

So 1 is a zero, and $P(x) = (x - 1)(2x^4 + 7x^3 - x^2 - 15x - 9)$. We continue by factoring the quotient. We still have the same list of possible zeros except that $\frac{1}{2}$ has been eliminated.

$$\begin{array}{r|rrrrr} 1 & 2 & 7 & -1 & -15 & -9 \\ & & 2 & 9 & 8 & -7 \\ \hline & 2 & 9 & 8 & -7 & -16 \end{array}$$

1 is not a zero

$$\begin{array}{r|rrrrr} \frac{3}{2} & 2 & 7 & -1 & -15 & -9 \\ & & 3 & 15 & 21 & 9 \\ \hline & 2 & 10 & 14 & 6 & 0 \end{array}$$

$P\left(\frac{3}{2}\right) = 0$, all entries nonnegative

We see that $\frac{3}{2}$ is both a zero and an upper bound for the zeros of $P(x)$, so we do not need to check any further for positive zeros, because all the remaining candidates are greater than $\frac{3}{2}$.

$$\begin{aligned} P(x) &= (x - 1)(x - \tfrac{3}{2})(2x^3 + 10x^2 + 14x + 6) && \text{From synthetic division} \\ &= (x - 1)(2x - 3)(x^3 + 5x^2 + 7x + 3) && \text{Factor 2 from last factor, multiply into second factor} \end{aligned}$$

By Descartes' Rule of Signs, $x^3 + 5x^2 + 7x + 3$ has no positive zero, so its only possible rational zeros are -1 and -3:

$$\begin{array}{r|rrrr} -1 & 1 & 5 & 7 & 3 \\ & & -1 & -4 & -3 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

$P(-1) = 0$

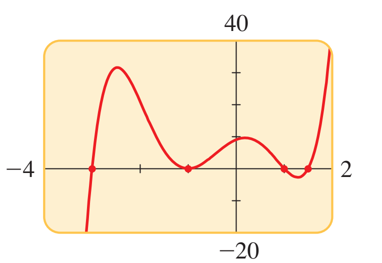


FIGURE 2
$P(x) = 2x^5 + 5x^4 - 8x^3 - 14x^2 + 6x + 9$
$= (x - 1)(2x - 3)(x + 1)^2(x + 3)$

Therefore,

$$\begin{aligned} P(x) &= (x - 1)(2x - 3)(x + 1)(x^2 + 4x + 3) && \text{From synthetic division} \\ &= (x - 1)(2x - 3)(x + 1)^2(x + 3) && \text{Factor quadratic} \end{aligned}$$

This means that the zeros of P are $1, \frac{3}{2}, -1$, and -3. The graph of the polynomial is shown in Figure 2.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **85**

Using Algebra and Graphing Devices to Solve Polynomial Equations

In Section 1.4 we used graphing devices to solve equations graphically. We can now use the algebraic techniques that we've learned to select an appropriate viewing rectangle when solving a polynomial equation graphically.

EXAMPLE 8 | Solving a Fourth-Degree Equation Graphically

Find all real solutions of the following equation, rounded to the nearest tenth:

$$3x^4 + 4x^3 - 7x^2 - 2x - 3 = 0$$

SOLUTION To solve the equation graphically, we graph

$$P(x) = 3x^4 + 4x^3 - 7x^2 - 2x - 3$$

We use the Upper and Lower Bounds Theorem to see where the solutions can be found.

First we use the Upper and Lower Bounds Theorem to find two numbers between which all the solutions must lie. This allows us to choose a viewing rectangle that is certain to contain all the x-intercepts of P. We use synthetic division and proceed by trial and error.

To find an upper bound, we try the whole numbers, 1, 2, 3, . . . , as potential candidates. We see that 2 is an upper bound for the solutions:

$$\begin{array}{r|rrrrr} 2 & 3 & 4 & -7 & -2 & -3 \\ & & 6 & 20 & 26 & 48 \\ \hline & 3 & 10 & 13 & 24 & 45 \end{array}$$

All positive

Now we look for a lower bound, trying the numbers -1, -2, and -3 as potential candidates. We see that -3 is a lower bound for the solutions:

$$\begin{array}{r|rrrrr} -3 & 3 & 4 & -7 & -2 & -3 \\ & & -9 & 15 & -24 & 78 \\ \hline & 3 & -5 & 8 & -26 & 75 \end{array}$$

Entries alternate in sign

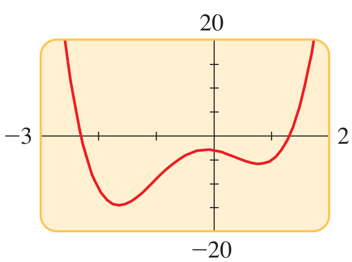


FIGURE 3
$y = 3x^4 + 4x^3 - 7x^2 - 2x - 3$

Thus all the solutions lie between -3 and 2. So the viewing rectangle $[-3, 2]$ by $[-20, 20]$ contains all the x-intercepts of P. The graph in Figure 3 has two x-intercepts, one between -3 and -2 and the other between 1 and 2. Zooming in, we find that the solutions of the equation, to the nearest tenth, are -2.3 and 1.3.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **99** ■

EXAMPLE 9 | Determining the Size of a Fuel Tank

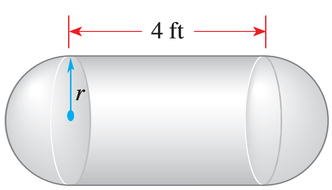


FIGURE 4

A fuel tank consists of a cylindrical center section that is 4 ft long and two hemispherical end sections, as shown in Figure 4. If the tank has a volume of 100 ft^3, what is the radius r shown in the figure, rounded to the nearest hundredth of a foot?

SOLUTION Using the volume formula listed on the inside back cover of this book, we see that the volume of the cylindrical section of the tank is

Volume of a cylinder: $V = \pi r^2 h$

$$\pi \cdot r^2 \cdot 4$$

The two hemispherical parts together form a complete sphere whose volume is

Volume of a sphere: $V = \frac{4}{3}\pi r^3$

$$\tfrac{4}{3}\pi r^3$$

Because the total volume of the tank is 100 ft^3, we get the following equation:

$$\tfrac{4}{3}\pi r^3 + 4\pi r^2 = 100$$

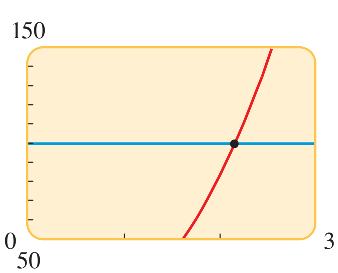


FIGURE 5
$y = \frac{4}{3}\pi x^3 + 4\pi x^2$ and $y = 100$

A negative solution for r would be meaningless in this physical situation, and by substitution we can verify that $r = 3$ leads to a tank that is over 226 ft^3 in volume, much larger than the required 100 ft^3. Thus, we know the correct radius lies somewhere between 0 and 3 ft, so we use a viewing rectangle of $[0, 3]$ by $[50, 150]$ to graph the function $y = \frac{4}{3}\pi x^3 + 4\pi x^2$, as shown in Figure 5. Since we want the value of this function to be 100, we also graph the horizontal line $y = 100$ in the same viewing rectangle. The correct radius will be the x-coordinate of the point of intersection of the curve and the line. Using the cursor and zooming in, we see that at the point of intersection $x \approx 2.15$, rounded to two decimal places. Thus the tank has a radius of about 2.15 ft.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **105** ■

Note that we also could have solved the equation in Example 9 by first writing it as

$$\tfrac{4}{3}\pi r^3 + 4\pi r^2 - 100 = 0$$

and then finding the x-intercept of the function $y = \frac{4}{3}\pi x^3 + 4\pi x^2 - 100$.

3.4 EXERCISES

CONCEPTS

1. If the polynomial function
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
has integer coefficients, then the only numbers that could possibly be rational zeros of P are all of the form $\frac{p}{q}$, where p is a factor of ________ and q is a factor of ________. The possible rational zeros of $P(x) = 6x^3 + 5x^2 - 19x - 10$ are

____

2. Using Descartes' Rule of Signs, we can tell that the polynomial $P(x) = x^5 - 3x^4 + 2x^3 - x^2 + 8x - 8$ has ________, ________, or ________ positive real zeros and ________ negative real zeros.

3. *True or false*? If c is a real zero of the polynomial P, then all the other zeros of P are zeros of $P(x)/(x - c)$.

4. *True or false*? If a is an upper bound for the real zeros of the polynomial P, then $-a$ is necessarily a lower bound for the real zeros of P.

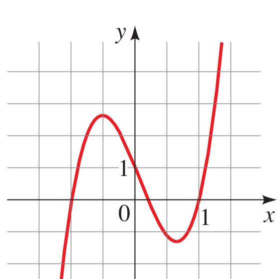
SKILLS

5–10 ■ List all possible rational zeros given by the Rational Zeros Theorem (but don't check to see which actually are zeros).

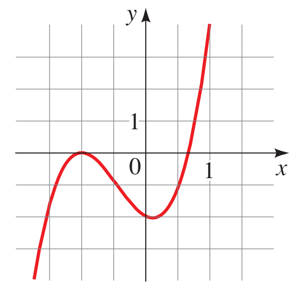
5. $P(x) = x^3 - 4x^2 + 3$
6. $Q(x) = x^4 - 3x^3 - 6x + 8$
7. $R(x) = 2x^5 + 3x^3 + 4x^2 - 8$
8. $S(x) = 6x^4 - x^2 + 2x + 12$
9. $T(x) = 4x^4 - 2x^2 - 7$
10. $U(x) = 12x^5 + 6x^3 - 2x - 8$

11–14 ■ A polynomial function P and its graph are given. **(a)** List all possible rational zeros of P given by the Rational Zeros Theorem. **(b)** From the graph, determine which of the possible rational zeros actually turn out to be zeros.

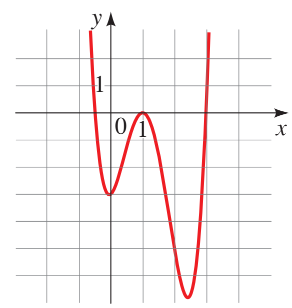
11. $P(x) = 5x^3 - x^2 - 5x + 1$



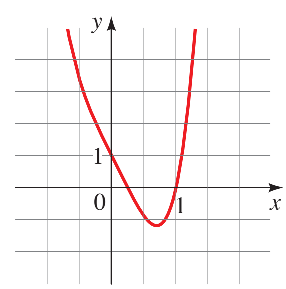
12. $P(x) = 3x^3 + 4x^2 - x - 2$



13. $P(x) = 2x^4 - 9x^3 + 9x^2 + x - 3$



14. $P(x) = 4x^4 - x^3 - 4x + 1$



15–30 ■ All the real zeros of the given polynomial are integers. Find the zeros, and write the polynomial in factored form.

15. $P(x) = x^3 - 4x^2 + x + 6$
16. $P(x) = x^3 - 7x^2 + 14x - 8$
17. $P(x) = x^3 + 3x^2 - 4$
18. $P(x) = x^3 - 3x - 2$
19. $P(x) = x^3 + 4x^2 - 3x - 18$
20. $P(x) = x^3 - x^2 - 8x + 12$
21. $P(x) = x^3 - 6x^2 + 12x - 8$
22. $P(x) = x^3 + 12x^2 + 48x + 64$
23. $P(x) = x^3 - 4x^2 + x + 6$
24. $P(x) = x^3 - 4x^2 - 7x + 10$
25. $P(x) = x^3 + 3x^2 - x - 3$
26. $P(x) = x^3 - 4x^2 - 11x + 30$
27. $P(x) = x^4 - 5x^2 + 4$

28. $P(x) = x^4 - 2x^3 - 3x^2 + 8x - 4$

29. $P(x) = x^4 + 6x^3 + 7x^2 - 6x - 8$

30. $P(x) = x^4 - x^3 - 23x^2 - 3x + 90$

31–48 ■ Find all rational zeros of the polynomial, and write the polynomial in factored form.

31. $P(x) = 4x^4 - 25x^2 + 36$

32. $P(x) = 2x^4 - x^3 - 19x^2 + 9x + 9$

33. $P(x) = 3x^4 - 10x^3 - 9x^2 + 40x - 12$

34. $P(x) = 2x^3 + 7x^2 + 4x - 4$

35. $P(x) = 4x^3 + 4x^2 - x - 1$

36. $P(x) = 2x^3 - 3x^2 - 2x + 3$

37. $P(x) = 4x^3 - 7x + 3$

38. $P(x) = 8x^3 + 10x^2 - x - 3$

39. $P(x) = 4x^3 + 8x^2 - 11x - 15$

40. $P(x) = 6x^3 + 11x^2 - 3x - 2$

41. $P(x) = 20x^3 - 8x^2 - 5x + 2$

42. $P(x) = 12x^3 - 20x^2 + x + 3$

43. $P(x) = 2x^4 - 7x^3 + 3x^2 + 8x - 4$

44. $P(x) = 6x^4 - 7x^3 - 12x^2 + 3x + 2$

45. $P(x) = x^5 + 3x^4 - 9x^3 - 31x^2 + 36$

46. $P(x) = x^5 - 4x^4 - 3x^3 + 22x^2 - 4x - 24$

47. $P(x) = 3x^5 - 14x^4 - 14x^3 + 36x^2 + 43x + 10$

48. $P(x) = 2x^6 - 3x^5 - 13x^4 + 29x^3 - 27x^2 + 32x - 12$

49–58 ■ Find all the real zeros of the polynomial. Use the Quadratic Formula if necessary, as in Example 3(a).

49. $P(x) = x^3 + 4x^2 + 3x - 2$

50. $P(x) = x^3 - 5x^2 + 2x + 12$

51. $P(x) = x^4 - 6x^3 + 4x^2 + 15x + 4$

52. $P(x) = x^4 + 2x^3 - 2x^2 - 3x + 2$

53. $P(x) = x^4 - 7x^3 + 14x^2 - 3x - 9$

54. $P(x) = x^5 - 4x^4 - x^3 + 10x^2 + 2x - 4$

55. $P(x) = 4x^3 - 6x^2 + 1$

56. $P(x) = 3x^3 - 5x^2 - 8x - 2$

57. $P(x) = 2x^4 + 15x^3 + 17x^2 + 3x - 1$

58. $P(x) = 4x^5 - 18x^4 - 6x^3 + 91x^2 - 60x + 9$

59–66 ■ A polynomial P is given. **(a)** Find all the real zeros of P. **(b)** Sketch the graph of P.

59. $P(x) = x^3 - 3x^2 - 4x + 12$

60. $P(x) = -x^3 - 2x^2 + 5x + 6$

61. $P(x) = 2x^3 - 7x^2 + 4x + 4$

62. $P(x) = 3x^3 + 17x^2 + 21x - 9$

63. $P(x) = x^4 - 5x^3 + 6x^2 + 4x - 8$

64. $P(x) = -x^4 + 10x^2 + 8x - 8$

65. $P(x) = x^5 - x^4 - 5x^3 + x^2 + 8x + 4$

66. $P(x) = x^5 - x^4 - 6x^3 + 14x^2 - 11x + 3$

67–72 ■ Use Descartes' Rule of Signs to determine how many positive and how many negative real zeros the polynomial can have. Then determine the possible total number of real zeros.

67. $P(x) = x^3 - x^2 - x - 3$

68. $P(x) = 2x^3 - x^2 + 4x - 7$

69. $P(x) = 2x^6 + 5x^4 - x^3 - 5x - 1$

70. $P(x) = x^4 + x^3 + x^2 + x + 12$

71. $P(x) = x^5 + 4x^3 - x^2 + 6x$

72. $P(x) = x^8 - x^5 + x^4 - x^3 + x^2 - x + 1$

73–80 ■ Show that the given values for a and b are lower and upper bounds for the real zeros of the polynomial.

73. $P(x) = 2x^3 + 5x^2 + x - 2;\quad a = -3, b = 1$

74. $P(x) = x^4 - 2x^3 - 9x^2 + 2x + 8;\quad a = -3, b = 5$

75. $P(x) = 8x^3 + 10x^2 - 39x + 9;\quad a = -3, b = 2$

76. $P(x) = 3x^4 - 17x^3 + 24x^2 - 9x + 1;\quad a = 0, b = 6$

77. $P(x) = x^4 + 2x^3 + 3x^2 + 5x - 1;\quad a = -2, b = 1$

78. $P(x) = x^4 + 3x^3 - 4x^2 - 2x - 7;\quad a = -4, b = 2$

79. $P(x) = 2x^4 - 6x^3 + x^2 - 2x + 3;\quad a = -1, b = 3$

80. $P(x) = 3x^4 - 5x^3 - 2x^2 + x - 1;\quad a = -1, b = 2$

81–84 ■ Find integers that are upper and lower bounds for the real zeros of the polynomial.

81. $P(x) = x^3 - 3x^2 + 4$

82. $P(x) = 2x^3 - 3x^2 - 8x + 12$

83. $P(x) = x^4 - 2x^3 + x^2 - 9x + 2$

84. $P(x) = x^5 - x^4 + 1$

85–90 ■ Find all rational zeros of the polynomial, and then find the irrational zeros, if any. Whenever appropriate, use the Rational Zeros Theorem, the Upper and Lower Bounds Theorem, Descartes' Rule of Signs, the quadratic formula, or other factoring techniques.

85. $P(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$

86. $P(x) = 2x^4 + 15x^3 + 31x^2 + 20x + 4$

87. $P(x) = 4x^4 - 21x^2 + 5$

88. $P(x) = 6x^4 - 7x^3 - 8x^2 + 5x$

89. $P(x) = x^5 - 7x^4 + 9x^3 + 23x^2 - 50x + 24$

90. $P(x) = 8x^5 - 14x^4 - 22x^3 + 57x^2 - 35x + 6$

91–94 ■ Show that the polynomial does not have any rational zeros.

91. $P(x) = x^3 - x - 2$

92. $P(x) = 2x^4 - x^3 + x + 2$

93. $P(x) = 3x^3 - x^2 - 6x + 12$

94. $P(x) = x^{50} - 5x^{25} + x^2 - 1$

95–98 ■ The real solutions of the given equation are rational. List all possible rational roots using the Rational Zeros Theorem, and then graph the polynomial in the given viewing rectangle to determine which values are actually solutions. (All solutions can be seen in the given viewing rectangle.)

95. $x^3 - 3x^2 - 4x + 12 = 0$; $[-4, 4]$ by $[-15, 15]$

96. $x^4 - 5x^2 + 4 = 0$; $[-4, 4]$ by $[-30, 30]$

97. $2x^4 - 5x^3 - 14x^2 + 5x + 12 = 0$; $[-2, 5]$ by $[-40, 40]$

98. $3x^3 + 8x^2 + 5x + 2 = 0$; $[-3, 3]$ by $[-10, 10]$

99–102 ■ Use a graphing device to find all real solutions of the equation, rounded to two decimal places.

99. $x^4 - x - 4 = 0$

100. $2x^3 - 8x^2 + 9x - 9 = 0$

101. $4.00x^4 + 4.00x^3 - 10.96x^2 - 5.88x + 9.09 = 0$

102. $x^5 + 2.00x^4 + 0.96x^3 + 5.00x^2 + 10.00x + 4.80 = 0$

103. Let $P(x)$ be a polynomial with real coefficients, and let $b > 0$. Use the Division Algorithm to write

$$P(x) = (x - b) \cdot Q(x) + r$$

Suppose that $r \geq 0$ and that all the coefficients in $Q(x)$ are nonnegative. Let $z > b$.

(a) Show that $P(z) > 0$.

(b) Prove the first part of the Upper and Lower Bounds Theorem.

(c) Use the first part of the Upper and Lower Bounds Theorem to prove the second part. [*Hint:* Show that if $P(x)$ satisfies the second part of the theorem, then $P(-x)$ satisfies the first part.]

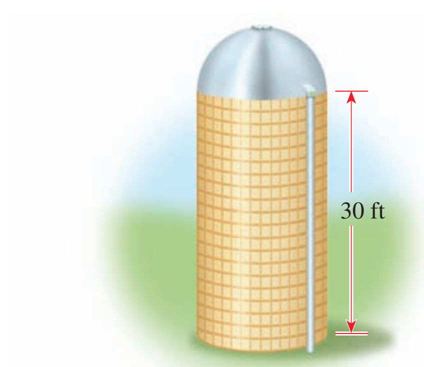
104. Show that the equation

$$x^5 - x^4 - x^3 - 5x^2 - 12x - 6 = 0$$

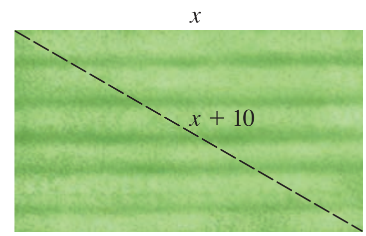
has exactly one rational root, and then prove that it must have either two or four irrational roots.

APPLICATIONS

105. Volume of a Silo A grain silo consists of a cylindrical main section and a hemispherical roof. If the total volume of the silo (including the part inside the roof section) is 15,000 ft^3 and the cylindrical part is 30 ft tall, what is the radius of the silo, rounded to the nearest tenth of a foot?



106. Dimensions of a Lot A rectangular parcel of land has an area of 5000 ft^2. A diagonal between opposite corners is measured to be 10 ft longer than one side of the parcel. What are the dimensions of the land, rounded to the nearest foot?



107. Depth of Snowfall Snow began falling at noon on Sunday. The amount of snow on the ground at a certain location at time t was given by the function

$$h(t) = 11.60t - 12.41t^2 + 6.20t^3 - 1.58t^4 + 0.20t^5 - 0.01t^6$$

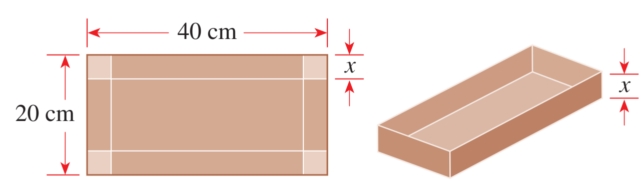
where t is measured in days from the start of the snowfall and $h(t)$ is the depth of snow in inches. Draw a graph of this function, and use your graph to answer the following questions.

(a) What happened shortly after noon on Tuesday?

(b) Was there ever more than 5 in. of snow on the ground? If so, on what day(s)?

(c) On what day and at what time (to the nearest hour) did the snow disappear completely?

108. Volume of a Box An open box with a volume of 1500 cm^3 is to be constructed by taking a piece of cardboard 20 cm by 40 cm, cutting squares of side length x cm from each corner, and folding up the sides. Show that this can be done in two different ways, and find the exact dimensions of the box in each case.



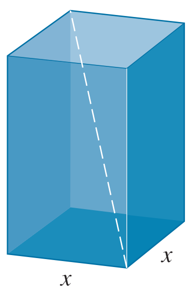
109. Volume of a Rocket A rocket consists of a right circular cylinder of height 20 m surmounted by a cone whose height and diameter are equal and whose radius is the same as that of the cylindrical section. What should this radius be (rounded to two decimal places) if the total volume is to be $500\pi/3$ m^3?

110. Volume of a Box A rectangular box with a volume of $2\sqrt{2}$ ft^3 has a square base as shown below. The diagonal of the box (between a pair of opposite corners) is 1 ft longer than each side of the base.

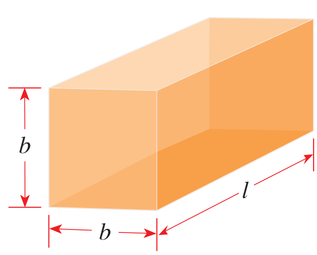
(a) If the base has sides of length x feet, show that

$$x^6 - 2x^5 - x^4 + 8 = 0$$

(b) Show that two different boxes satisfy the given conditions. Find the dimensions in each case, rounded to the nearest hundredth of a foot.



111. Girth of a Box A box with a square base has length plus girth of 108 in. (Girth is the distance "around" the box.) What is the length of the box if its volume is 2200 in^3?



DISCOVERY ▪ DISCUSSION ▪ WRITING

112. How Many Real Zeros Can a Polynomial Have? Give examples of polynomials that have the following properties, or explain why it is impossible to find such a polynomial.

(a) A polynomial of degree 3 that has no real zeros
(b) A polynomial of degree 4 that has no real zeros
(c) A polynomial of degree 3 that has three real zeros, only one of which is rational
(d) A polynomial of degree 4 that has four real zeros, none of which is rational

What must be true about the degree of a polynomial with integer coefficients if it has no real zeros?

113. The Depressed Cubic The most general cubic (third-degree) equation with rational coefficients can be written as

$$x^3 + ax^2 + bx + c = 0$$

(a) Show that if we replace x by $X - a/3$ and simplify, we end up with an equation that doesn't have an X^2 term, that is, an equation of the form

$$X^3 + pX + q = 0$$

This is called a *depressed cubic*, because we have "depressed" the quadratic term.

(b) Use the procedure described in part (a) to depress the equation $x^3 + 6x^2 + 9x + 4 = 0$.

114. The Cubic Formula The quadratic formula can be used to solve any quadratic (or second-degree) equation. You might have wondered whether similar formulas exist for cubic (third-degree), quartic (fourth-degree), and higher-degree equations. For the depressed cubic $x^3 + px + q = 0$, Cardano (page 308) found the following formula for one solution:

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

A formula for quartic equations was discovered by the Italian mathematician Ferrari in 1540. In 1824 the Norwegian mathematician Niels Henrik Abel proved that it is impossible to write a quintic formula, that is, a formula for fifth-degree equations. Finally, Galois (page 288) gave a criterion for determining which equations can be solved by a formula involving radicals.

Use the cubic formula to find a solution for the following equations. Then solve the equations using the methods you learned in this section. Which method is easier?

(a) $x^3 - 3x + 2 = 0$
(b) $x^3 - 27x - 54 = 0$
(c) $x^3 + 3x + 4 = 0$

DISCOVERY PROJECT **Zeroing in on a Zero**

In this project we explore a numerical method for approximating the zeros of a polynomial function.
You can find the project at the book companion website:
www.stewartmath.com

3.5 Complex Numbers

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Add and subtract complex numbers ▶ Multiply and divide complex numbers ▶ Work with square roots of negative numbers ▶ Find complex solutions of quadratic equations

In Section 1.6 we saw that if the discriminant of a quadratic equation is negative, the equation has no real solution. For example, the equation

$$x^2 + 4 = 0$$

has no real solution. If we try to solve this equation, we get $x^2 = -4$, so

$$x = \pm\sqrt{-4}$$

But this is impossible, since the square of any real number is positive. [For example, $(-2)^2 = 4$, a positive number.] Thus negative numbers don't have real square roots.

See the note on Cardano (page 308) for an example of how complex numbers are used to find real solutions of polynomial equations.

To make it possible to solve *all* quadratic equations, mathematicians invented an expanded number system, called the *complex number system*. First they defined the new number

$$i = \sqrt{-1}$$

This means that $i^2 = -1$. A complex number is then a number of the form $a + bi$, where a and b are real numbers.

DEFINITION OF COMPLEX NUMBERS

A **complex number** is an expression of the form

$$a + bi$$

where a and b are real numbers and $i^2 = -1$. The **real part** of this complex number is a and the **imaginary part** is b. Two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

Note that both the real and imaginary parts of a complex number are real numbers.

EXAMPLE 1 | Complex Numbers

The following are examples of complex numbers.

$3 + 4i$	Real part 3, imaginary part 4
$\frac{1}{2} - \frac{2}{3}i$	Real part $\frac{1}{2}$, imaginary part $-\frac{2}{3}$
$6i$	Real part 0, imaginary part 6
-7	Real part -7, imaginary part 0

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **5** AND **9** ■

A number such as $6i$, which has real part 0, is called a **pure imaginary number**. A real number such as -7 can be thought of as a complex number with imaginary part 0.

In the complex number system every quadratic equation has solutions. The numbers $2i$ and $-2i$ are solutions of $x^2 = -4$ because

$$(2i)^2 = 2^2i^2 = 4(-1) = -4 \qquad \text{and} \qquad (-2i)^2 = (-2)^2i^2 = 4(-1) = -4$$

Although we use the term *imaginary* in this context, imaginary numbers should not be thought of as any less "real" (in the ordinary rather than the mathematical sense of that word) than negative numbers or irrational numbers. All numbers (except possibly the positive integers) are creations of the human mind—the numbers -1 and $\sqrt{2}$ as well as the number i.

We study complex numbers because they complete, in a useful and elegant fashion, our study of the solutions of equations. In fact, imaginary numbers are useful not only in algebra and mathematics, but in the other sciences as well. To give just one example, in electrical theory the *reactance* of a circuit is a quantity whose measure is an imaginary number.

▼ Arithmetic Operations on Complex Numbers

Complex numbers are added, subtracted, multiplied, and divided just as we would any number of the form $a + b\sqrt{c}$. The only difference that we need to keep in mind is that $i^2 = -1$. Thus the following calculations are valid:

$$\begin{aligned}(a + bi)(c + di) &= ac + (ad + bc)i + bdi^2 && \text{Multiply and collect like terms}\\ &= ac + (ad + bc)i + bd(-1) && i^2 = -1\\ &= (ac - bd) + (ad + bc)i && \text{Combine real and imaginary parts}\end{aligned}$$

We therefore define the sum, difference, and product of complex numbers as follows.

ADDING, SUBTRACTING, AND MULTIPLYING COMPLEX NUMBERS

Definition	Description
Addition	
$(a + bi) + (c + di) = (a + c) + (b + d)i$	To add complex numbers, add the real parts and the imaginary parts.
Subtraction	
$(a + bi) - (c + di) = (a - c) + (b - d)i$	To subtract complex numbers, subtract the real parts and the imaginary parts.
Multiplication	
$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$	Multiply complex numbers like binomials, using $i^2 = -1$.

EXAMPLE 2 | Adding, Subtracting, and Multiplying Complex Numbers

Graphing calculators can perform arithmetic operations on complex numbers.

```
(3+5i)+(4-2i)
                7+3i
(3+5i)*(4-2i)
              22+14i
```

Express the following in the form $a + bi$.

(a) $(3 + 5i) + (4 - 2i)$ **(b)** $(3 + 5i) - (4 - 2i)$

(c) $(3 + 5i)(4 - 2i)$ **(d)** i^{23}

SOLUTION

(a) According to the definition, we add the real parts and we add the imaginary parts:

$$(3 + 5i) + (4 - 2i) = (3 + 4) + (5 - 2)i = 7 + 3i$$

(b) $(3 + 5i) - (4 - 2i) = (3 - 4) + [5 - (-2)]i = -1 + 7i$

(c) $(3 + 5i)(4 - 2i) = [3 \cdot 4 - 5(-2)] + [3(-2) + 5 \cdot 4]i = 22 + 14i$

(d) $i^{23} = i^{22+1} = (i^2)^{11}i = (-1)^{11}i = (-1)i = -i$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **17**, **21**, **29**, AND **47** ■

Complex Conjugates

Number	Conjugate
$3 + 2i$	$3 - 2i$
$1 - i$	$1 + i$
$4i$	$-4i$
5	5

Division of complex numbers is much like rationalizing the denominator of a radical expression, which we considered in Section P.7. For the complex number $z = a + bi$ we define its **complex conjugate** to be $\overline{z} = a - bi$. Note that

$$z \cdot \overline{z} = (a + bi)(a - bi) = a^2 + b^2$$

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LEONHARD EULER (1707–1783) was born in Basel, Switzerland, the son of a pastor. When Euler was 13, his father sent him to the University at Basel to study theology, but Euler soon decided to devote himself to the sciences. Besides theology he studied mathematics, medicine, astronomy, physics, and Asian languages. It is said that Euler could calculate as effortlessly as "men breathe or as eagles fly." One hundred years before Euler, Fermat (see page 107) had conjectured that $2^{2^n} + 1$ is a prime number for all n. The first five of these numbers are 5, 17, 257, 65537, and 4,294,967,297. It is easy to show that the first four are prime. The fifth was also thought to be prime until Euler, with his phenomenal calculating ability, showed that it is the product $641 \times 6{,}700{,}417$ and so is not prime. Euler published more than any other mathematician in history. His collected works comprise 75 large volumes. Although he was blind for the last 17 years of his life, he continued to work and publish. In his writings he popularized the use of the symbols π, e, and i, which you will find in this textbook. One of Euler's most lasting contributions is his development of complex numbers.

So the product of a complex number and its conjugate is always a nonnegative real number. We use this property to divide complex numbers.

DIVIDING COMPLEX NUMBERS

To simplify the quotient $\dfrac{a + bi}{c + di}$, multiply the numerator and the denominator by the complex conjugate of the denominator:

$$\frac{a + bi}{c + di} = \left(\frac{a + bi}{c + di}\right)\left(\frac{c - di}{c - di}\right) = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

Rather than memorizing this entire formula, it is easier to just remember the first step and then multiply out the numerator and the denominator as usual.

EXAMPLE 3 | Dividing Complex Numbers

Express the following in the form $a + bi$.

(a) $\dfrac{3 + 5i}{1 - 2i}$ **(b)** $\dfrac{7 + 3i}{4i}$

SOLUTION We multiply both the numerator and denominator by the complex conjugate of the denominator to make the new denominator a real number.

(a) The complex conjugate of $1 - 2i$ is $\overline{1 - 2i} = 1 + 2i$. Therefore

$$\frac{3 + 5i}{1 - 2i} = \left(\frac{3 + 5i}{1 - 2i}\right)\left(\frac{1 + 2i}{1 + 2i}\right) = \frac{-7 + 11i}{5} = -\frac{7}{5} + \frac{11}{5}i$$

(b) The complex conjugate of $4i$ is $-4i$. Therefore

$$\frac{7 + 3i}{4i} = \left(\frac{7 + 3i}{4i}\right)\left(\frac{-4i}{-4i}\right) = \frac{12 - 28i}{16} = \frac{3}{4} - \frac{7}{4}i$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **37** AND **43**

Square Roots of Negative Numbers

Just as every positive real number r has two square roots ($\sqrt{r}$ and $-\sqrt{r}$), every negative number has two square roots as well. If $-r$ is a negative number, then its square roots are $\pm i\sqrt{r}$, because $(i\sqrt{r})^2 = i^2 r = -r$ and $(-i\sqrt{r})^2 = (-1)^2 i^2 r = -r$.

SQUARE ROOTS OF NEGATIVE NUMBERS

If $-r$ is negative, then the **principal square root** of $-r$ is

$$\sqrt{-r} = i\sqrt{r}$$

The two square roots of $-r$ are $i\sqrt{r}$ and $-i\sqrt{r}$.

We usually write $i\sqrt{b}$ instead of $\sqrt{b}\,i$ to avoid confusion with $\sqrt{bi}$.

EXAMPLE 4 | Square Roots of Negative Numbers

(a) $\sqrt{-1} = i\sqrt{1} = i$ **(b)** $\sqrt{-16} = i\sqrt{16} = 4i$ **(c)** $\sqrt{-3} = i\sqrt{3}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **53** AND **55**

Special care must be taken in performing calculations that involve square roots of negative numbers. Although $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ when a and b are positive, this is *not* true when both are negative. For example,

$$\sqrt{-2} \cdot \sqrt{-3} = i\sqrt{2} \cdot i\sqrt{3} = i^2\sqrt{6} = -\sqrt{6}$$

but

$$\sqrt{(-2)(-3)} = \sqrt{6}$$

so

$$\sqrt{-2} \cdot \sqrt{-3} \neq \sqrt{(-2)(-3)}$$

When multiplying radicals of negative numbers, express them first in the form $i\sqrt{r}$ (where $r > 0$) to avoid possible errors of this type.

EXAMPLE 5 | Using Square Roots of Negative Numbers

Evaluate $(\sqrt{12} - \sqrt{-3})(3 + \sqrt{-4})$, and express the result in the form $a + bi$.

SOLUTION

$$\begin{aligned}(\sqrt{12} - \sqrt{-3})(3 + \sqrt{-4}) &= (\sqrt{12} - i\sqrt{3})(3 + i\sqrt{4}) \\ &= (2\sqrt{3} - i\sqrt{3})(3 + 2i) \\ &= (6\sqrt{3} + 2\sqrt{3}) + i(2 \cdot 2\sqrt{3} - 3\sqrt{3}) \\ &= 8\sqrt{3} + i\sqrt{3}\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **57** ■

Complex Solutions of Quadratic Equations

We have already seen that if $a \neq 0$, then the solutions of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac < 0$, then the equation has no real solution. But in the complex number system this equation will always have solutions, because negative numbers have square roots in this expanded setting.

EXAMPLE 6 | Quadratic Equations with Complex Solutions

Solve each equation.

(a) $x^2 + 9 = 0$ **(b)** $x^2 + 4x + 5 = 0$

SOLUTION

(a) The equation $x^2 + 9 = 0$ means $x^2 = -9$, so

$$x = \pm\sqrt{-9} = \pm i\sqrt{9} = \pm 3i$$

The solutions are therefore $3i$ and $-3i$.

(b) By the Quadratic Formula we have

$$\begin{aligned}x &= \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} \\ &= \frac{-4 \pm \sqrt{-4}}{2} \\ &= \frac{-4 \pm 2i}{2} = \frac{2(-2 \pm i)}{2} = -2 \pm i\end{aligned}$$

So the solutions are $-2 + i$ and $-2 - i$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **63** AND **65** ■

We see from Example 6 that if a quadratic equation with real coefficients has complex solutions, then these solutions are complex conjugates of each other. So if $a + bi$ is a solution, then $a - bi$ is also a solution.

EXAMPLE 7 | Complex Conjugates as Solutions of a Quadratic

Show that the solutions of the equation

$$4x^2 - 24x + 37 = 0$$

are complex conjugates of each other.

SOLUTION We use the Quadratic Formula to get

$$\begin{aligned} x &= \frac{24 \pm \sqrt{(24)^2 - 4(4)(37)}}{2(4)} \\ &= \frac{24 \pm \sqrt{-16}}{8} = \frac{24 \pm 4i}{8} = 3 \pm \frac{1}{2}i \end{aligned}$$

So the solutions are $3 + \frac{1}{2}i$ and $3 - \frac{1}{2}i$, and these are complex conjugates.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **71**

3.5 EXERCISES

CONCEPTS

1. The imaginary number i has the property that $i^2 =$ ________.
2. For the complex number $3 + 4i$ the real part is ________, and the imaginary part is ________.
3. **(a)** The complex conjugate of $3 + 4i$ is $\overline{3 + 4i} =$ ________.
   (b) $(3 + 4i)(\overline{3 + 4i}) =$ ________.
4. If $3 + 4i$ is a solution of a quadratic equation with real coefficients, then ________ is also a solution of the equation.

SKILLS

5–14 ■ Find the real and imaginary parts of the complex number.

5. $5 - 7i$
6. $-6 + 4i$
7. $\dfrac{-2 - 5i}{3}$
8. $\dfrac{4 + 7i}{2}$
9. 3
10. $-\frac{1}{2}$
11. $-\frac{2}{3}i$
12. $i\sqrt{3}$
13. $\sqrt{3} + \sqrt{-4}$
14. $2 - \sqrt{-5}$

15–24 ■ Evaluate the sum or difference, and write the result in the form $a + bi$.

15. $(3 + 2i) + 5i$
16. $3i - (2 - 3i)$
17. $(2 - 5i) + (3 + 4i)$
18. $(2 + 5i) + (4 - 6i)$
19. $(-6 + 6i) + (9 - i)$
20. $(3 - 2i) + \left(-5 - \frac{1}{3}i\right)$
21. $\left(7 - \frac{1}{2}i\right) - \left(5 + \frac{3}{2}i\right)$
22. $(-4 + i) - (2 - 5i)$
23. $(-12 + 8i) - (7 + 4i)$
24. $6i - (4 - i)$

25–34 ■ Evaluate the product, and write the result in the form $a + bi$.

25. $4(-1 + 2i)$
26. $-2(3 - 4i)$
27. $-3i(5 - i)$
28. $2i\left(\frac{1}{2} - i\right)$
29. $(7 - i)(4 + 2i)$
30. $(5 - 3i)(1 + i)$
31. $(3 - 4i)(5 - 12i)$
32. $\left(\frac{2}{3} + 12i\right)\left(\frac{1}{6} + 24i\right)$
33. $(6 + 5i)(2 - 3i)$
34. $(-2 + i)(3 - 7i)$

35–46 ■ Evaluate the quotient, and write the result in the form $a + bi$.

35. $\dfrac{1}{i}$
36. $\dfrac{1}{1 + i}$
37. $\dfrac{2 - 3i}{1 - 2i}$
38. $\dfrac{5 - i}{3 + 4i}$
39. $\dfrac{26 + 39i}{2 - 3i}$
40. $\dfrac{25}{4 - 3i}$
41. $\dfrac{10i}{1 - 2i}$
42. $(2 - 3i)^{-1}$
43. $\dfrac{4 + 6i}{3i}$
44. $\dfrac{-3 + 5i}{15i}$
45. $\dfrac{1}{1 + i} - \dfrac{1}{1 - i}$
46. $\dfrac{(1 + 2i)(3 - i)}{2 + i}$

47–52 ■ Evaluate the power, and write the result in the form $a + bi$.

47. i^3
48. i^{10}
49. $(3i)^5$
50. $(2i)^4$
51. i^{1000}
52. i^{1002}

53–62 ■ Evaluate the radical expression, and express the result in the form $a + bi$.

53. $\sqrt{-25}$
54. $\sqrt{\dfrac{-9}{4}}$
55. $\sqrt{-3}\sqrt{-12}$
56. $\sqrt{\frac{1}{3}}\sqrt{-27}$

57. $(3 - \sqrt{-5})(1 + \sqrt{-1})$

58. $(\sqrt{3} - \sqrt{-4})(\sqrt{6} - \sqrt{-8})$

59. $\dfrac{2 + \sqrt{-8}}{1 + \sqrt{-2}}$

60. $\dfrac{1 - \sqrt{-1}}{1 + \sqrt{-1}}$

61. $\dfrac{\sqrt{-36}}{\sqrt{-2}\sqrt{-9}}$

62. $\dfrac{\sqrt{-7}\sqrt{-49}}{\sqrt{28}}$

63–78 ■ Find all solutions of the equation and express them in the form $a + bi$.

63. $x^2 + 49 = 0$

64. $9x^2 + 4 = 0$

65. $x^2 - 4x + 5 = 0$

66. $x^2 + 2x + 2 = 0$

67. $x^2 + 2x + 5 = 0$

68. $x^2 - 6x + 10 = 0$

69. $x^2 + x + 1 = 0$

70. $x^2 - 3x + 3 = 0$

71. $2x^2 - 2x + 1 = 0$

72. $2x^2 + 3 = 2x$

73. $t + 3 + \dfrac{3}{t} = 0$

74. $z + 4 + \dfrac{12}{z} = 0$

75. $6x^2 + 12x + 7 = 0$

76. $4x^2 - 16x + 19 = 0$

77. $\frac{1}{2}x^2 - x + 5 = 0$

78. $x^2 + \frac{1}{2}x + 1 = 0$

79–86 ■ Recall that the symbol $\overline{z}$ represents the complex conjugate of z. If $z = a + bi$ and $w = c + di$, prove each statement.

79. $\overline{z} + \overline{w} = \overline{z + w}$

80. $\overline{zw} = \overline{z} \cdot \overline{w}$

81. $(\overline{z})^2 = \overline{z^2}$

82. $\overline{\overline{z}} = z$

83. $z + \overline{z}$ is a real number.

84. $z - \overline{z}$ is a pure imaginary number.

85. $z \cdot \overline{z}$ is a real number.

86. $z = \overline{z}$ if and only if z is real.

DISCOVERY ■ DISCUSSION ■ WRITING

87. Complex Conjugate Roots Suppose that the equation $ax^2 + bx + c = 0$ has real coefficients and complex roots. Why must the roots be complex conjugates of each other? (Think about how you would find the roots using the Quadratic Formula.)

88. Powers of i Calculate the first 12 powers of i, that is, $i, i^2, i^3, \ldots, i^{12}$. Do you notice a pattern? Explain how you would calculate any whole number power of i, using the pattern that you have discovered. Use this procedure to calculate i^{4446}.

3.6 Complex Zeros and the Fundamental Theorem of Algebra

LEARNING OBJECTIVES **After completing this section, you will be able to:**

State the Fundamental Theorem of Algebra ▶ Factor a polynomial completely (into linear factors) over the complex numbers ▶ Use the Conjugate Zeros Theorem to find polynomials with specified zeros ▶ Factor a polynomial completely (into linear and quadratic factors) over the real numbers

We have already seen that an nth-degree polynomial can have at most n real zeros. In the complex number system an nth-degree polynomial has exactly n zeros and so can be factored into exactly n linear factors. This fact is a consequence of the Fundamental Theorem of Algebra, which was proved by the German mathematician C. F. Gauss in 1799 (see page 306).

▼ The Fundamental Theorem of Algebra and Complete Factorization

The following theorem is the basis for much of our work in factoring polynomials and solving polynomial equations.

FUNDAMENTAL THEOREM OF ALGEBRA

Every polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (n \ge 1,\ a_n \ne 0)$$

with complex coefficients has at least one complex zero.

Because any real number is also a complex number, the theorem applies to polynomials with real coefficients as well.

The Fundamental Theorem of Algebra and the Factor Theorem together show that a polynomial can be factored completely into linear factors, as we now prove.

COMPLETE FACTORIZATION THEOREM

If $P(x)$ is a polynomial of degree $n \geq 1$, then there exist complex numbers $a, c_1, c_2, \ldots, c_n$ (with $a \neq 0$) such that

$$P(x) = a(x - c_1)(x - c_2)\cdots(x - c_n)$$

PROOF By the Fundamental Theorem of Algebra, P has at least one zero. Let's call it c_1. By the Factor Theorem (see page 284), $P(x)$ can be factored as

$$P(x) = (x - c_1)\cdot Q_1(x)$$

where $Q_1(x)$ is of degree $n - 1$. Applying the Fundamental Theorem to the quotient $Q_1(x)$ gives us the factorization

$$P(x) = (x - c_1)\cdot(x - c_2)\cdot Q_2(x)$$

where $Q_2(x)$ is of degree $n - 2$ and c_2 is a zero of $Q_1(x)$. Continuing this process for n steps, we get a final quotient $Q_n(x)$ of degree 0, a nonzero constant that we will call a. This means that P has been factored as

$$P(x) = a(x - c_1)(x - c_2)\cdots(x - c_n)$$

■

To actually find the complex zeros of an nth-degree polynomial, we usually first factor as much as possible, then use the quadratic formula on parts that we can't factor further.

EXAMPLE 1 | Factoring a Polynomial Completely

Let $P(x) = x^3 - 3x^2 + x - 3$.

(a) Find all the zeros of P.

(b) Find the complete factorization of P.

SOLUTION

(a) We first factor P as follows.

$$\begin{aligned} P(x) &= x^3 - 3x^2 + x - 3 && \text{Given} \\ &= x^2(x - 3) + (x - 3) && \text{Group terms} \\ &= (x - 3)(x^2 + 1) && \text{Factor } x - 3 \end{aligned}$$

We find the zeros of P by setting each factor equal to 0:

$$P(x) = (x - 3)(x^2 + 1)$$

This factor is 0 when $x = 3$ | This factor is 0 when $x = i$ or $-i$

Setting $x - 3 = 0$, we see that $x = 3$ is a zero. Setting $x^2 + 1 = 0$, we get $x^2 = -1$, so $x = \pm i$. So the zeros of P are 3, i, and $-i$.

(b) Since the zeros are 3, i, and $-i$, the complete factorization of P is

$$\begin{aligned} P(x) &= (x - 3)(x - i)[x - (-i)] \\ &= (x - 3)(x - i)(x + i) \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5** ■

EXAMPLE 2 | Factoring a Polynomial Completely

Let $P(x) = x^3 - 2x + 4$.

(a) Find all the zeros of P.

(b) Find the complete factorization of P.

SOLUTION

$$\begin{array}{r|rrrr} -2 & 1 & -0 & -2 & -4 \\ & & -2 & 4 & -4 \\ \hline & 1 & -2 & -2 & 0 \end{array}$$

(a) The possible rational zeros are the factors of 4, which are $\pm 1, \pm 2, \pm 4$. Using synthetic division (see the margin), we find that -2 is a zero, and the polynomial factors as

$$P(x) = (x + 2)(x^2 - 2x + 2)$$

This factor is 0 when $x = -2$

Use the Quadratic Formula to find when this factor is 0

To find the zeros, we set each factor equal to 0. Of course, $x + 2 = 0$ means that $x = -2$. We use the quadratic formula to find when the other factor is 0:

$$x^2 - 2x + 2 = 0 \qquad \text{Set factor equal to 0}$$

$$x = \frac{2 \pm \sqrt{4 - 8}}{2} \qquad \text{Quadratic Formula}$$

$$x = \frac{2 \pm 2i}{2} \qquad \text{Take square root}$$

$$x = 1 \pm i \qquad \text{Simplify}$$

So the zeros of P are -2, $1 + i$, and $1 - i$.

(b) Since the zeros are -2, $1 + i$, and $1 - i$, the complete factorization of P is

$$\begin{aligned} P(x) &= [x - (-2)][x - (1 + i)][x - (1 - i)] \\ &= (x + 2)(x - 1 - i)(x - 1 + i) \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **7**

Zeros and Their Multiplicities

In the Complete Factorization Theorem the numbers $c_1, c_2, \ldots, c_n$ are the zeros of P. These zeros need not all be different. If the factor $x - c$ appears k times in the complete factorization of $P(x)$, then we say that c is a zero of **multiplicity *k*** (see page 274). For example, the polynomial

$$P(x) = (x - 1)^3(x + 2)^2(x + 3)^5$$

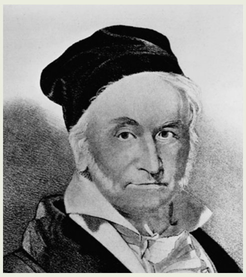
has the following zeros:

1 (multiplicity 3) $\qquad$ -2 (multiplicity 2) $\qquad$ -3 (multiplicity 5)

The polynomial P has the same number of zeros as its degree: It has degree 10 and has 10 zeros, provided that we count multiplicities. This is true for all polynomials, as we prove in the following theorem.

ZEROS THEOREM

Every polynomial of degree $n \geq 1$ has exactly n zeros, provided that a zero of multiplicity k is counted k times.



CARL FRIEDRICH GAUSS (1777–1855) is considered the greatest mathematician of modern times. His contemporaries called him the "Prince of Mathematics." He was born into a poor family; his father made a living as a mason. As a very small child, Gauss found a calculation error in his father's accounts, the first of many incidents that gave evidence of his mathematical precocity. (See also page 582.) At 19, Gauss demonstrated that the regular 17-sided polygon can be constructed with straight-edge and compass alone. This was remarkable because, since the time of Euclid, it had been thought that the only regular polygons constructible in this way were the triangle and pentagon. Because of this discovery Gauss decided to pursue a career in mathematics instead of languages, his other passion. In his doctoral dissertation, written at the age of 22, Gauss proved the Fundamental Theorem of Algebra: A polynomial of degree n with complex coefficients has n roots. His other accomplishments range over every branch of mathematics, as well as physics and astronomy.

PROOF Let P be a polynomial of degree n. By the Complete Factorization Theorem

$$P(x) = a(x - c_1)(x - c_2)\cdots(x - c_n)$$

Now suppose that c is a zero of P other than $c_1, c_2, \ldots, c_n$. Then

$$P(c) = a(c - c_1)(c - c_2)\cdots(c - c_n) = 0$$

Thus, by the Zero-Product Property, one of the factors $c - c_i$ must be 0, so $c = c_i$ for some i. It follows that P has exactly the n zeros $c_1, c_2, \ldots, c_n$. ■

EXAMPLE 3 | Factoring a Polynomial with Complex Zeros

Find the complete factorization and all five zeros of the polynomial

$$P(x) = 3x^5 + 24x^3 + 48x$$

SOLUTION Since $3x$ is a common factor, we have

$$\begin{aligned} P(x) &= 3x(x^4 + 8x^2 + 16) \\ &= 3x(x^2 + 4)^2 \end{aligned}$$

This factor is 0 when $x = 0$

This factor is 0 when $x = 2i$ or $x = -2i$

To factor $x^2 + 4$, note that $2i$ and $-2i$ are zeros of this polynomial. Thus $x^2 + 4 = (x - 2i)(x + 2i)$, so

$$\begin{aligned} P(x) &= 3x[(x - 2i)(x + 2i)]^2 \\ &= 3x(x - 2i)^2(x + 2i)^2 \end{aligned}$$

0 is a zero of multiplicity 1

$2i$ is a zero of multiplicity 2

$-2i$ is a zero of multiplicity 2

The zeros of P are 0, $2i$, and $-2i$. Since the factors $x - 2i$ and $x + 2i$ each occur twice in the complete factorization of P, the zeros $2i$ and $-2i$ are of multiplicity 2 (or *double* zeros). Thus we have found all five zeros.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **29** ■

The following table gives further examples of polynomials with their complete factorizations and zeros.

Degree	Polynomial	Zero(s)	Number of zeros
1	$P(x) = x - 4$	4	1
2	$P(x) = x^2 - 10x + 25$ $= (x - 5)(x - 5)$	5 (multiplicity 2)	2
3	$P(x) = x^3 + x$ $= x(x - i)(x + i)$	$0, i, -i$	3
4	$P(x) = x^4 + 18x^2 + 81$ $= (x - 3i)^2(x + 3i)^2$	$3i$ (multiplicity 2), $-3i$ (multiplicity 2)	4
5	$P(x) = x^5 - 2x^4 + x^3$ $= x^3(x - 1)^2$	0 (multiplicity 3), 1 (multiplicity 2)	5

EXAMPLE 4 | Finding Polynomials with Specified Zeros

(a) Find a polynomial $P(x)$ of degree 4, with zeros i, $-i$, 2, and -2, and with $P(3) = 25$.

(b) Find a polynomial $Q(x)$ of degree 4, with zeros -2 and 0, where -2 is a zero of multiplicity 3.

SOLUTION

(a) The required polynomial has the form

$$\begin{aligned} P(x) &= a(x - i)(x - (-i))(x - 2)(x - (-2)) \\ &= a(x^2 + 1)(x^2 - 4) && \text{Difference of squares} \\ &= a(x^4 - 3x^2 - 4) && \text{Multiply} \end{aligned}$$

We know that $P(3) = a(3^4 - 3 \cdot 3^2 - 4) = 50a = 25$, so $a = \frac{1}{2}$. Thus

$$P(x) = \tfrac{1}{2}x^4 - \tfrac{3}{2}x^2 - 2$$

(b) We require

$$\begin{aligned} Q(x) &= a[x - (-2)]^3(x - 0) \\ &= a(x + 2)^3 x \\ &= a(x^3 + 6x^2 + 12x + 8)x && \text{Special Product Formula 4 (Section P.5)} \\ &= a(x^4 + 6x^3 + 12x^2 + 8x) \end{aligned}$$

Since we are given no information about Q other than its zeros and their multiplicity, we can choose any number for a. If we use $a = 1$, we get

$$Q(x) = x^4 + 6x^3 + 12x^2 + 8x$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **35**

EXAMPLE 5 | Finding All the Zeros of a Polynomial

Find all four zeros of $P(x) = 3x^4 - 2x^3 - x^2 - 12x - 4$.

SOLUTION Using the Rational Zeros Theorem from Section 3.4, we obtain the following list of possible rational zeros: $\pm 1, \pm 2, \pm 4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$. Checking these using synthetic division, we find that 2 and $-\frac{1}{3}$ are zeros, and we get the following factorization:

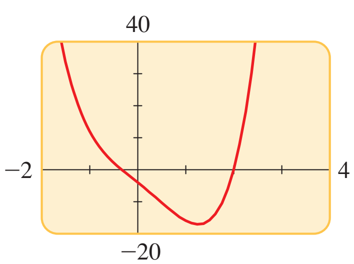


FIGURE 1
$P(x) = 3x^4 - 2x^3 - x^2 - 12x - 4$

Figure 1 shows the graph of the polynomial P in Example 5. The x-intercepts correspond to the real zeros of P. The imaginary zeros cannot be determined from the graph.

$$\begin{aligned} P(x) &= 3x^4 - 2x^3 - x^2 - 12x - 4 \\ &= (x - 2)(3x^3 + 4x^2 + 7x + 2) && \text{Factor } x - 2 \\ &= (x - 2)\left(x + \tfrac{1}{3}\right)(3x^2 + 3x + 6) && \text{Factor } x + \tfrac{1}{3} \\ &= 3(x - 2)\left(x + \tfrac{1}{3}\right)(x^2 + x + 2) && \text{Factor 3} \end{aligned}$$

The zeros of the quadratic factor are

$$x = \frac{-1 \pm \sqrt{1 - 8}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \qquad \text{Quadratic Formula}$$

so the zeros of $P(x)$ are

$$2, \quad -\frac{1}{3}, \quad -\frac{1}{2} + i\frac{\sqrt{7}}{2}, \quad \text{and} \quad -\frac{1}{2} - i\frac{\sqrt{7}}{2}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45**

GEROLAMO CARDANO (1501–1576) is certainly one of the most colorful figures in the history of mathematics. He was the best-known physician in Europe in his day, yet throughout his life he was plagued by numerous maladies, including ruptures, hemorrhoids, and an irrational fear of encountering rabid dogs. He was a doting father, but his beloved sons broke his heart—his favorite was eventually beheaded for murdering his own wife. Cardano was also a compulsive gambler; indeed, this vice might have driven him to write the *Book on Games of Chance,* the first study of probability from a mathematical point of view.

In Cardano's major mathematical work, the *Ars Magna,* he detailed the solution of the general third- and fourth-degree polynomial equations. At the time of its publication, mathematicians were uncomfortable even with negative numbers, but Cardano's formulas paved the way for the acceptance not just of negative numbers, but also of imaginary numbers, because they occurred naturally in solving polynomial equations. For example, for the cubic equation

$$x^3 - 15x - 4 = 0$$

one of his formulas gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

(See page 297, Exercise 114). This value for x actually turns out to be the *integer* 4, yet to find it, Cardano had to use the imaginary number $\sqrt{-121} = 11i$.

Complex Zeros Come in Conjugate Pairs

As you might have noticed from the examples so far, the complex zeros of polynomials with real coefficients come in pairs. Whenever $a + bi$ is a zero, its complex conjugate $a - bi$ is also a zero.

CONJUGATE ZEROS THEOREM

If the polynomial P has real coefficients and if the complex number z is a zero of P, then its complex conjugate $\overline{z}$ is also a zero of P.

PROOF Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where each coefficient is real. Suppose that $P(z) = 0$. We must prove that $P(\overline{z}) = 0$. We use the facts that the complex conjugate of a sum of two complex numbers is the sum of the conjugates and that the conjugate of a product is the product of the conjugates.

$$\begin{aligned}
P(\overline{z}) &= a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \cdots + a_1\overline{z} + a_0 \\
&= \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots + \overline{a_1}\,\overline{z} + \overline{a_0} && \text{Because the coefficients are real} \\
&= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} \\
&= \overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} \\
&= \overline{P(z)} = \overline{0} = 0
\end{aligned}$$

This shows that $\overline{z}$ is also a zero of $P(x)$,which proves the theorem. ■

EXAMPLE 6 | A Polynomial with a Specified Complex Zero

Find a polynomial $P(x)$ of degree 3 that has integer coefficients and zeros $\frac{1}{2}$ and $3 - i$.

SOLUTION Since $3 - i$ is a zero, then so is $3 + i$ by the Conjugate Zeros Theorem. This means that $P(x)$ must have the following form.

$$\begin{aligned}
P(x) &= a\left(x - \tfrac{1}{2}\right)[x - (3 - i)][x - (3 + i)] \\
&= a\left(x - \tfrac{1}{2}\right)[(x - 3) + i][(x - 3) - i] && \text{Regroup} \\
&= a\left(x - \tfrac{1}{2}\right)[(x - 3)^2 - i^2] && \text{Difference of Squares Formula} \\
&= a\left(x - \tfrac{1}{2}\right)(x^2 - 6x + 10) && \text{Expand} \\
&= a\left(x^3 - \tfrac{13}{2}x^2 + 13x - 5\right) && \text{Expand}
\end{aligned}$$

To make all coefficients integers, we set $a = 2$ and get

$$P(x) = 2x^3 - 13x^2 + 26x - 10$$

Any other polynomial that satisfies the given requirements must be an integer multiple of this one.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **39** ■

▼ Linear and Quadratic Factors

We have seen that a polynomial factors completely into linear factors if we use complex numbers. If we don't use complex numbers, then a polynomial with real coefficients can always be factored into linear and quadratic factors. We use this property in Section 5.3 when we study partial fractions. A quadratic polynomial with no real zeros is called **irreducible** over the real numbers. Such a polynomial cannot be factored without using complex numbers.

LINEAR AND QUADRATIC FACTORS THEOREM

Every polynomial with real coefficients can be factored into a product of linear and irreducible quadratic factors with real coefficients.

PROOF We first observe that if $c = a + bi$ is a complex number, then

$$\begin{aligned}(x - c)(x - \bar{c}) &= [x - (a + bi)][x - (a - bi)] \\ &= [(x - a) - bi][(x - a) + bi] \\ &= (x - a)^2 - (bi)^2 \\ &= x^2 - 2ax + (a^2 + b^2)\end{aligned}$$

The last expression is a quadratic with *real* coefficients.

Now, if P is a polynomial with real coefficients, then by the Complete Factorization Theorem

$$P(x) = a(x - c_1)(x - c_2) \cdots (x - c_n)$$

Since the complex roots occur in conjugate pairs, we can multiply the factors corresponding to each such pair to get a quadratic factor with real coefficients. This results in P being factored into linear and irreducible quadratic factors. ■

EXAMPLE 7 | Factoring a Polynomial into Linear and Quadratic Factors

Let $P(x) = x^4 + 2x^2 - 8$.

(a) Factor P into linear and irreducible quadratic factors with real coefficients.

(b) Factor P completely into linear factors with complex coefficients.

SOLUTION

(a)

$$\begin{aligned}P(x) &= x^4 + 2x^2 - 8 \\ &= (x^2 - 2)(x^2 + 4) \\ &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 4)\end{aligned}$$

The factor $x^2 + 4$ is irreducible, since it has no real zeros.

(b) To get the complete factorization, we factor the remaining quadratic factor:

$$\begin{aligned}P(x) &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 4) \\ &= (x - \sqrt{2})(x + \sqrt{2})(x - 2i)(x + 2i)\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **65** ■

3.6 EXERCISES

CONCEPTS

1. The polynomial $P(x) = 3(x-5)^3(x-3)(x+2)$ has degree ________. It has zeros 5, 3, and ________. The zero 5 has multiplicity ________, and the zero 3 has multiplicity ________.

2. **(a)** If a is a zero of the polynomial P, then ________ must be a factor of $P(x)$.
   (b) If a is a zero of multiplicity m of the polynomial P, then ________ must be a factor of $P(x)$ when we factor P completely.

3. A polynomial of degree $n \geq 1$ has exactly ________ zeros if a zero of multiplicity m is counted m times.

4. If the polynomial function P has real coefficients and if $a + bi$ is a zero of P, then ________ is also a zero of P.

SKILLS

5–16 ■ A polynomial P is given. **(a)** Find all zeros of P, real and complex. **(b)** Factor P completely.

5. $P(x) = x^4 + 4x^2$
6. $P(x) = x^5 + 9x^3$
7. $P(x) = x^3 - 2x^2 + 2x$
8. $P(x) = x^3 + x^2 + x$
9. $P(x) = x^4 + 2x^2 + 1$
10. $P(x) = x^4 - x^2 - 2$
11. $P(x) = x^4 - 16$
12. $P(x) = x^4 + 6x^2 + 9$
13. $P(x) = x^3 + 8$
14. $P(x) = x^3 - 8$
15. $P(x) = x^6 - 1$
16. $P(x) = x^6 - 7x^3 - 8$

17–34 ■ Factor the polynomial completely, and find all its zeros. State the multiplicity of each zero.

17. $P(x) = x^2 + 25$
18. $P(x) = 4x^2 + 9$
19. $Q(x) = x^2 + 2x + 2$
20. $Q(x) = x^2 - 8x + 17$
21. $P(x) = x^3 + 4x$
22. $P(x) = x^3 - x^2 + x$
23. $Q(x) = x^4 - 1$
24. $Q(x) = x^4 - 625$
25. $P(x) = 16x^4 - 81$
26. $P(x) = x^3 - 64$
27. $P(x) = x^3 + x^2 + 9x + 9$
28. $P(x) = x^6 - 729$
29. $Q(x) = x^4 + 2x^2 + 1$
30. $Q(x) = x^4 + 10x^2 + 25$
31. $P(x) = x^4 + 3x^2 - 4$
32. $P(x) = x^5 + 7x^3$
33. $P(x) = x^5 + 6x^3 + 9x$
34. $P(x) = x^6 + 16x^3 + 64$

35–44 ■ Find a polynomial with integer coefficients that satisfies the given conditions.

35. P has degree 2 and zeros $1 + i$ and $1 - i$.
36. P has degree 2 and zeros $1 + i\sqrt{2}$ and $1 - i\sqrt{2}$.
37. Q has degree 3 and zeros 3, $2i$, and $-2i$.
38. Q has degree 3 and zeros 0 and i.
39. P has degree 3 and zeros 2 and i.
40. Q has degree 3 and zeros -3 and $1 + i$.
41. R has degree 4 and zeros $1 - 2i$ and 1, with 1 a zero of multiplicity 2.
42. S has degree 4 and zeros $2i$ and $3i$.
43. T has degree 4, zeros i and $1 + i$, and constant term 12.
44. U has degree 5, zeros $\frac{1}{2}$, -1, and $-i$, and leading coefficient 4; the zero -1 has multiplicity 2.

45–62 ■ Find all zeros of the polynomial.

45. $P(x) = x^3 + 2x^2 + 4x + 8$
46. $P(x) = x^3 - 7x^2 + 17x - 15$
47. $P(x) = x^3 - 2x^2 + 2x - 1$
48. $P(x) = x^3 + 7x^2 + 18x + 18$
49. $P(x) = x^3 - 3x^2 + 3x - 2$
50. $P(x) = x^3 - x - 6$
51. $P(x) = 2x^3 + 7x^2 + 12x + 9$
52. $P(x) = 2x^3 - 8x^2 + 9x - 9$
53. $P(x) = x^4 + x^3 + 7x^2 + 9x - 18$
54. $P(x) = x^4 - 2x^3 - 2x^2 - 2x - 3$
55. $P(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$
56. $P(x) = x^5 + x^3 + 8x^2 + 8$ [*Hint:* Factor by grouping.]
57. $P(x) = x^4 - 6x^3 + 13x^2 - 24x + 36$
58. $P(x) = x^4 - x^2 + 2x + 2$
59. $P(x) = 4x^4 + 4x^3 + 5x^2 + 4x + 1$
60. $P(x) = 4x^4 + 2x^3 - 2x^2 - 3x - 1$
61. $P(x) = x^5 - 3x^4 + 12x^3 - 28x^2 + 27x - 9$
62. $P(x) = x^5 - 2x^4 + 2x^3 - 4x^2 + x - 2$

63–68 ■ A polynomial P is given. **(a)** Factor P into linear and irreducible quadratic factors with real coefficients. **(b)** Factor P completely into linear factors with complex coefficients.

63. $P(x) = x^3 - 5x^2 + 4x - 20$
64. $P(x) = x^3 - 2x - 4$
65. $P(x) = x^4 + 8x^2 - 9$
66. $P(x) = x^4 + 8x^2 + 16$
67. $P(x) = x^6 - 64$
68. $P(x) = x^5 - 16x$

69. By the Zeros Theorem, every nth-degree polynomial equation has exactly n solutions (including possibly some that are repeated). Some of these may be real, and some may be imaginary. Use a graphing device to determine how many real and imaginary solutions each equation has.
   (a) $x^4 - 2x^3 - 11x^2 + 12x = 0$
   (b) $x^4 - 2x^3 - 11x^2 + 12x - 5 = 0$
   (c) $x^4 - 2x^3 - 11x^2 + 12x + 40 = 0$

70–72 ■ So far, we have worked only with polynomials that have real coefficients. These exercises involve polynomials with real and imaginary coefficients.

70. Find all solutions of the equation.
   (a) $2x + 4i = 1$
   (b) $x^2 - ix = 0$
   (c) $x^2 + 2ix - 1 = 0$
   (d) $ix^2 - 2x + i = 0$

71. **(a)** Show that $2i$ and $1 - i$ are both solutions of the equation

$$x^2 - (1 + i)x + (2 + 2i) = 0$$

but that their complex conjugates $-2i$ and $1 + i$ are not.

(b) Explain why the result of part (a) does not violate the Conjugate Zeros Theorem.

72. **(a)** Find the polynomial with *real* coefficients of the smallest possible degree for which i and $1 + i$ are zeros and in which the coefficient of the highest power is 1.

(b) Find the polynomial with *complex* coefficients of the smallest possible degree for which i and $1 + i$ are zeros and in which the coefficient of the highest power is 1.

DISCOVERY ■ DISCUSSION ■ WRITING

73. Polynomials of Odd Degree The Conjugate Zeros Theorem says that the complex zeros of a polynomial with real coefficients occur in complex conjugate pairs. Explain how this fact proves that a polynomial with real coefficients and odd degree has at least one real zero.

74. Roots of Unity There are two square roots of 1, namely, 1 and -1. These are the solutions of $x^2 = 1$. The fourth roots of 1 are the solutions of the equation $x^4 = 1$ or $x^4 - 1 = 0$. How many fourth roots of 1 are there? Find them. The cube roots of 1 are the solutions of the equation $x^3 = 1$ or $x^3 - 1 = 0$. How many cube roots of 1 are there? Find them. How would you find the sixth roots of 1? How many are there? Make a conjecture about the number of nth roots of 1.

3.7 Rational Functions

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find vertical asymptotes of rational functions ▶ Find horizontal asymptotes of rational functions ▶ Graph transformations of the rational function $y = 1/x$ ▶ Use asymptotes to graph rational functions ▶ Find slant asymptotes of rational functions

GET READY **Prepare for this section by reviewing Section P.7 on rational expressions.**

A rational function is a function of the form

$$r(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. We assume that $P(x)$ and $Q(x)$ have no factor in common. Even though rational functions are constructed from polynomials, their graphs look quite different from the graphs of polynomial functions.

▼ Rational Functions and Asymptotes

Domains of rational expressions are discussed in Section P.7.

The *domain* of a rational function consists of all real numbers x except those for which the denominator is zero. When graphing a rational function, we must pay special attention to the behavior of the graph near those x-values. We begin by graphing a very simple rational function.

EXAMPLE 1 | A Simple Rational Function

Graph the rational function $f(x) = 1/x$, and state the domain and range.

SOLUTION The function f is not defined for $x = 0$. The following tables show that when x is close to zero, the value of $|f(x)|$ is large, and the closer x gets to zero, the larger $|f(x)|$ gets.

For positive real numbers,

$$\frac{1}{\text{BIG NUMBER}} = \text{small number}$$

$$\frac{1}{\text{small number}} = \text{BIG NUMBER}$$

x	$f(x)$
-0.1	-10
-0.01	-100
-0.00001	$-100{,}000$
Approaching 0^-	Approaching $-\infty$

x	$f(x)$
0.1	10
0.01	100
0.00001	$100{,}000$
Approaching 0^+	Approaching ∞

We describe this behavior in words and in symbols as follows. The first table shows that as x approaches 0 from the left, the values of $y = f(x)$ decrease without bound. In symbols,

$$f(x) \to -\infty \quad \text{as} \quad x \to 0^- \qquad \text{“}y\text{ approaches negative infinity as }x\text{ approaches 0 from the left”}$$

The second table shows that as x approaches 0 from the right, the values of $f(x)$ increase without bound. In symbols,

$$f(x) \to \infty \quad \text{as} \quad x \to 0^+ \qquad \text{“}y\text{ approaches infinity as }x\text{ approaches 0 from the right”}$$

The next two tables show how $f(x)$ changes as $|x|$ becomes large.

x	$f(x)$
-10	-0.1
-100	-0.01
$-100{,}000$	-0.00001

Approaching $-\infty$ Approaching 0

x	$f(x)$
10	0.1
100	0.01
100,000	0.00001

Approaching ∞ Approaching 0

These tables show that as $|x|$ becomes large, the value of $f(x)$ gets closer and closer to zero. We describe this situation in symbols by writing

$$f(x) \to 0 \quad \text{as} \quad x \to -\infty \qquad \text{and} \qquad f(x) \to 0 \quad \text{as} \quad x \to \infty$$

Using the information in these tables and plotting a few additional points, we obtain the graph shown in Figure 1.

x	$f(x) = 1/x$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$

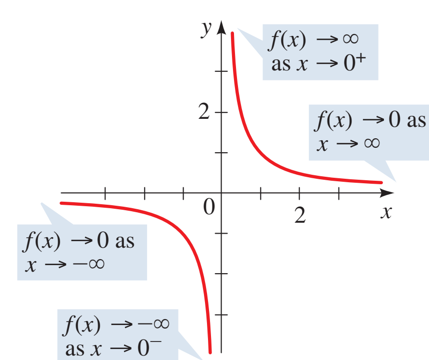


FIGURE 1
$f(x) = 1/x$

The function f is defined for all values of x other than 0, so the domain is $\{x \mid x \neq 0\}$. From the graph we see that the range is $\{y \mid y \neq 0\}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **7**

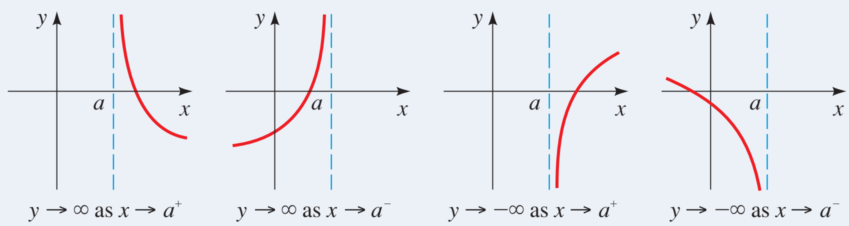
In Example 1 we used the following arrow notation.

Symbol	Meaning
$x \to a^-$	x approaches a from the left
$x \to a^+$	x approaches a from the right
$x \to -\infty$	x goes to negative infinity; that is, x decreases without bound
$x \to \infty$	x goes to infinity; that is, x increases without bound

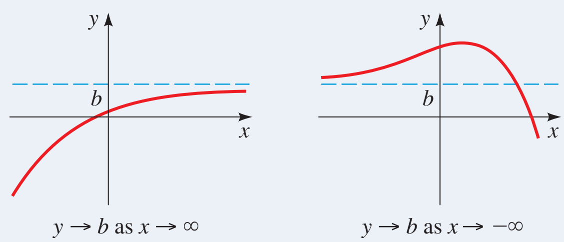
The line $x = 0$ is called a *vertical asymptote* of the graph in Figure 1, and the line $y = 0$ is a *horizontal asymptote*. Informally speaking, an asymptote of a function is a line to which the graph of the function gets closer and closer as one travels along that line.

DEFINITION OF VERTICAL AND HORIZONTAL ASYMPTOTES

1. The line $x = a$ is a **vertical asymptote** of the function $y = f(x)$ if y approaches $\pm\infty$ as x approaches a from the right or left.



2. The line $y = b$ is a **horizontal asymptote** of the function $y = f(x)$ if y approaches b as x approaches $\pm\infty$.



Recall that for a rational function $R(x) = P(x)/Q(x)$, we assume that $P(x)$ and $Q(x)$ have no factor in common. (See the definition of a rational function on page 311, and Exercise 91.)

A rational function has vertical asymptotes where the function is undefined, that is, where the denominator is zero.

Transformations of $y = 1/x$

A rational function of the form

$$r(x) = \frac{ax + b}{cx + d}$$

can be graphed by shifting, stretching, and/or reflecting the graph of $f(x) = 1/x$ shown in Figure 1, using the transformations studied in Section 2.5. (Such functions are called *linear fractional transformations*.)

EXAMPLE 2 | Using Transformations to Graph Rational Functions

Graph each rational function, and state the domain and range.

(a) $r(x) = \dfrac{2}{x - 3}$ **(b)** $s(x) = \dfrac{3x + 5}{x + 2}$

SOLUTION

(a) Let $f(x) = 1/x$. Then we can express r in terms of f as follows:

$$\begin{aligned} r(x) &= \frac{2}{x - 3} \\ &= 2\left(\frac{1}{x - 3}\right) && \text{Factor 2} \\ &= 2(f(x - 3)) && \text{Since } f(x) = 1/x \end{aligned}$$

From this form we see that the graph of r is obtained from the graph of f by shifting 3 units to the right and stretching vertically by a factor of 2. Thus r has vertical asymptote $x = 3$ and horizontal asymptote $y = 0$. The graph of r is shown in Figure 2.

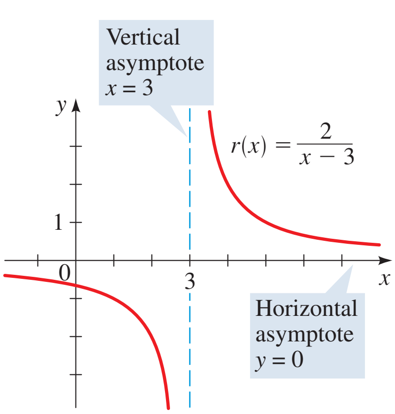


FIGURE 2

The function r is defined for all x other than 3, so the domain is $\{x \mid x \neq 3\}$. From the graph we see that the range is $\{y \mid y \neq 0\}$.

$$\begin{array}{r} 3 \\ x+2\overline{)3x+5} \\ \underline{3x+6} \\ -1 \end{array}$$

(b) Using long division (see the margin), we get $s(x) = 3 - \dfrac{1}{x+2}$. Thus we can express s in terms of f as follows:

$$\begin{aligned} s(x) &= 3 - \frac{1}{x+2} \\ &= -\frac{1}{x+2} + 3 && \text{Rearrange terms} \\ &= -f(x+2) + 3 && \text{Since } f(x) = 1/x \end{aligned}$$

From this form we see that the graph of s is obtained from the graph of f by shifting 2 units to the left, reflecting in the x-axis, and shifting upward 3 units. Thus s has vertical asymptote $x = -2$ and horizontal asymptote $y = 3$. The graph of s is shown in Figure 3.

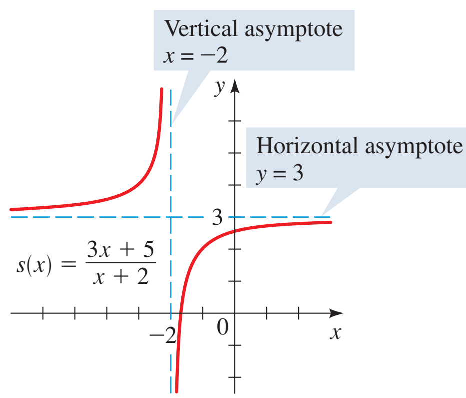


FIGURE 3

The function s is defined for all x other than -2, so the domain is $\{x \mid x \neq -2\}$. From the graph we see that the range is $\{y \mid y \neq 3\}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **13** AND **15**

Asymptotes of Rational Functions

The methods of Example 2 work only for simple rational functions. To graph more complicated ones, we need to take a closer look at the behavior of a rational function near its vertical and horizontal asymptotes.

EXAMPLE 3 | Asymptotes of a Rational Function

Graph $r(x) = \dfrac{2x^2 - 4x + 5}{x^2 - 2x + 1}$, and state the domain and range.

SOLUTION

Vertical asymptote: We first factor the denominator

$$r(x) = \frac{2x^2 - 4x + 5}{(x-1)^2}$$

The line $x = 1$ is a vertical asymptote because the denominator of r is zero when $x = 1$.

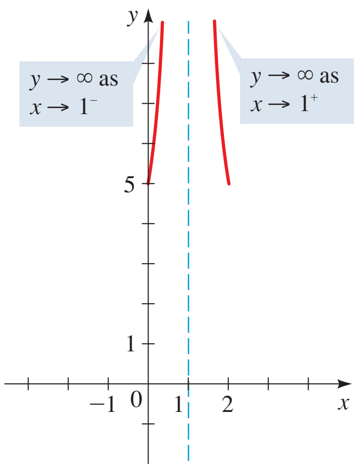


FIGURE 4

To see what the graph of r looks like near the vertical asymptote, we make tables of values for x-values to the left and to the right of 1. From the tables shown below we see that

$$y \to \infty \quad \text{as} \quad x \to 1^- \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to 1^+$$

$x \to 1^-$

x	y
0	5
0.5	14
0.9	302
0.99	30,002

Approaching 1^- Approaching ∞

$x \to 1^+$

x	y
2	5
1.5	14
1.1	302
1.01	30,002

Approaching 1^+ Approaching ∞

Thus, near the vertical asymptote $x = 1$, the graph of r has the shape shown in Figure 4.

Horizontal asymptote: The horizontal asymptote is the value that y approaches as $x \to \pm\infty$. To help us find this value, we divide both numerator and denominator by x^2, the highest power of x that appears in the expression:

$$y = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{2 - \frac{4}{x} + \frac{5}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}$$

The fractional expressions $\frac{4}{x}, \frac{5}{x^2}, \frac{2}{x}$, and $\frac{1}{x^2}$ all approach 0 as $x \to \pm\infty$ (see Exercise 95, page 17). So as $x \to \pm\infty$, we have

These terms approach 0

$$y = \frac{2 - \frac{4}{x} + \frac{5}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}} \longrightarrow \frac{2 - 0 + 0}{1 - 0 + 0} = 2$$

These terms approach 0

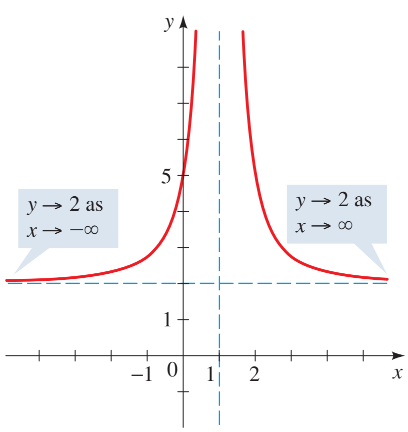


FIGURE 5

$$r(x) = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1}$$

Thus, the horizontal asymptote is the line $y = 2$.

Since the graph must approach the horizontal asymptote, we can complete it as in Figure 5.

Domain and range: The function r is defined for all values of x other than 1, so the domain is $\{x \mid x \neq 1\}$. From the graph we see that the range is $\{y \mid y > 2\}$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE 45 ■

From Example 3 we see that the horizontal asymptote is determined by the leading coefficients of the numerator and denominator, since after dividing through by x^2 (the highest power of x), all other terms approach zero. In general, if $r(x) = P(x)/Q(x)$ and the degrees of P and Q are the same (both n, say), then dividing both numerator and denominator by x^n shows that the horizontal asymptote is

$$y = \frac{\text{leading coefficient of } P}{\text{leading coefficient of } Q}$$

The following box summarizes the procedure for finding asymptotes.

FINDING ASYMPTOTES OF RATIONAL FUNCTIONS

Let r be the rational function

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

1. The vertical asymptotes of r are the lines $x = a$, where a is a zero of the denominator.

2. (a) If $n < m$, then r has horizontal asymptote $y = 0$.

(b) If $n = m$, then r has horizontal asymptote $y = \dfrac{a_n}{b_m}$.

(c) If $n > m$, then r has no horizontal asymptote.

Recall that for a rational function $R(x) = P(x)/Q(x)$, we assume that $P(x)$ and $Q(x)$ have no factor in common. (See Exercise 91.)

EXAMPLE 4 | Asymptotes of a Rational Function

Find the vertical and horizontal asymptotes of $r(x) = \dfrac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$.

SOLUTION

Vertical asymptotes: We first factor

$$r(x) = \frac{3x^2 - 2x - 1}{(2x - 1)(x + 2)}$$

This factor is 0 when $x = \frac{1}{2}$

This factor is 0 when $x = -2$

The vertical asymptotes are the lines $x = \frac{1}{2}$ and $x = -2$.

Horizontal asymptote: The degrees of the numerator and denominator are the same, and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{3}{2}$$

Thus the horizontal asymptote is the line $y = \frac{3}{2}$.

To confirm our results, we graph r using a graphing calculator (see Figure 6).

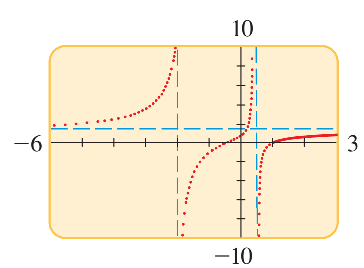


FIGURE 6
$r(x) = \dfrac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$
Graph is drawn using dot mode to avoid extraneous lines.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **31** AND **33**

Graphing Rational Functions

We have seen that asymptotes are important when graphing rational functions. In general, we use the following guidelines to graph rational functions.

A fraction is 0 if and only if its numerator is 0.

SKETCHING GRAPHS OF RATIONAL FUNCTIONS

1. **Factor.** Factor the numerator and denominator.
2. **Intercepts.** Find the x-intercepts by determining the zeros of the numerator and the y-intercept from the value of the function at $x = 0$.
3. **Vertical Asymptotes.** Find the vertical asymptotes by determining the zeros of the denominator, and then see whether $y \to \infty$ or $y \to -\infty$ on each side of each vertical asymptote by using test values.
4. **Horizontal Asymptote.** Find the horizontal asymptote (if any), using the procedure described in the box on page 316.
5. **Sketch the Graph.** Graph the information provided by the first four steps. Then plot as many additional points as needed to fill in the rest of the graph of the function.

EXAMPLE 5 | Graphing a Rational Function

Graph $r(x) = \dfrac{2x^2 + 7x - 4}{x^2 + x - 2}$, and state the domain and range.

SOLUTION We factor the numerator and denominator, find the intercepts and asymptotes, and sketch the graph.

Factor: $y = \dfrac{(2x - 1)(x + 4)}{(x - 1)(x + 2)}$

x-Intercepts: The x-intercepts are the zeros of the numerator, $x = \frac{1}{2}$ and $x = -4$.

y-Intercept: To find the y-intercept, we substitute $x = 0$ into the original form of the function.

$$r(0) = \frac{2(0)^2 + 7(0) - 4}{(0)2 + (0) - 2} = \frac{-4}{-2} = 2$$

The y-intercept is 2.

Vertical asymptotes: The vertical asymptotes occur where the denominator is 0, that is, where the function is undefined. From the factored form we see that the vertical asymptotes are the lines $x = 1$ and $x = -2$.

When choosing test values, we must make sure that there is no x-intercept between the test point and the vertical asymptote.

Behavior near vertical asymptotes: We need to know whether $y \to \infty$ or $y \to -\infty$ on each side of each vertical asymptote. To determine the sign of y for x-values near the vertical asymptotes, we use test values. For instance, as $x \to 1^-$, we use a test value close to and to the left of 1 ($x = 0.9$, say) to check whether y is positive or negative to the left of $x = 1$.

$$y = \frac{(2(0.9) - 1)((0.9) + 4)}{((0.9) - 1)((0.9) + 2)} \qquad \text{whose sign is} \qquad \frac{(+)(+)}{(-)(+)} \quad \text{(negative)}$$

So $y \to -\infty$ as $x \to 1^-$. On the other hand, as $x \to 1^+$, we use a test value close to and to the right of 1 ($x = 1.1$, say), to get

$$y = \frac{(2(1.1) - 1)((1.1) + 4)}{((1.1) - 1)((1.1) + 2)} \qquad \text{whose sign is} \qquad \frac{(+)(+)}{(+)(+)} \quad \text{(positive)}$$

So $y \to \infty$ as $x \to 1^+$. The other entries in the following table are calculated similarly.

As $x \to$	-2^-	-2^+	1^-	1^+
the sign of $y = \dfrac{(2x - 1)(x + 4)}{(x - 1)(x + 2)}$ is	$\dfrac{(-)(+)}{(-)(-)}$	$\dfrac{(-)(+)}{(-)(+)}$	$\dfrac{(+)(+)}{(-)(+)}$	$\dfrac{(+)(+)}{(+)(+)}$
so $y \to$	$-\infty$	∞	$-\infty$	∞

MATHEMATICS IN THE MODERN WORLD

Unbreakable Codes

If you read spy novels, you know about secret codes and how the hero "breaks" the code. Today secret codes have a much more common use. Most of the information that is stored on computers is coded to prevent unauthorized use. For example, your banking records, medical records, and school records are coded. Many cellular and cordless phones code the signal carrying your voice so that no one can listen in. Fortunately, because of recent advances in mathematics, today's codes are "unbreakable."

Modern codes are based on a simple principle: Factoring is much harder than multiplying. For example, try multiplying 78 and 93; now try factoring 9991. It takes a long time to factor 9991 because it is a product of two primes 97×103, so to factor it, we have to find one of these primes. Now imagine trying to factor a number N that is the product of two primes p and q, each about 200 digits long. Even the fastest computers would take many millions of years to factor such a number! But the same computer would take less than a second to multiply two such numbers. This fact was used by Ron Rivest, Adi Shamir, and Leonard Adleman in the 1970s to devise the RSA code. Their code uses an extremely large number to encode a message but requires us to know its factors to decode it. As you can see, such a code is practically unbreakable.

The RSA code is an example of a "public key encryption" code. In such codes, anyone can code a message using a publicly known procedure based on N, but to decode the message, they must know p and q, the factors of N. When the RSA code was developed, it was thought that a carefully selected 80-digit number would provide an unbreakable code. But interestingly, recent advances in the study of factoring have made much larger numbers necessary.

Horizontal asymptote: The degrees of the numerator and denominator are the same, and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{2}{1} = 2$$

Thus the horizontal asymptote is the line $y = 2$.

Graph: We use the information we have found, together with some additional values, to sketch the graph in Figure 7.

x	y
-6	0.93
-3	-1.75
-1	4.50
1.5	6.29
2	4.50
3	3.50

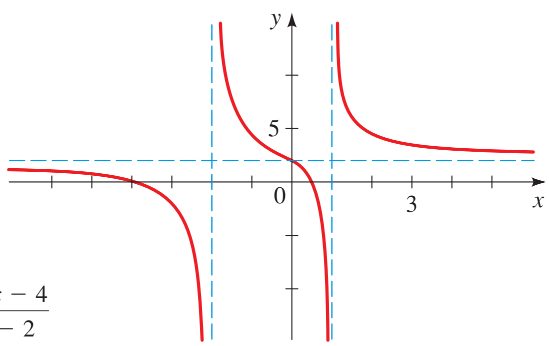


FIGURE 7

$$r(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2}$$

Domain and range: The domain is $\{x \mid x \neq 1, x \neq -2\}$. From the graph we see that the range is all real numbers.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **53**

EXAMPLE 6 | Graphing a Rational Function

Graph $r(x) = \dfrac{5x + 21}{x^2 + 10x + 25}$, and state the domain and range.

SOLUTION

Factor: $y = \dfrac{5x + 21}{(x + 5)^2}$

x-Intercept: $-\dfrac{21}{5}$, from $5x + 21 = 0$

y-Intercept: $\dfrac{21}{25}$, because $r(0) = \dfrac{5 \cdot 0 + 21}{0^2 + 10 \cdot 0 + 25}$

$$= \frac{21}{25}$$

Vertical asymptote: $x = -5$, from the zeros of the denominator

Behavior near vertical asymptote:

As $x \to$	-5^-	-5^+
the sign of $y = \dfrac{5x + 21}{(x + 5)^2}$ is	$\dfrac{(-)}{(-)(-)}$	$\dfrac{(-)}{(+)(+)}$
so $y \to$	$-\infty$	$-\infty$

Horizontal asymptote: $y = 0$, because the degree of the numerator is less than the degree of the denominator

Graph: We use the information we have found, together with some additional values, to sketch the graph in Figure 8.

x	y
-15	-0.5
-10	-1.2
-3	1.5
-1	1.0
3	0.6
5	0.5
10	0.3

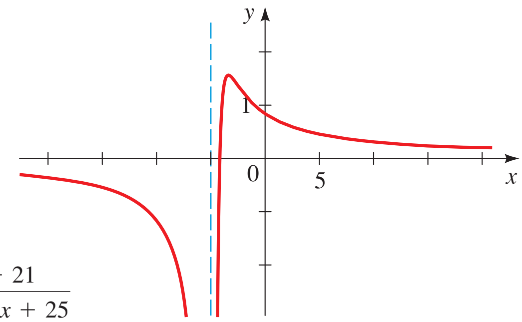


FIGURE 8

$r(x) = \dfrac{5x + 21}{x^2 + 10x + 25}$

Domain and range: The domain is $\{x \mid x \neq -5\}$. From the graph we see that the range is approximately the interval $(-\infty, 1.5]$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **55**

From the graph in Figure 8 we see that, contrary to common misconception, a graph may cross a horizontal asymptote. The graph in Figure 8 crosses the x-axis (the horizontal asymptote) from below, reaches a maximum value near $x = -3$, and then approaches the x-axis from above as $x \to \infty$.

EXAMPLE 7 | Graphing a Rational Function

Graph the rational function $r(x) = \dfrac{x^2 - 3x - 4}{2x^2 + 4x}$.

SOLUTION

Factor: $y = \dfrac{(x + 1)(x - 4)}{2x(x + 2)}$

x-Intercepts: -1 and 4, from $x + 1 = 0$ and $x - 4 = 0$

y-Intercept: None, because $r(0)$ is undefined

Vertical asymptotes: $x = 0$ and $x = -2$, from the zeros of the denominator

Behavior near vertical asymptotes:

As $x \to$	-2^-	-2^+	0^-	0^+
the sign of $y = \dfrac{(x+1)(x-4)}{2x(x+2)}$ is	$\dfrac{(-)(-)}{(-)(-)}$	$\dfrac{(-)(-)}{(-)(+)}$	$\dfrac{(+)(-)}{(-)(+)}$	$\dfrac{(+)(-)}{(+)(+)}$
so $y \to$	∞	$-\infty$	∞	$-\infty$

Horizontal asymptote: $y = \frac{1}{2}$, because the degree of the numerator and the degree of the denominator are the same and

$$\frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}} = \frac{1}{2}$$

Graph: We use the information we have found, together with some additional values, to sketch the graph in Figure 9.

x	y
-3	2.33
-2.5	3.90
-0.5	1.50
1	-1.00
3	-0.13
5	0.09

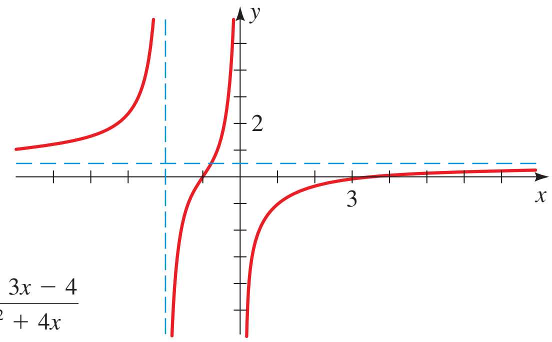


FIGURE 9
$r(x) = \dfrac{x^2 - 3x - 4}{2x^2 + 4x}$

Domain and range: The domain is $\{x \mid x \neq 0, x \neq -2\}$. From the graph we see that the range is all real numbers.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **57**

Slant Asymptotes and End Behavior

If $r(x) = P(x)/Q(x)$ is a rational function in which the degree of the numerator is one more than the degree of the denominator, we can use the Division Algorithm to express the function in the form

$$r(x) = ax + b + \frac{R(x)}{Q(x)}$$

where the degree of R is less than the degree of Q and $a \neq 0$. This means that as $x \to \pm\infty$, $R(x)/Q(x) \to 0$, so for large values of $|x|$ the graph of $y = r(x)$ approaches the graph of the line $y = ax + b$. In this situation we say that $y = ax + b$ is a **slant asymptote**, or an **oblique asymptote**.

EXAMPLE 8 | A Rational Function with a Slant Asymptote

Graph the rational function $r(x) = \dfrac{x^2 - 4x - 5}{x - 3}$.

SOLUTION

Factor: $y = \dfrac{(x + 1)(x - 5)}{x - 3}$

x-Intercepts: -1 and 5, from $x + 1 = 0$ and $x - 5 = 0$

y-Intercepts: $\frac{5}{3}$, because $r(0) = \dfrac{0^2 - 4 \cdot 0 - 5}{0 - 3} = \dfrac{5}{3}$

Horizontal asymptote: None, because the degree of the numerator is greater than the degree of the denominator

Vertical asymptote: $x = 3$, from the zero of the denominator

Behavior near vertical asymptote: $y \to \infty$ as $x \to 3^-$ and $y \to -\infty$ as $x \to 3^+$

$$\begin{array}{r} x - 1 \\ x - 3\overline{)x^2 - 4x - 5} \\ \underline{x^2 - 3x} \\ -x - 5 \\ \underline{-x + 3} \\ -8 \end{array}$$

Slant asymptote: Since the degree of the numerator is one more than the degree of the denominator, the function has a slant asymptote. Dividing (see the margin), we obtain

$$r(x) = x - 1 - \frac{8}{x - 3}$$

Thus $y = x - 1$ is the slant asymptote.

Graph: We use the information we have found, together with some additional values, to sketch the graph in Figure 10.

x	y
-2	-1.4
1	4
2	9
4	-5
6	2.33

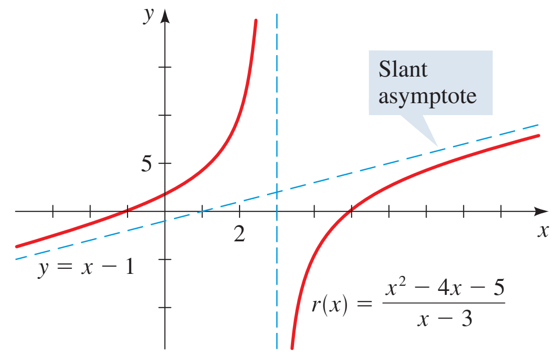


FIGURE 10

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **65**

So far, we have considered only horizontal and slant asymptotes as end behaviors for rational functions. In the next example we graph a function whose end behavior is like that of a parabola.

EXAMPLE 9 | End Behavior of a Rational Function

Graph the rational function

$$r(x) = \frac{x^3 - 2x^2 + 3}{x - 2}$$

and describe its end behavior.

SOLUTION

Factor: $y = \dfrac{(x + 1)(x^2 - 3x + 3)}{x - 2}$

x-Intercepts: -1, from $x + 1 = 0$ (The other factor in the numerator has no real zeros.)

y-Intercepts: $-\frac{3}{2}$, because $r(0) = \dfrac{0^3 - 2 \cdot 0^2 + 3}{0 - 2} = -\dfrac{3}{2}$

Vertical asymptote: $x = 2$, from the zero of the denominator

Behavior near vertical asymptote: $y \to -\infty$ as $x \to 2^-$ and $y \to \infty$ as $x \to 2^+$

Horizontal asymptote: None, because the degree of the numerator is greater than the degree of the denominator

$$\begin{array}{r} x^2 \\ x - 2\overline{)x^3 - 2x^2 + 0x + 3} \\ \underline{x^3 - 2x^2 } \\ 3 \end{array}$$

End behavior: Dividing (see the margin), we get

$$r(x) = x^2 + \frac{3}{x - 2}$$

This shows that the end behavior of r is like that of the parabola $y = x^2$ because $3/(x - 2)$ is small when $|x|$ is large. That is, $3/(x - 2) \to 0$ as $x \to \pm\infty$. This means that the graph of r will be close to the graph of $y = x^2$ for large $|x|$.

Graph: In Figure 11(a) we graph r in a small viewing rectangle; we can see the intercepts, the vertical asymptotes, and the local minimum. In Figure 11(b) we graph r in a larger viewing rectangle; here the graph looks almost like the graph of a parabola. In Figure 11(c) we graph both $y = r(x)$ and $y = x^2$; these graphs are very close to each other except near the vertical asymptote.

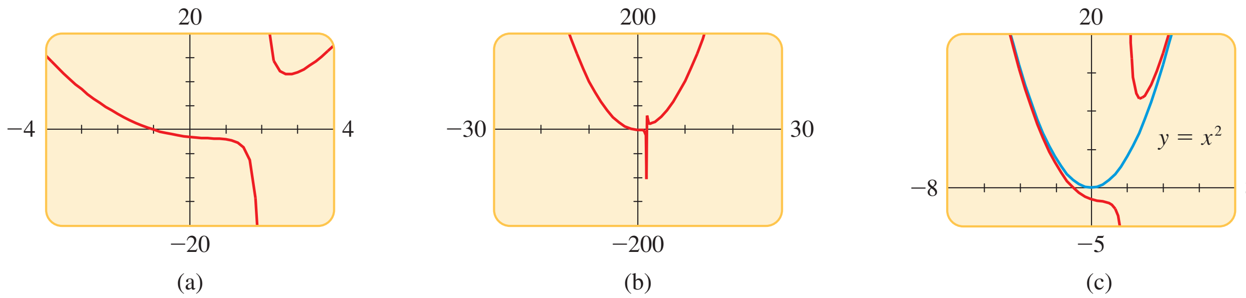


FIGURE 11
$r(x) = \dfrac{x^3 - 2x^2 + 3}{x - 2}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **73**

Applications

Rational functions occur frequently in scientific applications of algebra. In the next example we analyze the graph of a function from the theory of electricity.

EXAMPLE 10 | Electrical Resistance

When two resistors with resistances R_1 and R_2 are connected in parallel, their combined resistance R is given by the formula

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

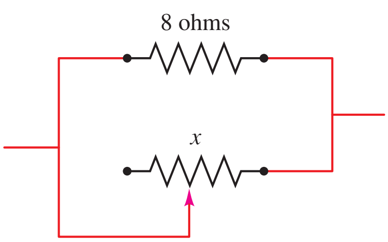


FIGURE 12

Suppose that a fixed 8-ohm resistor is connected in parallel with a variable resistor, as shown in Figure 12. If the resistance of the variable resistor is denoted by x, then the combined resistance R is a function of x. Graph R, and give a physical interpretation of the graph.

SOLUTION Substituting $R_1 = 8$ and $R_2 = x$ into the formula gives the function

$$R(x) = \frac{8x}{8 + x}$$

Since resistance cannot be negative, this function has physical meaning only when $x > 0$. The function is graphed in Figure 13(a) using the viewing rectangle $[0, 20]$ by $[0, 10]$. The function has no vertical asymptote when x is restricted to positive values. The combined resistance R increases as the variable resistance x increases. If we widen the viewing rectangle to $[0, 100]$ by $[0, 10]$, we obtain the graph in Figure 13(b). For large x the combined resistance R levels off, getting closer and closer to the horizontal asymptote $R = 8$. No matter how large the variable resistance x, the combined resistance is never greater than 8 ohms.

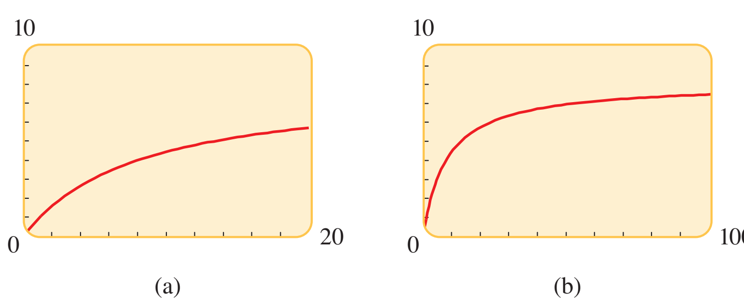


FIGURE 13
$R(x) = \dfrac{8x}{8 + x}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **83**

3.7 EXERCISES

CONCEPTS

1. If the rational function $y = r(x)$ has the vertical asymptote $x = 2$, then as $x \to 2^+$, either $y \to$ ________ or $y \to$ ________.

2. If the rational function $y = r(x)$ has the horizontal asymptote $y = 2$, then $y \to$ ________ as $x \to \pm\infty$.

3–6 ■ The following questions are about the rational function

$$r(x) = \frac{(x+1)(x-2)}{(x+2)(x-3)}$$

3. The function r has x-intercepts ________ and ________.

4. The function r has y-intercept ________.

5. The function r has vertical asymptotes $x =$ ________ and $x =$ ________.

6. The function r has horizontal asymptote $y =$ ________.

SKILLS

7–10 ■ A rational function is given. **(a)** Complete each table for the function. **(b)** Describe the behavior of the function near its vertical asymptote, based on Tables 1 and 2. **(c)** Determine the horizontal asymptote, based on Tables 3 and 4.

TABLE 1

x	$r(x)$
1.5	
1.9	
1.99	
1.999	

TABLE 2

x	$r(x)$
2.5	
2.1	
2.01	
2.001	

TABLE 3

x	$r(x)$
10	
50	
100	
1000	

TABLE 4

x	$r(x)$
−10	
−50	
−100	
−1000	

7. $r(x) = \dfrac{x}{x-2}$ **8.** $r(x) = \dfrac{4x+1}{x-2}$

9. $r(x) = \dfrac{3x-10}{(x-2)^2}$ **10.** $r(x) = \dfrac{3x^2+1}{(x-2)^2}$

11–18 ■ Use transformations of the graph of $y = 1/x$ to graph the rational function, and state the domain and range of r, as in Example 2.

11. $r(x) = \dfrac{1}{x-1}$ **12.** $r(x) = \dfrac{1}{x+4}$

13. $s(x) = \dfrac{3}{x+1}$ **14.** $s(x) = \dfrac{-2}{x-2}$

15. $t(x) = \dfrac{2x-3}{x-2}$ **16.** $t(x) = \dfrac{3x-3}{x+2}$

17. $r(x) = \dfrac{x+2}{x+3}$ **18.** $r(x) = \dfrac{2x-9}{x-4}$

19–24 ■ Find the x- and y-intercepts of the rational function.

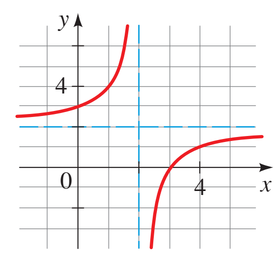
19. $r(x) = \dfrac{x-1}{x+4}$ **20.** $s(x) = \dfrac{3x}{x-5}$

21. $t(x) = \dfrac{x^2-x-2}{x-6}$ **22.** $r(x) = \dfrac{2}{x^2+3x-4}$

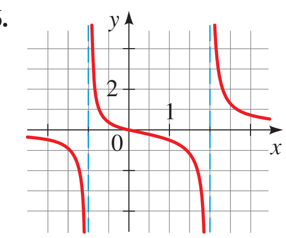
23. $r(x) = \dfrac{x^2-9}{x^2}$ **24.** $r(x) = \dfrac{x^3+8}{x^2+4}$

25–28 ■ From the graph, determine the x- and y-intercepts and the vertical and horizontal asymptotes.

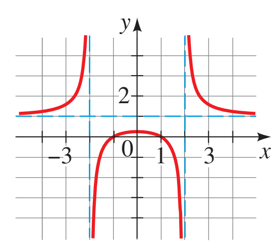
25.



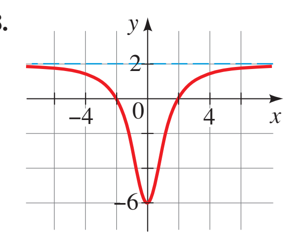
26.



27.



28.



29–40 ■ Find all horizontal and vertical asymptotes (if any).

29. $r(x) = \dfrac{5}{x-2}$ **30.** $r(x) = \dfrac{2x-3}{x^2-1}$

31. $r(x) = \dfrac{6x}{x^2+2}$ **32.** $r(x) = \dfrac{2x-4}{x^2+x+1}$

33. $s(x) = \dfrac{6x^2+1}{2x^2+x-1}$ **34.** $s(x) = \dfrac{8x^2+1}{4x^2+2x-6}$

35. $s(x) = \dfrac{(5x-1)(x+1)}{(3x-1)(x+2)}$ **36.** $s(x) = \dfrac{(2x-1)(x+3)}{(3x-1)(x-4)}$

37. $r(x) = \dfrac{6x^3-2}{2x^3+5x^2+6x}$ **38.** $r(x) = \dfrac{5x^3}{x^3+2x^2+5x}$

39. $t(x) = \dfrac{x^2+2}{x-1}$ **40.** $r(x) = \dfrac{x^3+3x^2}{x^2-4}$

41–64 ■ Find the intercepts and asymptotes, and then sketch a graph of the rational function and state the domain and range. Use a graphing device to confirm your answer.

41. $r(x) = \dfrac{4x-4}{x+2}$ **42.** $r(x) = \dfrac{2x+6}{-6x+3}$

43. $s(x) = \dfrac{4 - 3x}{x + 7}$

44. $s(x) = \dfrac{1 - 2x}{2x + 3}$

45. $r(x) = \dfrac{18}{(x - 3)^2}$

46. $r(x) = \dfrac{x - 2}{(x + 1)^2}$

47. $s(x) = \dfrac{4x - 8}{(x - 4)(x + 1)}$

48. $s(x) = \dfrac{x + 2}{(x + 3)(x - 1)}$

49. $s(x) = \dfrac{6}{x^2 - 5x - 6}$

50. $s(x) = \dfrac{2x - 4}{x^2 + x - 2}$

51. $t(x) = \dfrac{3x + 6}{x^2 + 2x - 8}$

52. $t(x) = \dfrac{x - 2}{x^2 - 4x}$

53. $r(x) = \dfrac{(x - 1)(x + 2)}{(x + 1)(x - 3)}$

54. $r(x) = \dfrac{2x(x + 2)}{(x - 1)(x - 4)}$

55. $r(x) = \dfrac{x^2 - 2x + 1}{x^2 + 2x + 1}$

56. $r(x) = \dfrac{4x^2}{x^2 - 2x - 3}$

57. $r(x) = \dfrac{2x^2 + 10x - 12}{x^2 + x - 6}$

58. $r(x) = \dfrac{2x^2 + 2x - 4}{x^2 + x}$

59. $r(x) = \dfrac{x^2 - x - 6}{x^2 + 3x}$

60. $r(x) = \dfrac{x^2 + 3x}{x^2 - x - 6}$

61. $r(x) = \dfrac{3x^2 + 6}{x^2 - 2x - 3}$

62. $r(x) = \dfrac{5x^2 + 5}{x^2 + 4x + 4}$

63. $s(x) = \dfrac{x^2 - 2x + 1}{x^3 - 3x^2}$

64. $t(x) = \dfrac{x^3 - x^2}{x^3 - 3x - 2}$

65–72 ■ Find the slant asymptote and the vertical asymptotes, and sketch a graph of the function.

65. $r(x) = \dfrac{x^2}{x - 2}$

66. $r(x) = \dfrac{x^2 + 2x}{x - 1}$

67. $r(x) = \dfrac{x^2 - 2x - 8}{x}$

68. $r(x) = \dfrac{3x - x^2}{2x - 2}$

69. $r(x) = \dfrac{x^2 + 5x + 4}{x - 3}$

70. $r(x) = \dfrac{x^3 + 4}{2x^2 + x - 1}$

71. $r(x) = \dfrac{x^3 + x^2}{x^2 - 4}$

72. $r(x) = \dfrac{2x^3 + 2x}{x^2 - 1}$

73–76 ■ Graph the rational function f, and determine all vertical asymptotes from your graph. Then graph f and g in a sufficiently large viewing rectangle to show that they have the same end behavior.

73. $f(x) = \dfrac{2x^2 + 6x + 6}{x + 3}, \quad g(x) = 2x$

74. $f(x) = \dfrac{-x^3 + 6x^2 - 5}{x^2 - 2x}, \quad g(x) = -x + 4$

75. $f(x) = \dfrac{x^3 - 2x^2 + 16}{x - 2}, \quad g(x) = x^2$

76. $f(x) = \dfrac{-x^4 + 2x^3 - 2x}{(x - 1)^2}, \quad g(x) = 1 - x^2$

77–82 ■ Graph the rational function, and find all vertical asymptotes, x- and y-intercepts, and local extrema, correct to the nearest decimal. Then use long division to find a polynomial that has the same end behavior as the rational function, and graph both functions in a sufficiently large viewing rectangle to verify that the end behaviors of the polynomial and the rational function are the same.

77. $y = \dfrac{2x^2 - 5x}{2x + 3}$

78. $y = \dfrac{x^4 - 3x^3 + x^2 - 3x + 3}{x^2 - 3x}$

79. $y = \dfrac{x^5}{x^3 - 1}$

80. $y = \dfrac{x^4}{x^2 - 2}$

81. $r(x) = \dfrac{x^4 - 3x^3 + 6}{x - 3}$

82. $r(x) = \dfrac{4 + x^2 - x^4}{x^2 - 1}$

APPLICATIONS

83. Population Growth Suppose that the rabbit population on Mr. Jenkins' farm follows the formula

$$p(t) = \frac{3000t}{t + 1}$$

where $t \geq 0$ is the time (in months) since the beginning of the year.

(a) Draw a graph of the rabbit population.

(b) What eventually happens to the rabbit population?

84. Drug Concentration After a certain drug is injected into a patient, the concentration c of the drug in the bloodstream is monitored. At time $t \geq 0$ (in minutes since the injection), the concentration (in mg/L) is given by

$$c(t) = \frac{30t}{t^2 + 2}$$

(a) Draw a graph of the drug concentration.

(b) What eventually happens to the concentration of drug in the bloodstream?

85. Drug Concentration A drug is administered to a patient, and the concentration of the drug in the bloodstream is monitored. At time $t \geq 0$ (in hours since giving the drug), the concentration (in mg/L) is given by

$$c(t) = \frac{5t}{t^2 + 1}$$

Graph the function c with a graphing device.

(a) What is the highest concentration of drug that is reached in the patient's bloodstream?

(b) What happens to the drug concentration after a long period of time?

(c) How long does it take for the concentration to drop below 0.3 mg/L?

86. Flight of a Rocket Suppose a rocket is fired upward from the surface of the earth with an initial velocity v (measured in meters per second). Then the maximum height h (in meters) reached by the rocket is given by the function

$$h(v) = \frac{Rv^2}{2gR - v^2}$$

where $R = 6.4 \times 10^6$ m is the radius of the earth and $g = 9.8 \text{ m/s}^2$ is the acceleration due to gravity. Use a graphing device to draw a graph of the function h. (Note that h and v must both be positive, so the viewing rectangle need not contain negative values.) What does the vertical asymptote represent physically?

87. The Doppler Effect As a train moves toward an observer (see the figure), the pitch of its whistle sounds higher to the observer than it would if the train were at rest, because the crests of the sound waves are compressed closer together. This phenomenon is called the *Doppler effect*. The observed pitch P is a function of the speed v of the train and is given by

$$P(v) = P_0\left(\frac{s_0}{s_0 - v}\right)$$

where P_0 is the actual pitch of the whistle at the source and $s_0 = 332$ m/s is the speed of sound in air. Suppose that a train has a whistle pitched at $P_0 = 440$ Hz. Graph the function $y = P(v)$ using a graphing device. How can the vertical asymptote of this function be interpreted physically?

88. Focusing Distance For a camera with a lens of fixed focal length F to focus on an object located a distance x from the lens, the film must be placed a distance y behind the lens, where F, x, and y are related by

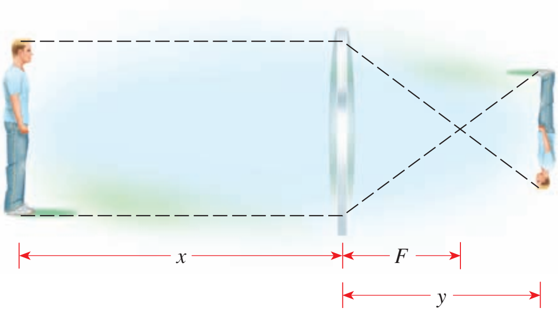
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{F}$$

(See the figure.) Suppose the camera has a 55-mm lens ($F = 55$).

(a) Express y as a function of x and graph the function.

(b) What happens to the focusing distance y as the object moves far away from the lens?

(c) What happens to the focusing distance y as the object moves close to the lens?



DISCOVERY ■ DISCUSSION ■ WRITING

89. Constructing a Rational Function from Its Asymptotes Give an example of a rational function that has vertical asymptote $x = 3$. Now give an example of one that has vertical asymptote $x = 3$ *and* horizontal asymptote $y = 2$. Now give an example of a rational function with vertical asymptotes $x = 1$ and $x = -1$, horizontal asymptote $y = 0$, and x-intercept 4.

90. A Rational Function with No Asymptote Explain how you can tell (without graphing it) that the function

$$r(x) = \frac{x^6 + 10}{x^4 + 8x^2 + 15}$$

has no x-intercept and no horizontal, vertical, or slant asymptote. What is its end behavior?'

91. Graphs with Holes In this chapter we adopted the convention that in rational functions, the numerator and denominator don't share a common factor. In this exercise we consider the graph of a rational function that does not satisfy this rule.

(a) Show that the graph of

$$r(x) = \frac{3x^2 - 3x - 6}{x - 2}$$

is the line $y = 3x + 3$ with the point $(2, 9)$ removed. [*Hint*: Factor. What is the domain of r?]

(b) Graph the following rational functions:

$$s(x) = \frac{x^2 + x - 20}{x + 5}$$

$$t(x) = \frac{2x^2 - x - 1}{x - 1}$$

$$u(x) = \frac{x - 2}{x^2 - 2x}$$

92. Transformations of $y = 1/x^2$ In Example 2 we saw that some simple rational functions can be graphed by shifting, stretching, or reflecting the graph of $y = 1/x$. In this exercise we consider rational functions that can be graphed by transforming the graph of $y = 1/x^2$, shown on the following page.

(a) Graph the function

$$r(x) = \frac{1}{(x - 2)^2}$$

by transforming the graph of $y = 1/x^2$.

(b) Use long division and factoring to show that the function

$$s(x) = \frac{2x^2 + 4x + 5}{x^2 + 2x + 1}$$

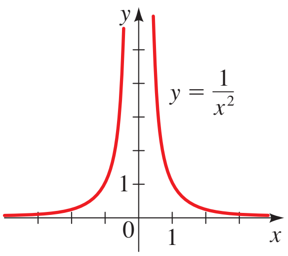
can be written as

$$s(x) = 2 + \frac{3}{(x + 1)^2}$$

Then graph s by transforming the graph of $y = 1/x^2$.

(c) One of the following functions can be graphed by transforming the graph of $y = 1/x^2$; the other cannot. Use transformations to graph the one that can be, and explain why this method doesn't work for the other one.

$$p(x) = \frac{2 - 3x^2}{x^2 - 4x + 4} \qquad q(x) = \frac{12x - 3x^2}{x^2 - 4x + 4}$$



3.8 Modeling Variation

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find equations for direct variation ▶ Find equations for inverse variation ▶ Find equations for combined variation

When scientists talk about a *mathematical model* for a real-world phenomenon, they often mean a function that describes the dependence of one physical quantity on another. For instance, the model may describe the population of an animal species as a function of time or the pressure of a gas as a function of its volume. In this section we study a kind of modeling that occurs frequently in the sciences, called *variation.*

▼ Direct Variation

One type of variation is called *direct variation*; it occurs when one quantity is a constant multiple of the other. We use a function of the form $f(x) = kx$ to model this dependence.

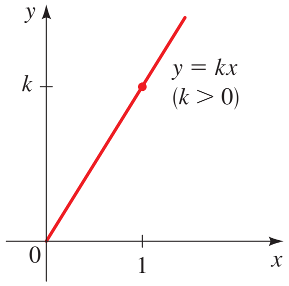


FIGURE 1

DIRECT VARIATION

If the quantities x and y are related by an equation

$$y = kx$$

for some constant $k \neq 0$, we say that y **varies directly as** x, or y is **directly proportional to** x, or simply y **is proportional to** x. The constant k is called the **constant of proportionality**.

Recall that the graph of an equation of the form $y = mx + b$ is a line with slope m and y-intercept b. So the graph of an equation $y = kx$ that describes direct variation is a line with slope k and y-intercept 0 (see Figure 1).

EXAMPLE 1 | Direct Variation

During a thunderstorm you see the lightning before you hear the thunder because light travels much faster than sound. The distance between you and the storm varies directly as the time interval between the lightning and the thunder.

(a) Suppose that the thunder from a storm 5400 ft away takes 5 s to reach you. Determine the constant of proportionality, and write the equation for the variation.

(b) Sketch the graph of this equation. What does the constant of proportionality represent?

(c) If the time interval between the lightning and thunder is now 8 s, how far away is the storm?

SOLUTION

(a) Let d be the distance from you to the storm, and let t be the length of the time interval. We are given that d varies directly as t, so

$$d = kt$$

where k is a constant. To find k, we use the fact that $t = 5$ when $d = 5400$. Substituting these values in the equation, we get

$$5400 = k(5) \qquad \text{Substitute}$$

$$k = \frac{5400}{5} = 1080 \qquad \text{Solve for } k$$

Substituting this value of k in the equation for d, we obtain

$$d = 1080t$$

as the equation for d as a function of t.

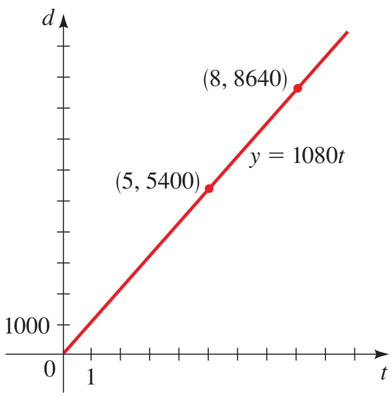


FIGURE 2

(b) The graph of the equation $d = 1080t$ is a line through the origin with slope 1080 and is shown in Figure 2. The constant $k = 1080$ is the approximate speed of sound (in ft/s).

(c) When $t = 8$, we have

$$d = 1080 \cdot 8 = 8640$$

So the storm is 8640 ft $\approx$ 1.6 mi away. ■

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **17** AND **33**

Inverse Variation

Another function that is frequently used in mathematical modeling is $f(x) = k/x$, where k is a constant.

INVERSE VARIATION

If the quantities x and y are related by the equation

$$y = \frac{k}{x}$$

for some constant $k \neq 0$, we say that y **is inversely proportional to** x or y **varies inversely as** x. The constant k is called the **constant of proportionality**.

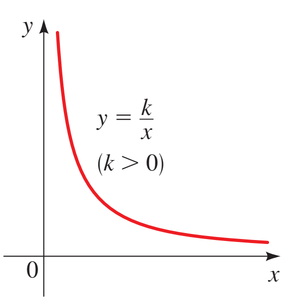


FIGURE 3 Inverse variation

The function $y = k/x$ is a rational function. We graphed this function (for $k = 1$) in Example 1 on page 311. The graph of $y = k/x$ for $x > 0$ is shown in Figure 3 for the case $k > 0$. It gives a picture of what happens when y is inversely proportional to x.

EXAMPLE 2 | Inverse Variation

Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure of the gas is inversely proportional to the volume of the gas.

(a) Suppose the pressure of a sample of air that occupies 0.106 m^3 at 25°C is 50 kPa. Find the constant of proportionality, and write the equation that expresses the inverse proportionality. Sketch a graph of this equation.

(b) If the sample expands to a volume of 0.3 m^3, find the new pressure.

SOLUTION

(a) Let P be the pressure of the sample of gas, and let V be its volume. Then, by the definition of inverse proportionality, we have

$$P = \frac{k}{V}$$

where k is a constant. To find k, we use the fact that $P = 50$ when $V = 0.106$. Substituting these values in the equation, we get

$$50 = \frac{k}{0.106} \qquad \text{Substitute}$$

$$k = (50)(0.106) = 5.3 \qquad \text{Solve for } k$$

Putting this value of k in the equation for P, we have

$$P = \frac{5.3}{V}$$

The function P is a rational function of V. We sketch the graph using the methods of Section 3.7. Since V represents volume (which is never negative), we sketch the part of the graph for which $V > 0$ only. The graph is shown in Figure 4.

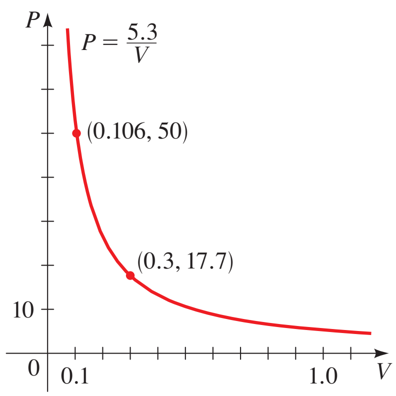


FIGURE 4

(b) When $V = 0.3$, we have

$$P = \frac{5.3}{0.3} \approx 17.7$$

So the new pressure is about 17.7 kPa.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **19** AND **41**

Combining Different Types of Variation

In the sciences, relationships between three or more variables are common, and any combination of the different types of proportionality that we have discussed is possible. For example, if the quantities x, y, and z are related by the equation

$$z = kxy$$

then we say that z is **proportional to the product** of x and y. We can also express this relationship by saying that z **varies jointly** as x and y, or that z **is jointly proportional to** x and y. If the quantities x, y, and z are related by the equation

$$z = k\frac{x}{y}$$

we say that z **is proportional to** x **and inversely proportional to** y or that z **varies directly as** x **and inversely as** y.

EXAMPLE 3 | Combining Variations

The apparent brightness B of a light source (measured in W/m^2) is directly proportional to the luminosity L (measured in W) of the light source and inversely proportional to the square of the distance d from the light source (measured in meters).

(a) Write an equation that expresses this variation.

(b) If the distance is doubled, by what factor will the brightness change?

(c) If the distance is cut in half and the luminosity is tripled, by what factor will the brightness change?

SOLUTION

(a) Since B is directly proportional to L and inversely proportional to d^2, we have

$$B = k\frac{L}{d^2} \qquad \text{Brightness at distance } d \text{ and luminosity } L$$

where k is a constant.

(b) To obtain the brightness at double the distance we replace d by $2d$ in the equation we obtained in part (a):

$$B = k\frac{L}{(2d)^2} = \frac{1}{4}\left(k\frac{L}{d^2}\right) \qquad \text{Brightness at distance } 2d$$

Comparing this expression with that obtained in part (a), we see that the brightness is $\frac{1}{4}$ of the original brightness.

(c) To obtain the brightness at half the distance d and triple the luminosity L, we replace d by $d/2$ and L by $3L$ in the equation we obtained in part (a):

$$B = k\frac{3L}{(\frac{1}{2}d)^2} = \frac{3}{\frac{1}{4}}\left(k\frac{L}{d^2}\right) = 12\left(k\frac{L}{d^2}\right) \qquad \text{Brightness at distance } \tfrac{1}{2}d \text{ and luminosity } 3L$$

Comparing this expression with that obtained in part (a), we see that the brightness is 12 times the original brightness.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **29** AND **43**

The relationship between apparent brightness, actual brightness (or luminosity), and distance is used in estimating distances to stars (see Exercise 54).

EXAMPLE 4 | Newton's Law of Gravitation

Newton's Law of Gravitation says that two objects with masses m_1 and m_2 attract each other with a force F that is jointly proportional to their masses and inversely proportional to the square of the distance r between the objects. Express Newton's Law of Gravitation as an equation.

SOLUTION Using the definitions of joint and inverse variation and the traditional notation G for the gravitational constant of proportionality, we have

$$F = G\frac{m_1 m_2}{r^2}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **21** AND **45**

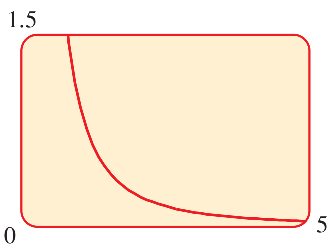


FIGURE 5 Graph of $F = \dfrac{1}{r^2}$

If m_1 and m_2 are fixed masses, then the gravitational force between them is $F = C/r^2$ (where $C = Gm_1m_2$ is a constant). Figure 5 shows the graph of this equation for $r > 0$ with $C = 1$. Observe how the gravitational attraction decreases with increasing distance.

Like the Law of Gravity, many laws of nature are *inverse square laws*. There is a geometric reason for this. Imagine a force or energy originating from a point source and spreading its influence equally in all directions, just like the light source in Example 3 or the gravitational force exerted by a planet in Example 4. The influence of the force or energy at a distance r from the source is spread out over the surface of a sphere of radius r, which has area $A = 4\pi r^2$ (see Figure 6). So the intensity I at a distance r from the source is the source strength S divided by the area A of the sphere:

$$I = \frac{S}{4\pi r^2} = \frac{k}{r^2}$$

where k is the constant $S/(4\pi)$. Thus point sources of light, sound, gravity, electromagnetic fields, and radiation must all obey inverse square laws, simply because of the geometry of space.

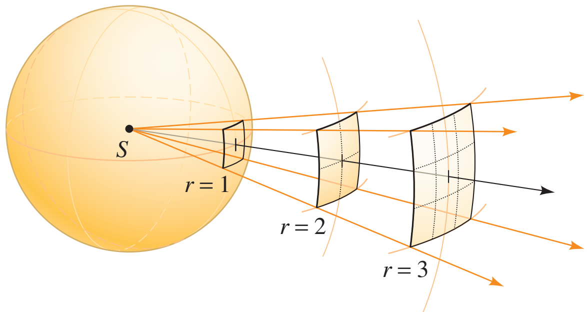


FIGURE 6 Energy from a point source S.

3.8 EXERCISES

CONCEPTS

1. If the quantities x and y are related by the equation $y = 3x$, then we say that y is ________ ________ to x and the constant of ________ is 3.

2. If the quantities x and y are related by the equation $y = \frac{3}{x}$, then we say that y is ________ ________ to x and the constant of ________ is 3.

3. If the quantities x, y, and z are related by the equation $z = 3\frac{x}{y}$, then we say that z is ________ ________ to x and ________ ________ to y.

4. If z is jointly proportional to x and y and if z is 10 when x is 4 and y is 5, then x, y, and z are related by the equation $z =$ ________.

SKILLS

5–16 ■ Write an equation that expresses the statement.

5. T varies directly as x.

6. P is directly proportional to w.

7. v is inversely proportional to z.

8. w is proportional to the product of m and n.

9. y is proportional to s and inversely proportional to t.

10. P varies inversely as T.

11. z is proportional to the square root of y.

12. A is proportional to the square of t and inversely proportional to the cube of x.

13. V is proportional to the product of l, w, and h.

14. S is proportional to the product of the squares of r and θ.

15. R is proportional to i and inversely proportional to P and t.

16. A is jointly proportional to the square roots of x and y.

17–28 ■ Express the statement as an equation. Use the given information to find the constant of proportionality.

17. y is directly proportional to x. If $x = 6$, then $y = 42$.

18. z varies inversely as t. If $t = 3$, then $z = 5$.

19. R is inversely proportional to s. If $s = 4$, then $R = 3$.

20. P is directly proportional to T. If $T = 300$, then $P = 20$.

21. M varies directly as x and inversely as y. If $x = 2$ and $y = 6$, then $M = 5$.

22. S varies jointly as p and q. If $p = 4$ and $q = 5$, then $S = 180$.

23. W is inversely proportional to the square of r. If $r = 6$, then $W = 10$.

24. t is jointly proportional to x and y, and inversely proportional to r. If $x = 2$, $y = 3$, and $r = 12$, then $t = 25$.

25. C is jointly proportional to l, w, and h. If $l = w = h = 2$, then $C = 128$.

26. H is jointly proportional to the squares of l and w. If $l = 2$ and $w = \frac{1}{3}$, then $H = 36$.

27. s is inversely proportional to the square root of t. If $s = 100$, then $t = 25$.

28. M is jointly proportional to a, b, and c and inversely proportional to d. If a and d have the same value and if b and c are both 2, then $M = 128$.

29–32 ■ A statement describing the relationship between the variables x, y, and z is given. **(a)** Express the statement as an equation. **(b)** If x is tripled and y is doubled, by what factor does z change? (See Example 3.)

29. z varies directly as the cube of x and inversely as the square of y.

30. z is directly proportional to the square of x and inversely proportional to the fourth power of y.

31. z is jointly proportional to the cube of x and the fifth power of y.

32. z is inversely proportional to the square of x and the cube of y.

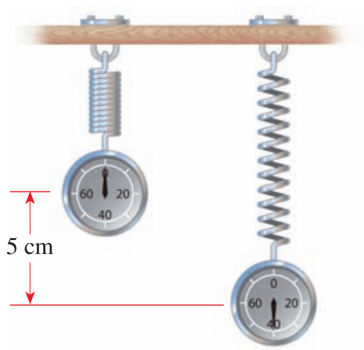
APPLICATIONS

33. Hooke's Law Hooke's Law states that the force needed to keep a spring stretched x units beyond its natural length is directly proportional to x. Here the constant of proportionality is called the **spring constant**.

(a) Write Hooke's Law as an equation.

(b) If a spring has a natural length of 10 cm and a force of 40 N is required to maintain the spring stretched to a length of 15 cm, find the spring constant.

(c) What force is needed to keep the spring stretched to a length of 14 cm?



34. Printing Costs The cost C of printing a magazine is jointly proportional to the number of pages p in the magazine and the number of magazines printed m.

(a) Write an equation that expresses this joint variation.

(b) Find the constant of proportionality if the printing cost is \$60,000 for 4000 copies of a 120-page magazine.

(c) How much would the printing cost be for 5000 copies of a 92-page magazine?

35. Power from a Windmill The power P that can be obtained from a windmill is directly proportional to the cube of the wind speed s.

(a) Write an equation that expresses this variation.

(b) Find the constant of proportionality for a windmill that produces 96 watts of power when the wind is blowing at 20 mi/h.

(c) How much power will this windmill produce if the wind speed increases to 30 mi/h?

36. Power Needed to Propel a Boat The power P (measured in horsepower, hp) needed to propel a boat is directly proportional to the cube of the speed s.

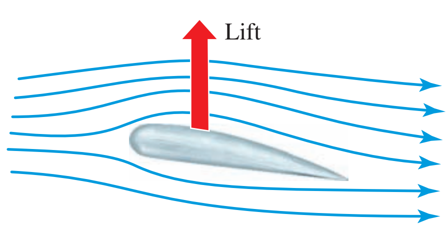
(a) Write an equation that expresses this variation.

(b) Find the constant of proportionality for a boat that needs an 80-hp engine to propel the boat at 10 knots.

(c) How much power is needed to drive this boat at 15 knots?

37. Stopping Distance The stopping distance D of a car after the brakes have been applied varies directly as the square of the speed s. A certain car traveling at 50 mi/h can stop in 240 ft. What is the maximum speed it can be traveling if it needs to stop in 160 ft?

38. Aerodynamic Lift The lift L on an airplane wing at takeoff varies jointly as the square of the speed s of the plane and the area A of its wings. A plane with a wing area of 500 ft^2 traveling at 50 mi/h experiences a lift of 1700 lb. How much lift would a plane with a wing area of 600 ft^2 traveling at 40 mi/h experience?



39. Drag Force on a Boat The drag force F on a boat is jointly proportional to the wetted surface area A on the hull and the square of the speed s of the boat. A boat experiences a drag force of 220 lb when traveling at 5 mi/h with a wetted surface area of 40 ft^2. How fast must a boat be traveling if it has 28 ft^2 of wetted surface area and is experiencing a drag force of 175 lb?

40. Kepler's Third Law Kepler's Third Law of planetary motion states that the square of the period T of a planet (the time it takes for the planet to make a complete revolution about the sun) is directly proportional to the cube of its average distance d from the sun.

(a) Express Kepler's Third Law as an equation.

(b) Find the constant of proportionality by using the fact that for our planet the period is about 365 days and the average distance is about 93 million miles.

(c) The planet Neptune is about 2.79×10^9 mi from the sun. Find the period of Neptune.

41. Ideal Gas Law The pressure P of a sample of gas is directly proportional to the temperature T and inversely proportional to the volume V.

(a) Write an equation that expresses this variation.

(b) Find the constant of proportionality if 100 L of gas exerts a pressure of 33.2 kPa at a temperature of 400 K (absolute temperature measured on the Kelvin scale).

(c) If the temperature is increased to 500 K and the volume is decreased to 80 L, what is the pressure of the gas?

42. Skidding in a Curve A car is traveling on a curve that forms a circular arc. The force F needed to keep the car from skidding is jointly proportional to the weight w of the car and the square of its speed s, and is inversely proportional to the radius r of the curve.

(a) Write an equation that expresses this variation.

(b) A car weighing 1600 lb travels around a curve at 60 mi/h. The next car to round this curve weighs 2500 lb and requires the same force as the first car to keep from skidding. How fast is the second car traveling?

43. Loudness of Sound The loudness L of a sound (measured in decibels, dB) is inversely proportional to the square of the distance d from the source of the sound.

(a) Write an equation that expresses this variation.

(b) Find the constant of proportionality if a person 10 ft from a lawn mower experiences a sound level of 70 dB.

(c) If the distance in part (b) is doubled, by what factor is the loudness changed?

(d) If the distance in part (b) is cut in half, by what factor is the loudness changed?

44. A Jet of Water The power P of a jet of water is jointly proportional to the cross-sectional area A of the jet and to the cube of the velocity v.

(a) Write an equation that expresses this variation.

(b) If the velocity is doubled and the cross-sectional area is halved, by what factor is the power changed?

(c) If the velocity is halved and the cross-sectional area is tripled, by what factor is the power changed?

45. Electrical Resistance The resistance R of a wire varies directly as its length L and inversely as the square of its diameter d.

(a) Write an equation that expresses this joint variation.

(b) Find the constant of proportionality if a wire 1.2 m long and 0.005 m in diameter has a resistance of 140 ohms.

(c) Find the resistance of a wire made of the same material that is 3 m long and has a diameter of 0.008 m.

(d) If the diameter is doubled and the length is tripled, by what factor is the resistance changed?

46. Growing Cabbages In the short growing season of the Canadian arctic territory of Nunavut, some gardeners find it possible to grow gigantic cabbages in the midnight sun. Assume that the final size of a cabbage is proportional to the amount of nutrients it receives and inversely proportional to the number of other cabbages surrounding it. A cabbage that received 20 oz of nutrients and had 12 other cabbages around it grew to 30 lb. What size would it grow to if it received 10 oz of nutrients and had only 5 cabbage "neighbors"?

47. Radiation Energy The total radiation energy E emitted by a heated surface per unit area varies as the fourth power of its absolute temperature T. The temperature is 6000 K at the surface of the sun and 300 K at the surface of the earth.

(a) How many times more radiation energy per unit area is produced by the sun than by the earth?

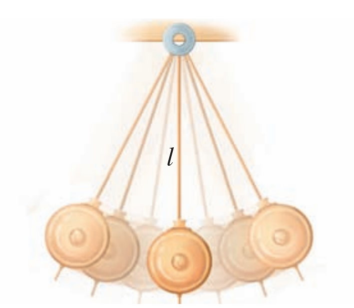
(b) The radius of the earth is 3960 mi and the radius of the sun is 435,000 mi. How many times more total radiation does the sun emit than the earth?

48. Value of a Lot The value of a building lot on Galiano Island is jointly proportional to its area and the quantity of water produced by a well on the property. A 200 ft by 300 ft lot has a well producing 10 gallons of water per minute, and is valued at \$48,000. What is the value of a 400 ft by 400 ft lot if the well on the lot produces 4 gallons of water per minute?

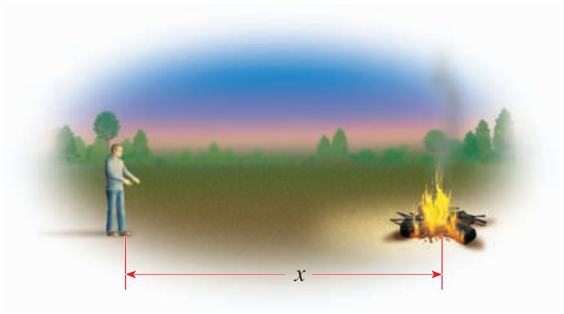
49. Law of the Pendulum The period of a pendulum (the time elapsed during one complete swing of the pendulum) varies directly with the square root of the length of the pendulum.

(a) Express this relationship by writing an equation.

(b) To double the period, how would we have to change the length l?



50. Heat of a Campfire The heat experienced by a hiker at a campfire is proportional to the amount of wood on the fire and inversely proportional to the cube of his distance from the fire. If the hiker is 20 ft from the fire and someone doubles the amount of wood burning, how far from the fire would he have to be so that he feels the same heat as before?



51. Frequency of Vibration The frequency f of vibration of a violin string is inversely proportional to its length L. The constant of proportionality k is positive and depends on the tension and density of the string.

(a) Write an equation that represents this variation.

(b) What effect does doubling the length of the string have on the frequency of its vibration?

52. Spread of a Disease The rate r at which a disease spreads in a population of size P is jointly proportional to the number x of infected people and the number $P - x$ who are not infected. An infection erupts in a small town that has population $P = 5000$.

(a) Write an equation that expresses r as a function of x.

(b) Compare the rate of spread of this infection when 10 people are infected to the rate of spread when 1000 people are infected. Which rate is larger? By what factor?

(c) Calculate the rate of spread when the entire population is infected. Why does this answer make intuitive sense?

53–54 ■ Solve the problem using the relationship between brightness B, luminosity L, and distance d derived in Example 3. The proportionality constant is $k = 0.080$.

53. Brightness of a Star The luminosity of a star is $L = 2.5 \times 10^{26}$ W, and its distance to the earth is $d = 2.4 \times 10^{19}$ m. How bright does the star appear on the earth?

54. Distance to a Star The luminosity of a star is $L = 5.8 \times 10^{30}$ W and its brightness as viewed from the earth is $B = 8.2 \times 10^{-16}$ W/m^2. Find the distance of the star from the earth.

DISCOVERY ■ DISCUSSION ■ WRITING

55. Is Proportionality Everything? A great many laws of physics and chemistry are expressible as proportionalities. Give at least one example of a function that occurs in the sciences that is *not* a proportionality.

CHAPTER 3 | REVIEW

■ PROPERTIES AND FORMULAS

Quadratic Functions (pp. 258–261)

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c$$

It can be expressed in the **standard form**

$$f(x) = a(x - h)^2 + k$$

by completing the square.

The graph of a quadratic function in standard form is a **parabola** with **vertex** (h, k).

If $a > 0$, then the quadratic function f has the **minimum value** k at $x = h = -b/(2a)$.

If $a < 0$, then the quadratic function f has the **maximum value** k at $x = h = -b/(2a)$.

Polynomial Functions (p. 266)

A **polynomial function** of **degree** n is a function P of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

The numbers a_i are the **coefficients** of the polynomial; a_n is the **leading coefficient**, and a_0 is the **constant coefficient** (or **constant term**).

The graph of a polynomial function is a smooth, continuous curve.

Real Zeros of Polynomials (p. 271)

A **zero** of a polynomial P is a number c for which $P(c) = 0$. The following are equivalent ways of describing real zeros of polynomials:

1. c is a real zero of P.
2. $x = c$ is a solution of the equation $P(x) = 0$.
3. $x - c$ is a factor of $P(x)$.
4. c is an x-intercept of the graph of P.

Multiplicity of a Zero (pp. 274–275)

A zero c of a polynomial P has multiplicity m if m is the highest power for which $(x - c)^m$ is a factor of $P(x)$.

Local Maxima and Minima (pp. 275–276)

A polynomial function P of degree n has $n - 1$ or fewer **local extrema** (i.e., local maxima and minima).

Division of Polynomials (pp. 280–281)

If P and D are any polynomials (with $D(x) \neq 0$), then we can divide P by D using either **long division** or (if D is linear) **synthetic division**. The result of the division can be expressed in one of the following equivalent forms:

$$P(x) = D(x) \cdot Q(x) + R(x)$$

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

In this division, P is the **dividend**, D is the **divisor**, Q is the **quotient**, and R is the **remainder**. When the division is continued to its completion, the degree of R will be less than the degree of D (or $R(x) = 0$).

Remainder Theorem (p. 283)

When $P(x)$ is divided by the linear divisor $D(x) = x - c$, the **remainder** is the constant $P(c)$. So one way to **evaluate** a polynomial function P at c is to use synthetic division to divide $P(x)$ by $x - c$ and observe the value of the remainder.

Rational Zeros of Polynomials (pp. 287–288)

If the polynomial P given by

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has integer coefficients, then all the **rational zeros** of P have the form

$$x = \pm\frac{p}{q}$$

where p is a divisor of the constant term a_0 and q is a divisor of the leading coefficient a_n.

So to find all the rational zeros of a polynomial, we list all the *possible* rational zeros given by this principle and then check to see which *actually* are zeros by using synthetic division.

Descartes' Rule of Signs (p. 290)

Let P be a polynomial with real coefficients. Then:

The number of positive real zeros of P either is the number of **changes of sign** in the coefficients of $P(x)$ or is less than that by an even number.

The number of negative real zeros of P either is the number of **changes of sign** in the coefficients of $P(-x)$ or is less than that by an even number.

Upper and Lower Bounds Theorem (p. 291)

Suppose we divide the polynomial P by the linear expression $x - c$ and arrive at the result

$$P(x) = (x - c) \cdot Q(x) + r$$

If $c > 0$ and the coefficients of Q, followed by r, are all nonnegative, then c is an **upper bound** for the zeros of P.

If $c < 0$ and the coefficients of Q, followed by r (including zero coefficients), are alternately nonnegative and nonpositive, then c is a **lower bound** for the zeros of P.

Complex Numbers (pp. 298–300)

A **complex number** is a number of the form $a + bi$, where $i = \sqrt{-1}$.

The **complex conjugate** of $a + bi$ is

$$\overline{a + bi} = a - bi$$

To **multiply** complex numbers, treat them as binomials and use $i^2 = -1$ to simplify the result.

To **divide** complex numbers, multiply numerator and denominator by the complex conjugate of the denominator:

$$\frac{a + bi}{c + di} = \left(\frac{a + bi}{c + di}\right) \cdot \left(\frac{c - di}{c - di}\right) = \frac{(a + bi)(c - di)}{c^2 + d^2}$$

The Fundamental Theorem of Algebra, Complete Factorization, and the Zeros Theorem (pp. 303–304)

Every polynomial P of degree n with complex coefficients has exactly n complex zeros, provided that each zero of multiplicity m is counted m times. P factors into n linear factors as follows:

$$P(x) = a(x - c_1)(x - c_2) \cdots (x - c_n)$$

where a is the leading coefficient of P and $c_1, c_1, \ldots, c_n$ are the zeros of P.

Conjugate Zeros Theorem (p. 308)

If the polynomial P has real coefficients and if $a + bi$ is a zero of P, then its complex conjugate $a - bi$ is also a zero of P.

Linear and Quadratic Factors Theorem (p. 309)

Every polynomial with real coefficients can be factored into linear and irreducible quadratic factors with real coefficients.

Rational Functions (p. 311)

A **rational function** r is a quotient of polynomial functions:

$$r(x) = \frac{P(x)}{Q(x)}$$

We generally assume that the polynomials P and Q have no factors in common.

Asymptotes (pp. 312–313)

The line $x = a$ is a **vertical asymptote** of the function $y = f(x)$ if

$$y \to \infty \quad \text{or} \quad y \to -\infty \quad \text{as} \quad x \to a^+ \quad \text{or} \quad x \to a^-$$

The line $y = b$ is a **horizontal asymptote** of the function $y = f(x)$ if

$$y \to b \quad \text{as} \quad x \to \infty \quad \text{or} \quad x \to -\infty$$

Asymptotes of Rational Functions (pp. 313–316)

Let $r(x) = \dfrac{P(x)}{Q(x)}$ be a rational function.

The vertical asymptotes of r are the lines $x = a$ where a is a zero of Q.

If the degree of P is less than the degree of Q, then the horizontal asymptote of r is the line $y = 0$.

If the degrees of P and Q are the same, then the horizontal asymptote of r is the line $y = b$, where

$$b = \frac{\text{leading coefficient of } P}{\text{leading coefficient of } Q}$$

If the degree of P is greater than the degree of Q, then r has no horizontal asymptote.

Variation (pp. 326–328)

If y is **directly proportional** to x, then

$$y = kx$$

If y is **inversely proportional** to x, then

$$y = \frac{k}{x}$$

LEARNING OBJECTIVES SUMMARY

Section	After completing this chapter, you should be able to . . .	Review Exercises
3.1	▪ Express quadratic functions in standard form	1–4
	▪ Graph quadratic functions using the standard form	1–4
	▪ Find maximum or minimum values of quadratic functions	5–6
	▪ Model with quadratic functions	7–8
3.2	▪ Graph basic polynomial functions	9–14
	▪ Describe the end behavior of a polynomial function	15–24, 43–50
	▪ Graph a polynomial function using its zeros	15–24, 43–50
	▪ Use multiplicity to help graph a polynomial function	17–20, 43–50
	▪ Find local maxima and minima of polynomial functions	21–26
3.3	▪ Use long division to divide polynomials	33–34
	▪ Use synthetic division to divide polynomials	27–32
	▪ Use the Remainder Theorem to find values of a polynomial	35–36, 39–40
	▪ Use the Factor Theorem to factor a polynomial	37–38
	▪ Find a polynomial with specified zeros	61–64
3.4	▪ Use the Rational Zeros Theorem to find the rational zeroes of polynomials	41–42, 65–76
	▪ Use Descartes' Rule of Signs to determine the possible number of positive and negative zeros of a polynomial	41–42, 65–76
	▪ Use the Upper and Lower Bounds Theorem to find upper and lower bounds for the zeros of a polynomial	63–64, 65–76
	▪ Use algebra and graphing devices to solve polynomial equations	77–80
3.5	▪ Add and subtract complex numbers	51–52
	▪ Multiply and divide complex numbers	53–58
	▪ Work with square roots of negative numbers	59–60
	▪ Find complex solutions of quadratic equations	67–68, 69, 72–73, 75–76
3.6	▪ State the Fundamental Theorem of Algebra	
	▪ Factor a polynomial completely (into linear factors) over the complex numbers	43–50, 65–76
	▪ Use the Conjugate Zeros Theorem to find polynomials with specified zeros	62–63
	▪ Factor a polynomial completely (into linear and quadratic factors) over the real numbers	81–82
3.7	▪ Find vertical asymptotes of rational functions	83–92
	▪ Find horizontal asymptotes of rational functions	83–92
	▪ Graph transformations of the rational function $y = 1/x$	83–86
	▪ Use asymptotes to graph rational functions	83–92
	▪ Find slant asymptote of rational functions	93–96
3.8	▪ Find functions that model direct variation	98
	▪ Find functions that model inverse variation	99, 101
	▪ Find functions that model combined variation	100, 102–103

EXERCISES

1–4 ▪ A quadratic function is given. **(a)** Express the function in standard form. **(b)** Graph the function.

1. $f(x) = x^2 + 4x + 1$

2. $f(x) = -2x^2 + 12x + 12$

3. $g(x) = 1 + 8x - x^2$

4. $g(x) = 6x - 3x^2$

5–6 ▪ Find the maximum or minimum value of the quadratic function.

5. $f(x) = 2x^2 + 4x - 5$

6. $g(x) = 1 - x - x^2$

7. A stone is thrown upward from the top of a building. Its height (in feet) above the ground after t seconds is given by the function $h(t) = -16t^2 + 48t + 32$. What maximum height does the stone reach?

8. The profit P (in dollars) generated by selling x units of a certain commodity is given by the function

$$P(x) = -1500 + 12x - 0.004x^2$$

What is the maximum profit, and how many units must be sold to generate it?

9–14 ■ Graph the polynomial by transforming an appropriate graph of the form $y = x^n$. Show clearly all x- and y-intercepts.

9. $P(x) = -x^3 + 64$
10. $P(x) = 2x^3 - 16$
11. $P(x) = 2(x + 1)^4 - 32$
12. $P(x) = 81 - (x - 3)^4$
13. $P(x) = 32 + (x - 1)^5$
14. $P(x) = -3(x + 2)^5 + 96$

15–18 ■ A polynomial function P is given. **(a)** Describe the end behavior of P. **(b)** Sketch a graph of P. Make sure your graph shows all intercepts.

15. $P(x) = (x - 3)(x + 1)(x - 5)$
16. $P(x) = -(x - 5)(x^2 - 9)(x + 2)$
17. $P(x) = -(x - 1)^2(x - 4)(x + 2)^2$
18. $P(x) = x^2(x^2 - 4)(x^2 - 9)$

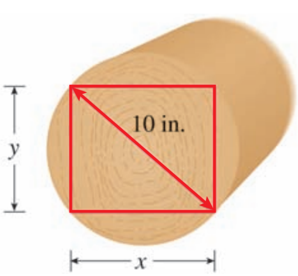
19–20 ■ A polynomial function P is given. **(a)** Determine the multiplicity of each zero of P. **(b)** Sketch a graph of P.

19. $P(x) = x^3(x - 2)^2$
20. $P(x) = x(x + 1)^3(x - 1)^2$

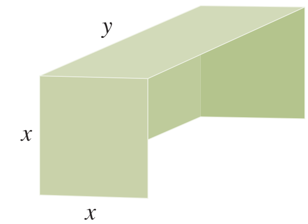
21–24 ■ Use a graphing device to graph the polynomial. Find the x- and y-intercepts and the coordinates of all local extrema, correct to the nearest decimal. Describe the end behavior of the polynomial.

21. $P(x) = x^3 - 4x + 1$
22. $P(x) = -2x^3 + 6x^2 - 2$
23. $P(x) = 3x^4 - 4x^3 - 10x - 1$
24. $P(x) = x^5 + x^4 - 7x^3 - x^2 + 6x + 3$

25. The strength S of a wooden beam of width x and depth y is given by the formula $S = 13.8xy^2$. A beam is to be cut from a log of diameter 10 in., as shown in the figure.
(a) Express the strength S of this beam as a function of x only.
(b) What is the domain of the function S?
(c) Draw a graph of S.
(d) What width will make the beam the strongest?



26. A small shelter for delicate plants is to be constructed of thin plastic material. It will have square ends and a rectangular top and back, with an open bottom and front, as shown in the figure. The total area of the four plastic sides is to be 1200 in^2.
(a) Express the volume V of the shelter as a function of the depth x.
(b) Draw a graph of V.
(c) What dimensions will maximize the volume of the shelter?



27–34 ■ Find the quotient and remainder.

27. $\dfrac{x^2 - 3x + 5}{x - 2}$
28. $\dfrac{x^2 + x - 12}{x - 3}$
29. $\dfrac{x^3 - x^2 + 11x + 2}{x - 4}$
30. $\dfrac{x^3 + 2x^2 - 10}{x + 3}$
31. $\dfrac{x^4 - 8x^2 + 2x + 7}{x + 5}$
32. $\dfrac{2x^4 + 3x^3 - 12}{x + 4}$
33. $\dfrac{2x^3 + x^2 - 8x + 15}{x^2 + 2x - 1}$
34. $\dfrac{x^4 - 2x^2 + 7x}{x^2 - x + 3}$

35–36 ■ Find the indicated value of the polynomial using the Remainder Theorem.

35. $P(x) = 2x^3 - 9x^2 - 7x + 13$; find $P(5)$
36. $Q(x) = x^4 + 4x^3 + 7x^2 + 10x + 15$; find $Q(-3)$

37. Show that $\frac{1}{2}$ is a zero of the polynomial

$$P(x) = 2x^4 + x^3 - 5x^2 + 10x - 4$$

38. Use the Factor Theorem to show that $x + 4$ is a factor of the polynomial

$$P(x) = x^5 + 4x^4 - 7x^3 - 23x^2 + 23x + 12$$

39. What is the remainder when the polynomial

$$P(x) = x^{500} + 6x^{201} - x^2 - 2x + 4$$

is divided by $x - 1$?

40. What is the remainder when $x^{101} - x^4 + 2$ is divided by $x + 1$?

41–42 ■ A polynomial P is given. **(a)** List all possible rational zeros (without testing to see whether they actually are zeros). **(b)** Determine the possible number of positive and negative real zeros using Descartes' Rule of Signs.

41. $P(x) = x^5 - 6x^3 - x^2 + 2x + 18$
42. $P(x) = 6x^4 + 3x^3 + x^2 + 3x + 4$

43–50 ■ A polynomial P is given. **(a)** Find all real zeros of P, and state their multiplicities. **(b)** Sketch the graph of P.

43. $P(x) = x^3 - 16x$
44. $P(x) = x^3 - 3x^2 - 4x$
45. $P(x) = x^4 + x^3 - 2x^2$
46. $P(x) = x^4 - 5x^2 + 4$

47. $P(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$

48. $P(x) = x^4 - 2x^3 - 2x^2 + 8x - 8$

49. $P(x) = 2x^4 + x^3 + 2x^2 - 3x - 2$

50. $P(x) = 9x^5 - 21x^4 + 10x^3 + 6x^2 - 3x - 1$

51–60 ■ Evaluate the expression and write in the form $a + bi$.

51. $(2 - 3i) + (1 + 4i)$ **52.** $(3 - 6i) - (6 - 4i)$

53. $(2 + i)(3 - 2i)$ **54.** $4i(2 - \frac{1}{2}i)$

55. $\dfrac{4 + 2i}{2 - i}$ **56.** $\dfrac{8 + 3i}{4 + 3i}$

57. i^{25} **58.** $(1 + i)^3$

59. $(1 - \sqrt{-1})(1 + \sqrt{-1})$ **60.** $\sqrt{-10} \cdot \sqrt{-40}$

61. Find a polynomial of degree 3 with constant coefficient 12 and zeros $-\frac{1}{2}$, 2, and 3.

62. Find a polynomial of degree 4 that has integer coefficients and zeros $3i$ and 4, with 4 a double zero.

63. Does there exist a polynomial of degree 4 with integer coefficients that has zeros i, $2i$, $3i$, and $4i$? If so, find it. If not, explain why.

64. Prove that the equation $3x^4 + 5x^2 + 2 = 0$ has no real root.

65–76 ■ Find all rational, irrational, and complex zeros (and state their multiplicities). Use Descartes' Rule of Signs, the Upper and Lower Bounds Theorem, the Quadratic Formula, or other factoring techniques to help you whenever possible.

65. $P(x) = x^3 - x^2 + x - 1$ **66.** $P(x) = x^3 - 8$

67. $P(x) = x^3 - 3x^2 - 13x + 15$

68. $P(x) = 2x^3 + 5x^2 - 6x - 9$

69. $P(x) = x^4 + 6x^3 + 17x^2 + 28x + 20$

70. $P(x) = x^4 + 7x^3 + 9x^2 - 17x - 20$

71. $P(x) = x^5 - 3x^4 - x^3 + 11x^2 - 12x + 4$

72. $P(x) = x^4 - 81$

73. $P(x) = x^6 - 64$

74. $P(x) = 18x^3 + 3x^2 - 4x - 1$

75. $P(x) = 6x^4 - 18x^3 + 6x^2 - 30x + 36$

76. $P(x) = x^4 + 15x^2 + 54$

77–80 ■ Use a graphing device to find all real solutions of the equation.

77. $2x^2 = 5x + 3$

78. $x^3 + x^2 - 14x - 24 = 0$

79. $x^4 - 3x^3 - 3x^2 - 9x - 2 = 0$

80. $x^5 = x + 3$

81–82 ■ A polynomial function P is given. Find all the real zeros of P, and factor P completely into linear and irreducible quadratic factors with real coefficients.

81. $P(x) = x^3 - 2x - 4$ **82.** $P(x) = x^4 + 3x^2 - 4$

83–86 ■ A rational function is given. **(a)** Find all vertical and horizontal asymptotes, all x- and y-intercepts, and state the domain and range. **(b)** Use transformations of the graph of $y = 1/x$ to sketch a graph of the rational function.

83. $r(x) = \dfrac{3}{x + 4}$ **84.** $r(x) = \dfrac{-1}{x - 5}$

85. $r(x) = \dfrac{3x - 4}{x - 1}$ **86.** $r(x) = \dfrac{2x + 5}{x + 2}$

87–92 ■ Graph the rational function. Show clearly all x- and y-intercepts and asymptotes.

87. $r(x) = \dfrac{3x - 12}{x + 1}$ **88.** $r(x) = \dfrac{1}{(x + 2)^2}$

89. $r(x) = \dfrac{x - 2}{x^2 - 2x - 8}$ **90.** $r(x) = \dfrac{2x^2 - 6x - 7}{x - 4}$

91. $r(x) = \dfrac{x^2 - 9}{2x^2 + 1}$ **92.** $r(x) = \dfrac{x^3 + 27}{x + 4}$

93–96 ■ Use a graphing device to analyze the graph of the rational function. Find all x- and y-intercepts and all vertical, horizontal, and slant asymptotes. If the function has no horizontal or slant asymptote, find a polynomial that has the same end behavior as the rational function.

93. $r(x) = \dfrac{x - 3}{2x + 6}$ **94.** $r(x) = \dfrac{2x - 7}{x^2 + 9}$

95. $r(x) = \dfrac{x^3 + 8}{x^2 - x - 2}$ **96.** $r(x) = \dfrac{2x^3 - x^2}{x + 1}$

97. Find the coordinates of all points of intersection of the graphs of

$$y = x^4 + x^2 + 24x \quad \text{and} \quad y = 6x^3 + 20$$

98. Suppose that M varies directly as z and that $M = 120$ when $z = 15$. Write an equation that expresses this variation.

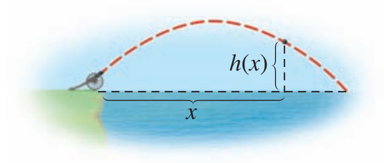
99. Suppose that z is inversely proportional to y and that $z = 12$ when $y = 16$. Write an equation that expresses z in terms of y.

100. The intensity of illumination I from a light varies inversely as the square of the distance d from the light.
(a) Write this statement as an equation.
(b) Determine the constant of proportionality if it is known that a lamp has an intensity of 1000 candles at a distance of 8 m.
(c) What is the intensity of this lamp at a distance of 20 m?

101. The frequency of a vibrating string under constant tension is inversely proportional to its length. If a violin string 12 inches long vibrates 440 times per second, to what length must it be shortened to vibrate 660 times per second?

102. The terminal velocity of a parachutist is directly proportional to the square root of his weight. A 160-lb parachutist attains a terminal velocity of 9 mi/h. What is the terminal velocity for a parachutist weighing 240 lb?

103. The maximum range of a projectile is directly proportional to the square of its velocity. A baseball pitcher throws a ball at 60 mi/h, with a maximum range of 242 ft. What is his maximum range if he throws the ball at 70 mi/h?



1. Express the quadratic function $f(x) = x^2 - x - 6$ in standard form, and sketch its graph.

2. Find the maximum or minimum value of the quadratic function $g(x) = 2x^2 + 6x + 3$.

3. A cannonball fired out to sea from a shore battery follows a parabolic trajectory given by the graph of the equation

$$h(x) = 10x - 0.01x^2$$

where $h(x)$ is the height of the cannonball above the water when it has traveled a horizontal distance of x feet.

(a) What is the maximum height that the cannonball reaches?

(b) How far does the cannonball travel horizontally before splashing into the water?

4. Graph the polynomial $P(x) = -(x + 2)^3 + 27$, showing clearly all x- and y-intercepts.

5. **(a)** Use synthetic division to find the quotient and remainder when $x^4 - 4x^2 + 2x + 5$ is divided by $x - 2$.

(b) Use long division to find the quotient and remainder when $2x^5 + 4x^4 - x^3 - x^2 + 7$ is divided by $2x^2 - 1$.

6. Let $P(x) = 2x^3 - 5x^2 - 4x + 3$.

(a) List all possible rational zeros of P.

(b) Find the complete factorization of P.

(c) Find the zeros of P.

(d) Sketch the graph of P.

7. Perform the indicated operations, and write the result in the form $a + bi$.

(a) $(3 - 2i) + (4 + 3i)$ **(b)** $(3 - 2i) - (4 + 3i)$

(c) $(3 - 2i)(4 + 3i)$ **(d)** $\dfrac{3 - 2i}{4 + 3i}$

(e) i^{48} **(f)** $(\sqrt{2} - \sqrt{-2})(\sqrt{8} + \sqrt{-2})$

8. Find all real and complex zeros of $P(x) = x^3 - x^2 - 4x - 6$.

9. Find the complete factorization of $P(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$.

10. Find a fourth-degree polynomial with integer coefficients that has zeros $3i$ and -1, with -1 a zero of multiplicity 2.

11. Let $P(x) = 2x^4 - 7x^3 + x^2 - 18x + 3$.

(a) Use Descartes' Rule of Signs to determine how many positive and how many negative real zeros P can have.

(b) Show that 4 is an upper bound and -1 is a lower bound for the real zeros of P.

(c) Draw a graph of P, and use it to estimate the real zeros of P, correct to two decimal places.

(d) Find the coordinates of all local extrema of P, correct to two decimals.

12. Consider the following rational functions:

$$r(x) = \frac{2x - 1}{x^2 - x - 2} \qquad s(x) = \frac{x^3 + 27}{x^2 + 4} \qquad t(x) = \frac{x^3 - 9x}{x + 2} \qquad u(x) = \frac{x^2 + x - 6}{x^2 - 25}$$

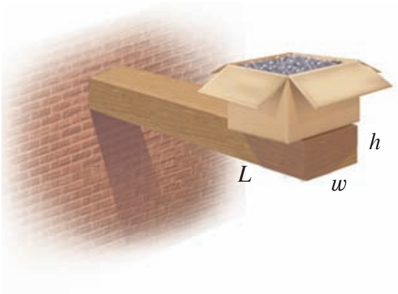
(a) Which of these rational functions has a horizontal asymptote?

(b) Which of these functions has a slant asymptote?

(c) Which of these functions has no vertical asymptote?

(d) Graph $y = u(x)$, showing clearly any asymptotes and x- and y-intercepts the function may have.

(e) Use long division to find a polynomial P that has the same end behavior as t. Graph both P and t on the same screen to verify that they have the same end behavior.



13. The maximum weight M that can be supported by a beam is jointly proportional to its width w and the square of its height h, and inversely proportional to its length L.

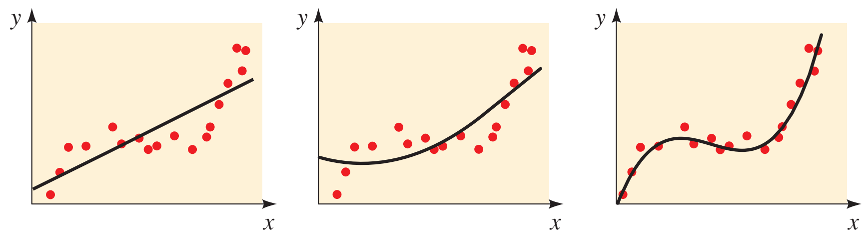
(a) Write an equation that expresses this proportionality.

(b) Determine the constant of proportionality if a beam 4 in. wide, 6 in. high, and 12 ft long can support a weight of 4800 lb.

(c) If a 10-ft beam made of the same material is 3 in. wide and 10 in. high, what is the maximum weight it can support?

Fitting Polynomial Curves to Data

We have learned how to fit a line to data (see *Focus on Modeling*, page 162). The line models the increasing or decreasing trend in the data. If the data exhibit more variability, such as an increase followed by a decrease, then to model the data, we need to use a curve rather than a line. Figure 1 shows a scatter plot with three possible models that appear to fit the data. Which model fits the data best?

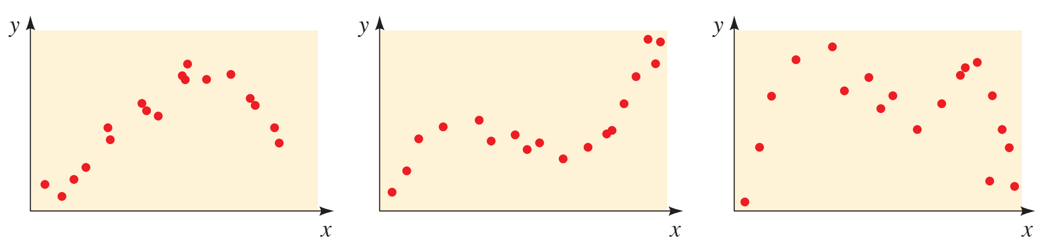


Linear model   Quadratic model   Cubic model

FIGURE 1

Polynomial Functions as Models

Polynomial functions are ideal for modeling data for which the scatter plot has peaks or valleys (that is, local maxima or minima). For example, if the data have a single peak as in Figure 2(a), then it may be appropriate to use a quadratic polynomial to model the data. The more peaks or valleys the data exhibit, the higher the degree of the polynomial needed to model the data (see Figure 2).



(a)   (b)   (c)

FIGURE 2

Graphing calculators are programmed to find the **polynomial of best fit** of a specified degree. As is the case for lines (see page 163), a polynomial of a given degree fits the data *best* if the sum of the squares of the distances between the graph of the polynomial and the data points is minimized.

EXAMPLE 1 | Rainfall and Crop Yield

Rain is essential for crops to grow, but too much rain can diminish crop yields. The data give rainfall and cotton yield per acre for several seasons in a certain county.

(a) Make a scatter plot of the data. What degree polynomial seems appropriate for modeling the data?

(b) Use a graphing calculator to find the polynomial of best fit. Graph the polynomial on the scatter plot.

(c) Use the model that you found to estimate the yield if there are 25 in. of rainfall.

Ted Wood/The Image Bank/Getty Images

Season	Rainfall (in.)	Yield (kg/acre)
1	23.3	5311
2	20.1	4382
3	18.1	3950
4	12.5	3137
5	30.9	5113
6	33.6	4814
7	35.8	3540
8	15.5	3850
9	27.6	5071
10	34.5	3881

SOLUTION

(a) The scatter plot is shown in Figure 3. The data appear to have a peak, so it is appropriate to model the data by a quadratic polynomial (degree 2).

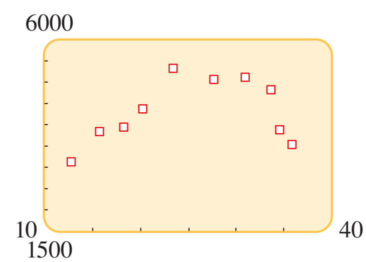
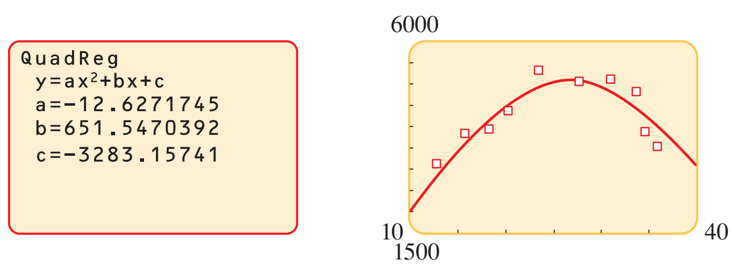


FIGURE 3 Scatter plot of yield vs. rainfall data

(b) Using a graphing calculator, we find that the quadratic polynomial of best fit is

$$y = -12.6x^2 + 651.5x - 3283.2$$

The calculator output and the scatter plot, together with the graph of the quadratic model, are shown in Figure 4.



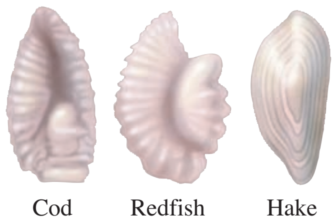
(a) (b)

FIGURE 4

(c) Using the model with $x = 25$, we get

$$y = -12.6(25)^2 + 651.5(25) - 3283.2 \approx 5129.3$$

We estimate the yield to be about 5130 kg/acre.



Otoliths for several fish species

EXAMPLE 2 | Length-at-Age Data for Fish

Otoliths ("earstones") are tiny structures that are found in the heads of fish. Microscopic growth rings on the otoliths, not unlike growth rings on a tree, record the age of a fish. The table gives the lengths of rock bass caught at different ages, as determined by the otoliths. Scientists have proposed a cubic polynomial to model this data.

(a) Use a graphing calculator to find the cubic polynomial of best fit for the data.

(b) Make a scatter plot of the data, and graph the polynomial from part (a).

(c) A fisherman catches a rock bass 20 in. long. Use the model to estimate its age.

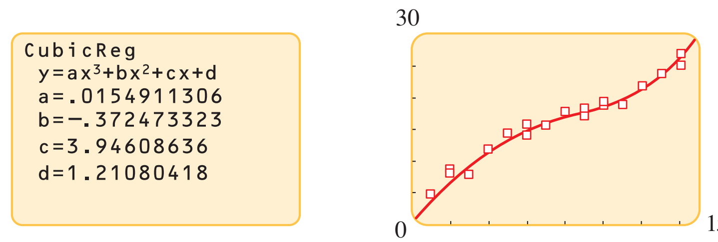
Age (yr)	Length (in.)	Age (yr)	Length (in.)
1	4.8	9	18.2
2	8.8	9	17.1
2	8.0	10	18.8
3	7.9	10	19.5
4	11.9	11	18.9
5	14.4	12	21.7
6	14.1	12	21.9
6	15.8	13	23.8
7	15.6	14	26.9
8	17.8	14	25.1

SOLUTION

(a) Using a graphing calculator (see Figure 5(a)), we find the cubic polynomial of best fit:

$$y = 0.0155x^3 - 0.372x^2 + 3.95x + 1.21$$

(b) The scatter plot of the data and the cubic polynomial are graphed in Figure 5(b).



(a) (b)

FIGURE 5

(c) Moving the cursor along the graph of the polynomial, we find that $y = 20$ when $x \approx 10.8$. Thus the fish is about 11 years old. ■

PROBLEMS

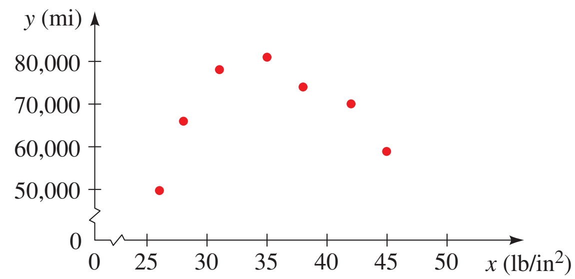
1. **Tire Inflation and Treadwear** Car tires need to be inflated properly. Overinflation or underinflation can cause premature treadwear. The data and scatter plot on the next page show tire life for different inflation values for a certain type of tire.

   (a) Find the quadratic polynomial that best fits the data.

   (b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

   (c) Use your result from part (b) to estimate the pressure that gives the longest tire life.

Pressure (lb/in^2)	Tire life (mi)
26	50,000
28	66,000
31	78,000
35	81,000
38	74,000
42	70,000
45	59,000



2. Too Many Corn Plants per Acre? The more corn a farmer plants per acre, the greater is the yield the farmer can expect, but only up to a point. Too many plants per acre can cause overcrowding and decrease yields. The data give crop yields per acre for various densities of corn plantings, as found by researchers at a university test farm.

(a) Find the quadratic polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Use your result from part (b) to estimate the yield for 37,000 plants per acre.

Density (plants/acre)	Crop yield (bushels/acre)
15,000	43
20,000	98
25,000	118
30,000	140
35,000	142
40,000	122
45,000	93
50,000	67

3. How Fast Can You List Your Favorite Things? If you are asked to make a list of objects in a certain category, how fast you can list them follows a predictable pattern. For example, if you try to name as many vegetables as you can, you'll probably think of several right away—for example, carrots, peas, beans, corn, and so on. Then after a pause you might think of ones you eat less frequently—perhaps zucchini, eggplant, and asparagus. Finally, a few more exotic vegetables might come to mind—artichokes, jicama, bok choy, and the like. A psychologist performs this experiment on a number of subjects. The table below gives the average number of vegetables that the subjects named by a given number of seconds.

(a) Find the cubic polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Use your result from part (b) to estimate the number of vegetables that subjects would be able to name in 40 seconds.

(d) According to the model, how long (to the nearest 0.1 second) would it take a person to name five vegetables?

Seconds	Number of vegetables
1	2
2	6
5	10
10	12
15	14
20	15
25	18
30	21

4. Clothing Sales Are Seasonal Clothing sales tend to vary by season, with more clothes sold in spring and fall. The table gives sales figures for each month at a certain clothing store.

(a) Find the quartic (fourth-degree) polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Do you think that a quartic polynomial is a good model for these data? Explain.

Month	Sales ($)
January	8,000
February	18,000
March	22,000
April	31,000
May	29,000
June	21,000
July	22,000
August	26,000
September	38,000
October	40,000
November	27,000
December	15,000

5. Height of a Baseball A baseball is thrown upward, and its height measured at 0.5-second intervals using a strobe light. The resulting data are given in the table.

(a) Draw a scatter plot of the data. What degree polynomial is appropriate for modeling the data?

(b) Find a polynomial model that best fits the data, and graph it on the scatter plot.

(c) Find the times when the ball is 20 ft above the ground.

(d) What is the maximum height attained by the ball?

Time (s)	Height (ft)
0	4.2
0.5	26.1
1.0	40.1
1.5	46.0
2.0	43.9
2.5	33.7
3.0	15.8

6. Torricelli's Law Water in a tank will flow out of a small hole in the bottom faster when the tank is nearly full than when it is nearly empty. According to Torricelli's Law, the height $h(t)$ of water remaining at time t is a quadratic function of t.

A certain tank is filled with water and allowed to drain. The height of the water is measured at different times as shown in the table.

(a) Find the quadratic polynomial that best fits the data.

(b) Draw a graph of the polynomial from part (a) together with a scatter plot of the data.

(c) Use your graph from part (b) to estimate how long it takes for the tank to drain completely.

Time (min)	Height (ft)
0	5.0
4	3.1
8	1.9
12	0.8
16	0.2

Andrew F Kazmierski/Shutterstock.com

CHAPTER 4

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Modeling Growth and Decay In this chapter we study *exponential functions*. These are functions like $f(x) = 2^x$, where the independent variable is in the exponent. Exponential functions are used in modeling many real-world phenomena, such as the growth of a population, the growth of an investment that earns compound interest, or the decay of a radioactive substance. Once an exponential model has been obtained, we can use the model to predict the size of a population, calculate the amount of an investment, or find the amount of a radioactive substance that remains. To find out *when* a population will grow to a certain number, *when* an investment will reach a certain amount, or *when* a radioactive substance will be reduced to a certain level, we use the inverse functions of exponential functions, called *logarithmic functions*. With exponential models and logarithmic functions we can answer questions such as these: When will my city be as crowded as the New York City street pictured here? When will my bank account have a million dollars? When will radiation from a radioactive spill decay to a safe level?

4.1 Exponential Functions

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Evaluate exponential functions ▶ Graph exponential functions ▶ Calculate compound interest

GET READY **Prepare for this section by reviewing the properties of exponents in Sections P.3 and P.4.**

In this chapter we study a new class of functions called *exponential functions*. For example,

$$f(x) = 2^x$$

is an exponential function (with base 2). Notice how quickly the values of this function increase:

$$\begin{aligned} f(3) &= 2^3 = 8 \\ f(10) &= 2^{10} = 1024 \\ f(30) &= 2^{30} = 1{,}073{,}741{,}824 \end{aligned}$$

Compare this with the function $g(x) = x^2$, where $g(30) = 30^2 = 900$. The point is that when the variable is in the exponent, even a small change in the variable can cause a dramatic change in the value of the function.

▼ Exponential Functions

To study exponential functions, we must first define what we mean by the exponential expression a^x when x is any real number. In Section P.4 we defined a^x for $a > 0$ and x a rational number, but we have not yet defined irrational powers. So what is meant by $5^{\sqrt{3}}$ or 2^{π}? To define a^x when x is irrational, we approximate x by rational numbers.

For example, since

$$\sqrt{3} \approx 1.73205\ldots$$

is an irrational number, we successively approximate $a^{\sqrt{3}}$ by the following rational powers:

$$a^{1.7}, a^{1.73}, a^{1.732}, a^{1.7320}, a^{1.73205}, \ldots$$

Intuitively, we can see that these rational powers of a are getting closer and closer to $a^{\sqrt{3}}$. It can be shown by using advanced mathematics that there is exactly one number that these powers approach. We define $a^{\sqrt{3}}$ to be this number.

For example, using a calculator, we find

$$\begin{aligned} 5^{\sqrt{3}} &\approx 5^{1.732} \\ &\approx 16.2411\ldots \end{aligned}$$

The more decimal places of $\sqrt{3}$ we use in our calculation, the better our approximation of $5^{\sqrt{3}}$.

The Laws of Exponents are listed on page 19.

It can be proved that the *Laws of Exponents are still true when the exponents are real numbers*.

> **EXPONENTIAL FUNCTIONS**
>
> The **exponential function with base *a*** is defined for all real numbers x by
>
> $$f(x) = a^x$$
>
> where $a > 0$ and $a \neq 1$.

We assume that $a \neq 1$ because the function $f(x) = 1^x = 1$ is just a constant function. Here are some examples of exponential functions:

$$\underbrace{f(x) = 2^x}_{\text{Base 2}} \qquad \underbrace{g(x) = 3^x}_{\text{Base 3}} \qquad \underbrace{h(x) = 10^x}_{\text{Base 10}}$$

EXAMPLE 1 | Evaluating Exponential Functions

Let $f(x) = 3^x$, and evaluate the following:

(a) $f(5)$ **(b)** $f(-\frac{2}{3})$

(c) $f(\pi)$ **(d)** $f(\sqrt{2})$

SOLUTION We use a calculator to obtain the values of f.

	Calculator keystrokes	Output
(a) $f(5) = 3^5 = 243$	3 ^ 5 ENTER	243
(b) $f(-\frac{2}{3}) = 3^{-2/3} \approx 0.4807$	3 ^ ((−) 2 ÷ 3) ENTER	0.4807498
(c) $f(\pi) = 3^\pi \approx 31.544$	3 ^ π ENTER	31.5442807
(d) $f(\sqrt{2}) = 3^{\sqrt{2}} \approx 4.7288$	3 ^ √ 2 ENTER	4.7288043

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5**

Graphs of Exponential Functions

We first graph exponential functions by plotting points. We will see that the graphs of such functions have an easily recognizable shape.

EXAMPLE 2 | Graphing Exponential Functions by Plotting Points

Draw the graph of each function.

(a) $f(x) = 3^x$ **(b)** $g(x) = \left(\frac{1}{3}\right)^x$

SOLUTION We calculate values of $f(x)$ and $g(x)$ and plot points to sketch the graphs in Figure 1.

x	$f(x) = 3^x$	$g(x) = \left(\frac{1}{3}\right)^x$
−3	$\frac{1}{27}$	27
−2	$\frac{1}{9}$	9
−1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$

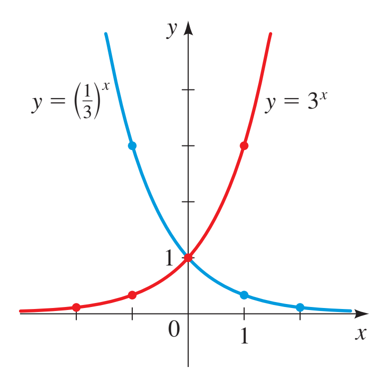


FIGURE 1

Notice that

$$g(x) = \left(\frac{1}{3}\right)^x = \frac{1}{3^x} = 3^{-x} = f(-x)$$

Reflecting graphs is explained in Section 2.5.

so we could have obtained the graph of g from the graph of f by reflecting in the y-axis.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **15**

To see just how quickly $f(x) = 2^x$ increases, let's perform the following thought experiment. Suppose we start with a piece of paper that is a thousandth of an inch thick, and we fold it in half 50 times. Each time we fold the paper, the thickness of the paper stack doubles, so the thickness of the resulting stack would be $2^{50}/1000$ inches. How thick do you think that is? It works out to be more than 17 million miles!

Figure 2 shows the graphs of the family of exponential functions $f(x) = a^x$ for various values of the base a. All of these graphs pass through the point $(0, 1)$ because $a^0 = 1$ for $a \neq 0$. You can see from Figure 2 that there are two kinds of exponential functions: If $0 < a < 1$, the exponential function decreases rapidly. If $a > 1$, the function increases rapidly (see the margin note).

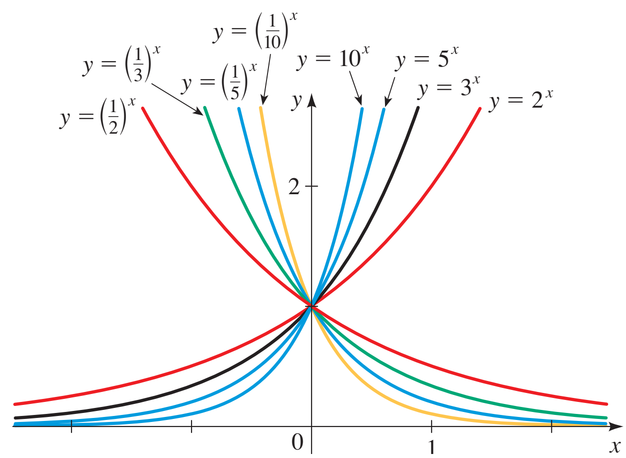


FIGURE 2 A family of exponential functions

See Section 3.7, page 312, where the "arrow notation" used here is explained.

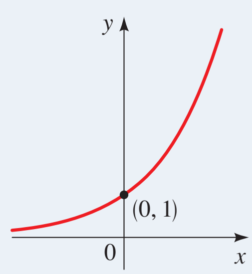
The x-axis is a horizontal asymptote for the exponential function $f(x) = a^x$. This is because when $a > 1$, we have $a^x \to 0$ as $x \to -\infty$, and when $0 < a < 1$, we have $a^x \to 0$ as $x \to \infty$ (see Figure 2). Also, $a^x > 0$ for all $x \in \mathbb{R}$, so the function $f(x) = a^x$ has domain $\mathbb{R}$ and range $(0, \infty)$. These observations are summarized in the following box.

GRAPHS OF EXPONENTIAL FUNCTIONS

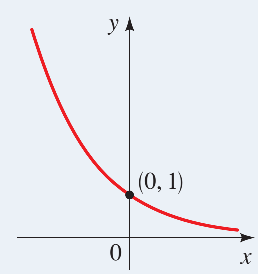
The exponential function

$$f(x) = a^x \qquad (a > 0,\ a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The line $y = 0$ (the x-axis) is a horizontal asymptote of f. The graph of f has one of the following shapes.



$f(x) = a^x$ for $a > 1$

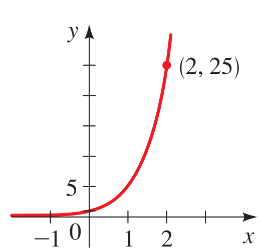


$f(x) = a^x$ for $0 < a < 1$

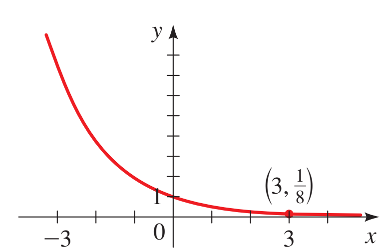
EXAMPLE 3 | Identifying Graphs of Exponential Functions

Find the exponential function $f(x) = a^x$ whose graph is given.

(a)



(b)



SOLUTION

(a) Since $f(2) = a^2 = 25$, we see that the base is $a = 5$. So $f(x) = 5^x$.

(b) Since $f(3) = a^3 = \frac{1}{8}$, we see that the base is $a = \frac{1}{2}$. So $f(x) = \left(\frac{1}{2}\right)^x$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **19**

In the next example we see how to graph certain functions, not by plotting points, but by taking the basic graphs of the exponential functions in Figure 2 and applying the shifting and reflecting transformations of Section 2.5.

EXAMPLE 4 | Transformations of Exponential Functions

Use the graph of $f(x) = 2^x$ to sketch the graph of each function.

(a) $g(x) = 1 + 2^x$ **(b)** $h(x) = -2^x$ **(c)** $k(x) = 2^{x-1}$

SOLUTION

Shifting and reflecting of graphs are explained in Section 2.5.

(a) To obtain the graph of $g(x) = 1 + 2^x$, we start with the graph of $f(x) = 2^x$ and shift it upward 1 unit. Notice from Figure 3(a) that the line $y = 1$ is now a horizontal asymptote.

(b) Again we start with the graph of $f(x) = 2^x$, but here we reflect in the x-axis to get the graph of $h(x) = -2^x$ shown in Figure 3(b).

(c) This time we start with the graph of $f(x) = 2^x$ and shift it to the right by 1 unit to get the graph of $k(x) = 2^{x-1}$ shown in Figure 3(c).

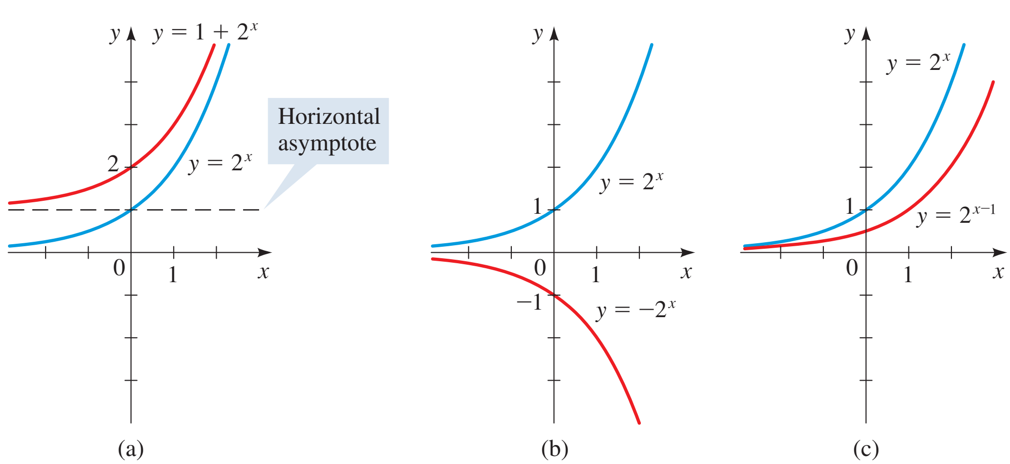


FIGURE 3

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **25**, **27**, AND **29**

EXAMPLE 5 | Comparing Exponential and Power Functions

Compare the rates of growth of the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$ by drawing the graphs of both functions in the following viewing rectangles.

(a) $[0, 3]$ by $[0, 8]$

(b) $[0, 6]$ by $[0, 25]$

(c) $[0, 20]$ by $[0, 1000]$

SOLUTION

(a) Figure 4(a) shows that the graph of $g(x) = x^2$ catches up with, and becomes higher than, the graph of $f(x) = 2^x$ at $x = 2$.

(b) The larger viewing rectangle in Figure 4(b) shows that the graph of $f(x) = 2^x$ overtakes that of $g(x) = x^2$ when $x = 4$.

(c) Figure 4(c) gives a more global view and shows that when x is large, $f(x) = 2^x$ is much larger than $g(x) = x^2$.

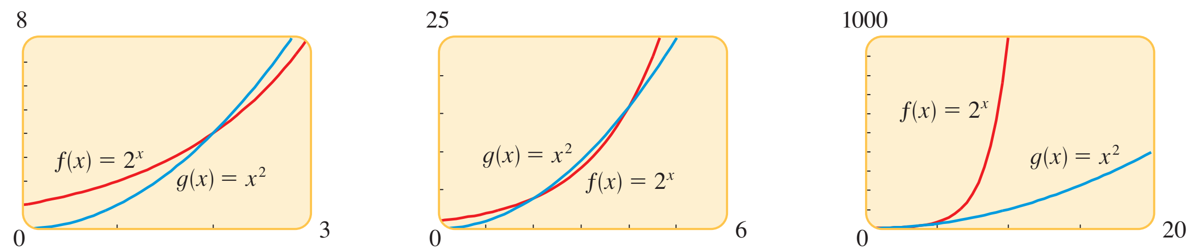


FIGURE 4 (a) (b) (c)

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41**

Compound Interest

Exponential functions occur in calculating compound interest. If an amount of money P, called the **principal**, is invested at an interest rate i per time period, then after one time period the interest is Pi, and the amount A of money is

$$A = P + Pi = P(1 + i)$$

If the interest is reinvested, then the new principal is $P(1 + i)$, and the amount after another time period is $A = P(1 + i)(1 + i) = P(1 + i)^2$. Similarly, after a third time period the amount is $A = P(1 + i)^3$. In general, after k periods the amount is

$$A = P(1 + i)^k$$

Notice that this is an exponential function with base $1 + i$.

If the annual interest rate is r and if interest is compounded n times per year, then in each time period the interest rate is $i = r/n$, and there are nt time periods in t years. This leads to the following formula for the amount after t years.

COMPOUND INTEREST

Compound interest is calculated by the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where $A(t) =$ amount after t years
$P =$ principal
$r =$ interest rate per year
$n =$ number of times interest is compounded per year
$t =$ number of years

r is often referred to as the nominal annual interest rate.

EXAMPLE 6 | Calculating Compound Interest

A sum of \$1000 is invested at an interest rate of 12% per year. Find the amounts in the account after 3 years if interest is compounded annually, semiannually, quarterly, monthly, and daily.

SOLUTION We use the compound interest formula with $P = \$1000$, $r = 0.12$, and $t = 3$.

Compounding	n	Amount after 3 years
Annual	1	$1000\left(1 + \frac{0.12}{1}\right)^{1(3)} = \$1404.93$
Semiannual	2	$1000\left(1 + \frac{0.12}{2}\right)^{2(3)} = \$1418.52$
Quarterly	4	$1000\left(1 + \frac{0.12}{4}\right)^{4(3)} = \$1425.76$
Monthly	12	$1000\left(1 + \frac{0.12}{12}\right)^{12(3)} = \$1430.77$
Daily	365	$1000\left(1 + \frac{0.12}{365}\right)^{365(3)} = \$1433.24$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **51** ■

If an investment earns compound interest, then the **annual percentage yield** (APY) is the *simple* interest rate that yields the same amount at the end of one year.

EXAMPLE 7 | Calculating the Annual Percentage Yield

Find the annual percentage yield for an investment that earns interest at a rate of 6% per year, compounded daily.

SOLUTION After one year, a principal P will grow to the amount

$$A = P\left(1 + \frac{0.06}{365}\right)^{365} = P(1.06183)$$

Simple interest is studied in Section 1.5.

The formula for simple interest is

$$A = P(1 + r)$$

Comparing, we see that $1 + r = 1.06183$, so $r = 0.06183$. Thus the annual percentage yield is 6.183%.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **57** ■

4.1 EXERCISES

CONCEPTS

1. The function $f(x) = 5^x$ is an exponential function with base ________; $f(-2) =$ ________, $f(0) =$ ________, $f(2) =$ ________, and $f(6) =$ ________.

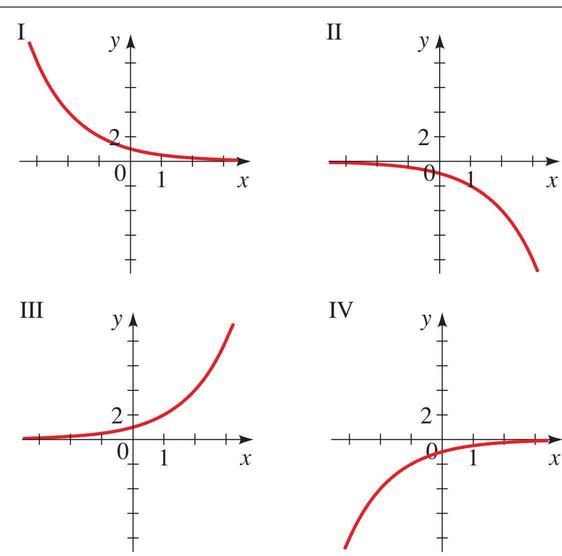
2. Match the exponential function with one of the graphs labeled I, II, III, or IV, shown on the right.

(a) $f(x) = 2^x$

(b) $f(x) = 2^{-x}$

(c) $f(x) = -2^x$

(d) $f(x) = -2^{-x}$



3. **(a)** To obtain the graph of $g(x) = 2^x - 1$, we start with the graph of $f(x) = 2^x$ and shift it ____________ (upward/downward) 1 unit.
   (b) To obtain the graph of $h(x) = 2^{x-1}$, we start with the graph of $f(x) = 2^x$ and shift it to the ____________ (left/right) 1 unit.

4. In the formula $A(t) = P(1 + \frac{r}{n})^{nt}$ for compound interest the letters P, r, n, and t stand for ____________, ____________, ____________, and ____________, respectively, and $A(t)$ stands for ____________. So if \$100 is invested at an interest rate of 6% compounded quarterly, then the amount after 2 years is ____________.

SKILLS

5–8 ■ Use a calculator to evaluate the function at the indicated values. Round your answers to three decimals.

5. $f(x) = 4^x$; $f(0.5), f(\sqrt{2}), f(-1), f(\frac{1}{3})$

6. $f(x) = 3^{x+1}$; $f(-1.5), f(\sqrt{3}), f(3.12), f(-\frac{5}{4})$

7. $g(x) = (\frac{2}{3})^{x-1}$; $g(1.3), g(\sqrt{5}), g(2\pi), g(-\frac{1}{2})$

8. $g(x) = (\frac{3}{4})^{2x}$; $g(0.7), g(\sqrt{7}/2), g(1/\pi), g(\frac{2}{3})$

9–14 ■ Sketch the graph of the function by making a table of values. Use a calculator if necessary.

9. $f(x) = 2^x$ **10.** $g(x) = 8^x$

11. $f(x) = (\frac{1}{3})^x$ **12.** $h(x) = (1.1)^x$

13. $g(x) = 3(1.3)^x$ **14.** $h(x) = 2(\frac{1}{4})^x$

15–18 ■ Graph both functions on one set of axes.

15. $f(x) = 2^x$ and $g(x) = 2^{-x}$

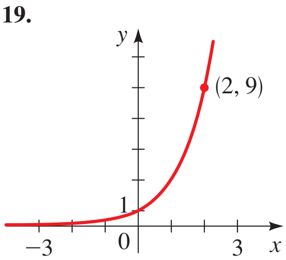
16. $f(x) = 3^{-x}$ and $g(x) = (\frac{1}{3})^x$

17. $f(x) = 4^x$ and $g(x) = 7^x$

18. $f(x) = (\frac{2}{3})^x$ and $g(x) = (\frac{4}{3})^x$

19–22 ■ Find the exponential function $f(x) = a^x$ whose graph is given.

19.



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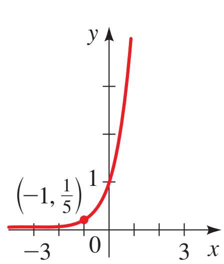

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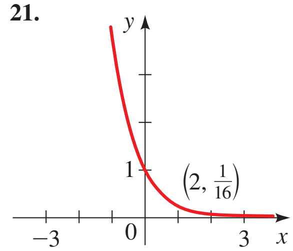

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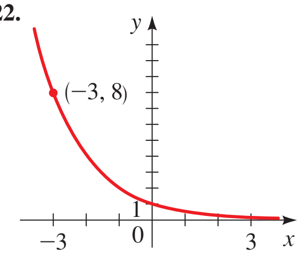
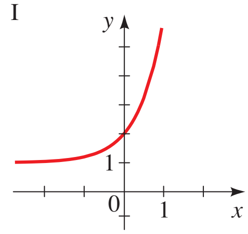


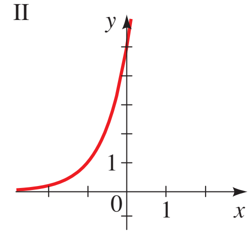
23–24 ■ Match the exponential function with one of the graphs labeled I or II.

23. $f(x) = 5^{x+1}$ **24.** $f(x) = 5^x + 1$

I



II



25–36 ■ Graph the function, not by plotting points, but by starting from the graphs in Figure 2. State the domain, range, and asymptote.

25. $g(x) = 2^x - 3$ **26.** $h(x) = 4 + (\frac{1}{2})^x$

27. $f(x) = -3^x$ **28.** $f(x) = 10^{-x}$

29. $f(x) = 10^{x+3}$ **30.** $g(x) = 2^{x-3}$

31. $y = 5^{-x} + 1$ **32.** $h(x) = 6 - 3^x$

33. $h(x) = 2^{x-4} + 1$ **34.** $y = 3 - 10^{x-1}$

35. $g(x) = 1 - 3^{-x}$ **36.** $y = 3 - (\frac{1}{5})^x$

37. **(a)** Sketch the graphs of $f(x) = 2^x$ and $g(x) = 3(2^x)$.
(b) How are the graphs related?

38. **(a)** Sketch the graphs of $f(x) = 9^{x/2}$ and $g(x) = 3^x$.
(b) Use the Laws of Exponents to explain the relationship between these graphs.

39. Compare the functions $f(x) = x^3$ and $g(x) = 3^x$ by evaluating both of them for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15$, and 20. Then draw the graphs of f and g on the same set of axes.

40. If $f(x) = 10^x$, show that $\dfrac{f(x+h) - f(x)}{h} = 10^x\left(\dfrac{10^h - 1}{h}\right)$.

41. **(a)** Compare the rates of growth of the functions $f(x) = 2^x$ and $g(x) = x^5$ by drawing the graphs of both functions in the following viewing rectangles.
(i) $[0, 5]$ by $[0, 20]$
(ii) $[0, 25]$ by $[0, 10^7]$
(iii) $[0, 50]$ by $[0, 10^8]$
(b) Find the solutions of the equation $2^x = x^5$, rounded to one decimal place.

42. **(a)** Compare the rates of growth of the functions $f(x) = 3^x$ and $g(x) = x^4$ by drawing the graphs of both functions in the following viewing rectangles:
(i) $[-4, 4]$ by $[0, 20]$
(ii) $[0, 10]$ by $[0, 5000]$
(iii) $[0, 20]$ by $[0, 10^5]$
(b) Find the solutions of the equation $3^x = x^4$, rounded to two decimal places.

43–44 ■ Draw graphs of the given family of functions for $c = 0.25, 0.5, 1, 2, 4$. How are the graphs related?

43. $f(x) = c2^x$ **44.** $f(x) = 2^{cx}$

45–46 ■ Find, rounded to two decimal places, **(a)** the intervals on which the function is increasing or decreasing and **(b)** the range of the function.

45. $y = 10^{x-x^2}$ **46.** $y = x2^x$

APPLICATIONS

47. Bacteria Growth A bacteria culture contains 1500 bacteria initially and doubles every hour.
(a) Find a function that models the number of bacteria after t hours.
(b) Find the number of bacteria after 24 hours.

48. Mouse Population A certain breed of mouse was introduced onto a small island with an initial population of 320 mice, and scientists estimate that the mouse population is doubling every year.
(a) Find a function that models the number of mice after t years.
(b) Estimate the mouse population after 8 years.

49–50 ■ **Compound Interest** An investment of $5000 is deposited into an account in which interest is compounded monthly. Complete the table by filling in the amounts to which the investment grows at the indicated times or interest rates.

49. $r = 4\%$ **50.** $t = 5$ years

Time (years)	Amount
1	
2	
3	
4	
5	
6	

Rate per year	Amount
1%	
2%	
3%	
4%	
5%	
6%	

51. Compound Interest If $10,000 is invested at an interest rate of 3% per year, compounded semiannually, find the value of the investment after the given number of years.
(a) 5 years **(b)** 10 years **(c)** 15 years

52. Compound Interest If $2500 is invested at an interest rate of 2.5% per year, compounded daily, find the value of the investment after the given number of years.
(a) 2 years **(b)** 3 years **(c)** 6 years

53. Compound Interest If $500 is invested at an interest rate of 3.75% per year, compounded quarterly, find the value of the investment after the given number of years.
(a) 1 year **(b)** 2 years **(c)** 10 years

54. Compound Interest If $4000 is borrowed at a rate of 5.75% interest per year, compounded quarterly, find the amount due at the end of the given number of years.
(a) 4 years **(b)** 6 years **(c)** 8 years

55–56 ■ **Present Value** The **present value** of a sum of money is the amount that must be invested now, at a given rate of interest, to produce the desired sum at a later date.

55. Find the present value of $10,000 if interest is paid at a rate of 9% per year, compounded semiannually, for 3 years.

56. Find the present value of $100,000 if interest is paid at a rate of 8% per year, compounded monthly, for 5 years.

57. Annual Percentage Yield Find the annual percentage yield for an investment that earns 8% per year, compounded monthly.

58. Annual Percentage Yield Find the annual percentage yield for an investment that earns $5\frac{1}{2}\%$ per year, compounded quarterly.

DISCOVERY ■ DISCUSSION ■ WRITING

59. Growth of an Exponential Function Suppose you are offered a job that lasts one month, and you are to be very well paid. Which of the following methods of payment is more profitable for you?
(a) One million dollars at the end of the month
(b) Two cents on the first day of the month, 4 cents on the second day, 8 cents on the third day, and, in general, 2^n cents on the nth day

60. The Height of the Graph of an Exponential Function Your mathematics instructor asks you to sketch a graph of the exponential function

$$f(x) = 2^x$$

for x between 0 and 40, using a scale of 10 units to one inch. What are the dimensions of the sheet of paper you will need to sketch this graph?

DISCOVERY PROJECT **Exponential Explosion**

In this project we explore an example about collecting pennies that helps us experience how exponential growth works. You can find the project at the book companion website: **www.stewartmath.com**

4.2 The Natural Exponential Function

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Evaluate the natural exponential function ▶ Graph the natural exponential function ▶ Calculate continuously compounded interest

Any positive number can be used as a base for an exponential function. In this section we study the special base e, which is convenient for applications involving calculus.

The **Gateway Arch** in St. Louis, Missouri, is shaped in the form of the graph of a combination of exponential functions (*not* a parabola, as it might first appear). Specifically, it is a **catenary**, which is the graph of an equation of the form

$$y = a(e^{bx} + e^{-bx})$$

(see Exercises 15 and 17). This shape was chosen because it is optimal for distributing the internal structural forces of the arch. Chains and cables suspended between two points (for example, the stretches of cable between pairs of telephone poles) hang in the shape of a catenary.

The Number e

The number e is defined as the value that $(1 + 1/n)^n$ approaches as n becomes large. (In calculus this idea is made more precise through the concept of a limit.) The table shows the values of the expression $(1 + 1/n)^n$ for increasingly large values of n.

n	$\left(1 + \frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

It appears that, rounded to five decimal places, $e \approx 2.71828$; in fact, the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

It can be shown that e is an irrational number, so we cannot write its exact value in decimal form.

The notation e was chosen by Leonhard Euler (see page 300), probably because it is the first letter of the word *exponential*.

The Natural Exponential Function

The number e is the base for the natural exponential function. Why use such a strange base for an exponential function? It might seem at first that a base such as 10 is easier to work with. We will see, however, that in certain applications the number e is the best possible base. In this section we study how e occurs in the description of compound interest.

THE NATURAL EXPONENTIAL FUNCTION

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

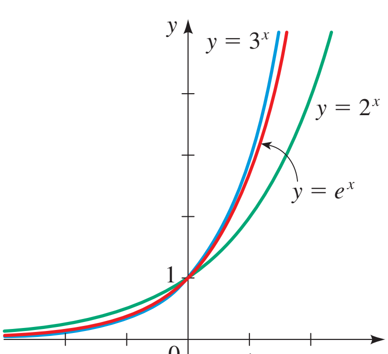


FIGURE 1 Graph of the natural exponential function

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$, as shown in Figure 1.

Scientific calculators have a special key for the function $f(x) = e^x$. We use this key in the next example.

EXAMPLE 1 | Evaluating the Exponential Function

Evaluate each expression rounded to five decimal places.

(a) e^3 **(b)** $2e^{-0.53}$ **(c)** $e^{4.8}$

SOLUTION We use the $\boxed{e^x}$ key on a calculator to evaluate the exponential function.

(a) $e^3 \approx 20.08554$ **(b)** $2e^{-0.53} \approx 1.17721$ **(c)** $e^{4.8} \approx 121.51042$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **3** ■

EXAMPLE 2 | Transformations of the Exponential Function

Sketch the graph of each function.

(a) $f(x) = e^{-x}$ **(b)** $g(x) = 3e^{0.5x}$

SOLUTION

(a) We start with the graph of $y = e^x$ and reflect in the y-axis to obtain the graph of $y = e^{-x}$ as in Figure 2.

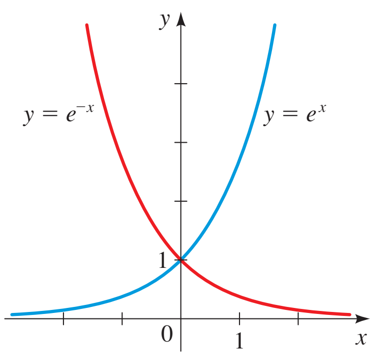


FIGURE 2

(b) We calculate several values, plot the resulting points, then connect the points with a smooth curve. The graph is shown in Figure 3.

x	$f(x) = 3e^{0.5x}$
−3	0.67
−2	1.10
−1	1.82
0	3.00
1	4.95
2	8.15
3	13.45

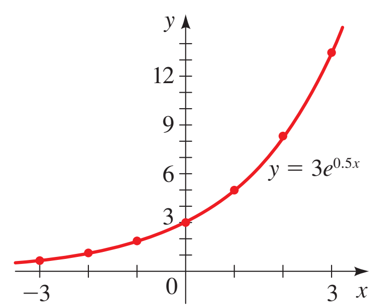


FIGURE 3

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **5** AND **7** ■

EXAMPLE 3 | An Exponential Model for the Spread of a Virus

An infectious disease begins to spread in a small city of population 10,000. After t days, the number of people who have succumbed to the virus is modeled by the function

$$v(t) = \frac{10{,}000}{5 + 1245e^{-0.97t}}$$

(a) How many infected people are there initially (at time $t = 0$)?

(b) Find the number of infected people after one day, two days, and five days.

(c) Graph the function v, and describe its behavior.

SOLUTION

(a) Since $v(0) = 10{,}000/(5 + 1245e^0) = 10{,}000/1250 = 8$, we conclude that 8 people initially have the disease.

(b) Using a calculator, we evaluate $v(1)$, $v(2)$, and $v(5)$ and then round off to obtain the following values.

Days	Infected people
1	21
2	54
5	678

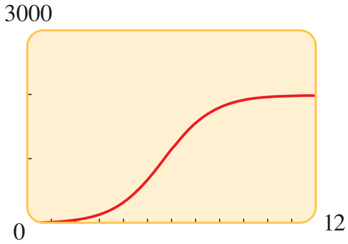


FIGURE 4

$$v(t) = \frac{10,000}{5 + 1245e^{-0.97t}}$$

(c) From the graph in Figure 4 we see that the number of infected people first rises slowly, then rises quickly between day 3 and day 8, and then levels off when about 2000 people are infected.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **25**

The graph in Figure 4 is called a *logistic curve* or a *logistic growth model*. Curves like it occur frequently in the study of population growth. (See Exercises 25–28.)

Continuously Compounded Interest

In Example 6 of Section 4.1 we saw that the interest paid increases as the number of compounding periods n increases. Let's see what happens as n increases indefinitely. If we let $m = n/r$, then

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$$

Recall that as m becomes large, the quantity $(1 + 1/m)^m$ approaches the number e. Thus the amount approaches $A = Pe^{rt}$. This expression gives the amount when the interest is compounded at "every instant."

CONTINUOUSLY COMPOUNDED INTEREST

Continuously compounded interest is calculated by the formula

$$A(t) = Pe^{rt}$$

where $A(t)$ = amount after t years
P = principal
r = interest rate per year
t = number of years

EXAMPLE 4 | Calculating Continuously Compounded Interest

Find the amount after 3 years if $1000 is invested at an interest rate of 12% per year, compounded continuously.

SOLUTION We use the formula for continuously compounded interest with $P = \$1000$, $r = 0.12$, and $t = 3$ to get

$$A(3) = 1000e^{(0.12)3} = 1000e^{0.36} = \$1433.33$$

Compare this amount with the amounts in Example 6 of Section 4.1.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **31**

4.2 EXERCISES

CONCEPTS

1. The function $f(x) = e^x$ is called the ________ exponential function. The number e is approximately equal to ________.

2. In the formula $A(t) = Pe^{rt}$ for continuously compound interest, the letters P, r, and t stand for ________, ________, and ________, respectively, and $A(t)$ stands for ________. So if $100 is invested at an interest rate of 6% compounded continuously, then the amount after 2 years is ________.

SKILLS

3–4 ■ Use a calculator to evaluate the function at the indicated values. Round your answers to three decimals.

3. $h(x) = e^x$; $h(3), h(0.23), h(1), h(-2)$

4. $h(x) = e^{-2x}$; $h(1), h(\sqrt{2}), h(-3), h(\frac{1}{2})$

5–6 ■ Complete the table of values, rounded to two decimal places, and sketch a graph of the function.

5.

x	$f(x) = 3e^x$
-2	
-1	
-0.5	
0	
0.5	
1	
2	

6.

x	$f(x) = 2e^{-0.5x}$
-3	
-2	
-1	
0	
1	
2	
3	

7–14 ■ Graph the function, not by plotting points, but by starting from the graph of $y = e^x$ in Figure 1. State the domain, range, and asymptote.

7. $f(x) = -e^x$

8. $y = 1 - e^x$

9. $y = e^{-x} - 1$

10. $f(x) = -e^{-x}$

11. $f(x) = e^{x-2}$

12. $y = e^{x-3} + 4$

13. $h(x) = e^{x+1} - 3$

14. $g(x) = -e^{x-1} - 2$

15. The *hyperbolic cosine function* is defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

(a) Sketch the graphs of the functions $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ on the same axes, and use graphical addition (see Section 2.6) to sketch the graph of $y = \cosh(x)$.
(b) Use the definition to show that $\cosh(-x) = \cosh(x)$.

16. The *hyperbolic sine function* is defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

(a) Sketch the graph of this function using graphical addition as in Exercise 15.
(b) Use the definition to show that $\sinh(-x) = -\sinh(x)$

17. (a) Draw the graphs of the family of functions

$$f(x) = \frac{a}{2}(e^{x/a} + e^{-x/a})$$

for $a = 0.5$, 1, 1.5, and 2.
(b) How does a larger value of a affect the graph?

18–19 ■ Find the local maximum and minimum values of the function and the value of x at which each occurs. State each answer rounded to two decimal places.

18. $g(x) = x^x \quad (x > 0)$

19. $g(x) = e^x + e^{-3x}$

APPLICATIONS

20. Medical Drugs When a certain medical drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream after t hours is modeled by

$$D(t) = 50e^{-0.2t}$$

How many milligrams of the drug remain in the patient's bloodstream after 3 hours?

21. Radioactive Decay A radioactive substance decays in such a way that the amount of mass remaining after t days is given by the function

$$m(t) = 13e^{-0.015t}$$

where $m(t)$ is measured in kilograms.
(a) Find the mass at time $t = 0$.
(b) How much of the mass remains after 45 days?

22. Radioactive Decay Doctors use radioactive iodine as a tracer in diagnosing certain thyroid gland disorders. This type of iodine decays in such a way that the mass remaining after t days is given by the function

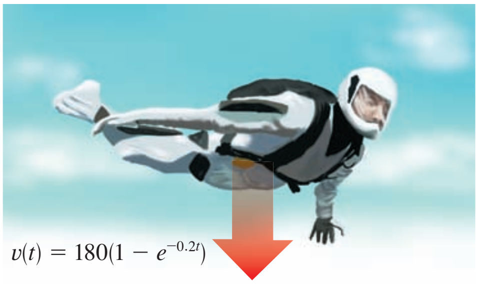
$$m(t) = 6e^{-0.087t}$$

where $m(t)$ is measured in grams.
(a) Find the mass at time $t = 0$.
(b) How much of the mass remains after 20 days?

23. Sky Diving A sky diver jumps from a reasonable height above the ground. The air resistance she experiences is proportional to her velocity, and the constant of proportionality is 0.2. It can be shown that the downward velocity of the sky diver at time t is given by

$$v(t) = 180(1 - e^{-0.2t})$$

where t is measured in seconds and $v(t)$ is measured in feet per second (ft/s).
(a) Find the initial velocity of the sky diver.
(b) Find the velocity after 5 s and after 10 s.
(c) Draw a graph of the velocity function $v(t)$.
(d) The maximum velocity of a falling object with wind resistance is called its *terminal velocity*. From the graph in part (c) find the terminal velocity of this sky diver.



24. Mixtures and Concentrations A 50-gallon barrel is filled completely with pure water. Salt water with a concentration of 0.3 lb/gal is then pumped into the barrel, and the resulting mixture overflows at the same rate. The amount of salt in the barrel at time t is given by

$$Q(t) = 15(1 - e^{-0.04t})$$

where t is measured in minutes and $Q(t)$ is measured in pounds.
(a) How much salt is in the barrel after 5 min?
(b) How much salt is in the barrel after 10 min?
(c) Draw a graph of the function $Q(t)$.

(d) Use the graph in part (c) to determine the value that the amount of salt in the barrel approaches as t becomes large. Is this what you would expect?

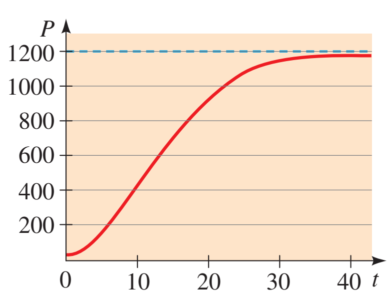
$$Q(t) = 15(1 - e^{-0.04t})$$

25. Logistic Growth Animal populations are not capable of unrestricted growth because of limited habitat and food supplies. Under such conditions the population follows a *logistic growth model*:

$$P(t) = \frac{d}{1 + ke^{-ct}}$$

where c, d, and k are positive constants. For a certain fish population in a small pond $d = 1200$, $k = 11$, $c = 0.2$, and t is measured in years. The fish were introduced into the pond at time $t = 0$.

(a) How many fish were originally put in the pond?
(b) Find the population after 10, 20, and 30 years.
(c) Evaluate $P(t)$ for large values of t. What value does the population approach as $t \to \infty$? Does the graph shown confirm your calculations?



26. Bird Population The population of a certain species of bird is limited by the type of habitat required for nesting. The population behaves according to the logistic growth model

$$n(t) = \frac{5600}{0.5 + 27.5e^{-0.044t}}$$

where t is measured in years.

(a) Find the initial bird population.
(b) Draw a graph of the function $n(t)$.
(c) What size does the population approach as time goes on?

27. World Population The relative growth rate of world population has been decreasing steadily in recent years. On the basis of this, some population models predict that world population will eventually stabilize at a level that the planet can support. One such logistic model is

$$P(t) = \frac{73.2}{6.1 + 5.9e^{-0.02t}}$$

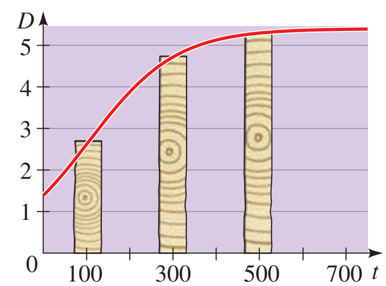
where $t = 0$ is the year 2000 and population is measured in billions.

(a) What world population does this model predict for the year 2200? For 2300?
(b) Sketch a graph of the function P for the years 2000 to 2500.
(c) According to this model, what size does the world population seem to approach as time goes on?

28. Tree Diameter For a certain type of tree the diameter D (in feet) depends on the tree's age t (in years) according to the logistic growth model

$$D(t) = \frac{5.4}{1 + 2.9e^{-0.01t}}$$

Find the diameter of a 20-year-old tree.



29–30 ■ Compound Interest An investment of \$7,000 is deposited into an account in which interest is compounded continuously. Complete the table by filling in the amounts to which the investment grows at the indicated times or interest rates.

29. $r = 3\%$

Time (years)	Amount
1	
2	
3	
4	
5	
6	

30. $t = 10$ years

Rate per year	Amount
1%	
2%	
3%	
4%	
5%	
6%	

31. Compound Interest If \$2000 is invested at an interest rate of 3.5% per year, compounded continuously, find the value of the investment after the given number of years.
(a) 2 years **(b)** 4 years **(c)** 12 years

32. Compound Interest If \$3500 is invested at an interest rate of 6.25% per year, compounded continuously, find the value of the investment after the given number of years.
(a) 3 years **(b)** 6 years **(c)** 9 years

33. Compound Interest If \$600 is invested at an interest rate of 2.5% per year, find the amount of the investment at the end of 10 years for the following compounding methods.
(a) Annually **(b)** Semiannually
(c) Quarterly **(d)** Continuously

34. Compound Interest If \$8000 is invested in an account for which interest is compounded continuously, find the amount of the investment at the end of 12 years for the following interest rates.
(a) 2% **(b)** 3% **(c)** 4.5% **(d)** 7%

35. Compound Interest Which of the given interest rates and compounding periods would provide the best investment?
(a) $2\frac{1}{2}\%$ per year, compounded semiannually
(b) $2\frac{1}{4}\%$ per year, compounded monthly
(c) 2% per year, compounded continuously

36. Compound Interest Which of the given interest rates and compounding periods would provide the better investment?
(a) $5\frac{1}{8}\%$ per year, compounded semiannually
(b) 5% per year, compounded continuously

37. Investment A sum of \$5000 is invested at an interest rate of 9% per year, compounded continuously.
(a) Find the value $A(t)$ of the investment after t years.
(b) Draw a graph of $A(t)$.
(c) Use the graph of $A(t)$ to determine when this investment will amount to \$25,000.

DISCOVERY ▪ DISCUSSION ▪ WRITING

38. The Definition of *e* Illustrate the definition of the number e by graphing the curve $y = (1 + 1/x)^x$ and the line $y = e$ on the same screen, using the viewing rectangle $[0, 40]$ by $[0, 4]$.

4.3 Logarithmic Functions

LEARNING OBJECTIVES After completing this section, you will be able to:

Evaluate logarithmic functions ▶ Graph logarithmic functions ▶ Change between logarithmic and exponential forms of an expression ▶ Use basic properties of logarithms ▶ Use common and natural logarithms

GET READY Prepare for this section by reviewing the definition and properties of inverse functions in Section 2.7.

In this section we study the inverses of exponential functions.

▼ Logarithmic Functions

Every exponential function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is a one-to-one function by the Horizontal Line Test (see Figure 1 for the case $a > 1$) and therefore has an inverse function. The inverse function f^{-1} is called the *logarithmic function with base a* and is denoted by $\log_a$. Recall from Section 2.7 that f^{-1} is defined by

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

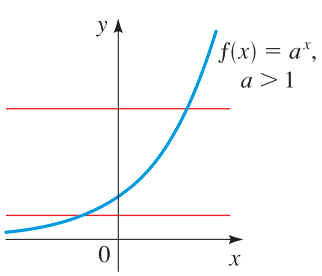


FIGURE 1 $f(x) = a^x$ is one-to-one.

This leads to the following definition of the logarithmic function.

DEFINITION OF THE LOGARITHMIC FUNCTION

Let a be a positive number with $a \neq 1$. The **logarithmic function with base *a***, denoted by $\log_a$, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

So $\log_a x$ is the *exponent* to which the base a must be raised to give x.

We read $\log_a x = y$ as "log base a of x is y."

By tradition the name of the logarithmic function is $\log_a$, not just a single letter. Also, we usually omit the parentheses in the function notation and write

$$\log_a(x) = \log_a x$$

When we use the definition of logarithms to switch back and forth between the **logarithmic form** $\log_a x = y$ and the **exponential form** $a^y = x$, it is helpful to notice that, in both forms, the base is the same:

Logarithmic form	Exponential form
Exponent	Exponent
$\log_a x = y$	$a^y = x$
Base	Base

EXAMPLE 1 | Logarithmic and Exponential Forms

The logarithmic and exponential forms are equivalent equations: If one is true, then so is the other. So we can switch from one form to the other as in the following illustrations.

Logarithmic form	Exponential form
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2\left(\frac{1}{8}\right) = -3$	$2^{-3} = \frac{1}{8}$
$\log_5 s = r$	$5^r = s$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5**

x	$\log_{10} x$
10^4	4
10^3	3
10^2	2
10	1
1	0
10^{-1}	-1
10^{-2}	-2
10^{-3}	-3
10^{-4}	-4

It is important to understand that $\log_a x$ is an *exponent*. For example, the numbers in the right column of the table in the margin are the logarithms (base 10) of the numbers in the left column. This is the case for all bases, as the following example illustrates.

EXAMPLE 2 | Evaluating Logarithms

(a) $\log_{10} 1000 = 3$ because $10^3 = 1000$

(b) $\log_2 32 = 5$ because $2^5 = 32$

(c) $\log_{10} 0.1 = -1$ because $10^{-1} = 0.1$

(d) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **7** AND **9**

Inverse Function Property:

$$f^{-1}(f(x)) = x$$
$$f(f^{-1}(x)) = x$$

When we apply the Inverse Function Property described on page 234 to $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, we get

$$\log_a(a^x) = x, \qquad x \in \mathbb{R}$$
$$a^{\log_a x} = x, \qquad x > 0$$

We list these and other properties of logarithms discussed in this section.

PROPERTIES OF LOGARITHMS

Property	Reason
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a a^x = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

EXAMPLE 3 | Applying Properties of Logarithms

We illustrate the properties of logarithms when the base is 5.

$\log_5 1 = 0$ Property 1 $\qquad \log_5 5 = 1$ Property 2

$\log_5 5^8 = 8$ Property 3 $\qquad 5^{\log_5 12} = 12$ Property 4

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **23** AND **29**

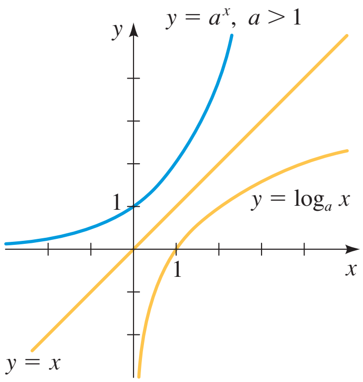


FIGURE 2 Graph of the logarithmic function $f(x) = \log_a x$

Graphs of Logarithmic Functions

Recall that if a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A. Since the exponential function $f(x) = a^x$ with $a \neq 1$ has domain $\mathbb{R}$ and range $(0, \infty)$, we conclude that its inverse function, $f^{-1}(x) = \log_a x$, has domain $(0, \infty)$ and range $\mathbb{R}$.

The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$. Figure 2 shows the case $a > 1$. The fact that $y = a^x$ (for $a > 1$) is a very rapidly increasing function for $x > 0$ implies that $y = \log_a x$ is a very slowly increasing function for $x > 1$ (see Exercise 100).

Since $\log_a 1 = 0$, the x-intercept of the function $y = \log_a x$ is 1. The y-axis is a vertical asymptote of $y = \log_a x$ because $\log_a x \to -\infty$ as $x \to 0^+$.

EXAMPLE 4 | Graphing a Logarithmic Function by Plotting Points

Sketch the graph of $f(x) = \log_2 x$.

SOLUTION To make a table of values, we choose the x-values to be powers of 2 so that we can easily find their logarithms. We plot these points and connect them with a smooth curve as in Figure 3.

x	$\log_2 x$
2^3	3
2^2	2
2	1
1	0
2^{-1}	-1
2^{-2}	-2
2^{-3}	-3
2^{-4}	-4

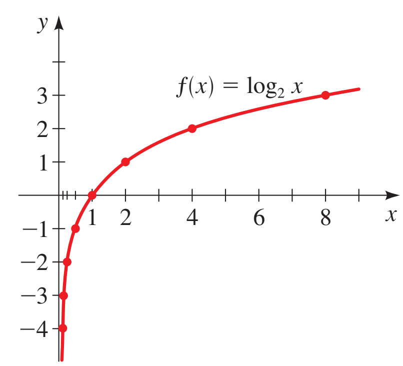


FIGURE 3

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **47**

Figure 4 shows the graphs of the family of logarithmic functions with bases 2, 3, 5, and 10. These graphs are drawn by reflecting the graphs of $y = 2^x$, $y = 3^x$, $y = 5^x$, and $y = 10^x$ (see Figure 2 in Section 4.1) in the line $y = x$. We can also plot points as an aid to sketching these graphs, as illustrated in Example 4.

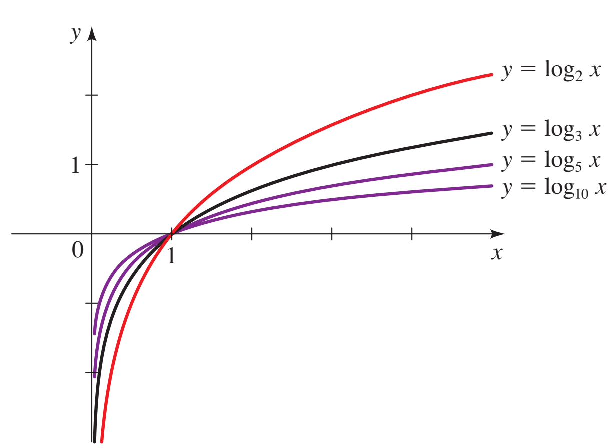


FIGURE 4 A family of logarithmic functions

In the next two examples we graph logarithmic functions by starting with the basic graphs in Figure 4 and using the transformations of Section 2.5.

EXAMPLE 5 | Reflecting Graphs of Logarithmic Functions

Sketch the graph of each function.

(a) $g(x) = -\log_2 x$

(b) $h(x) = \log_2(-x)$

SOLUTION

(a) We start with the graph of $f(x) = \log_2 x$ and reflect in the x-axis to get the graph of $g(x) = -\log_2 x$ in Figure 5(a).

(b) We start with the graph of $f(x) = \log_2 x$ and reflect in the y-axis to get the graph of $h(x) = \log_2(-x)$ in Figure 5(b).

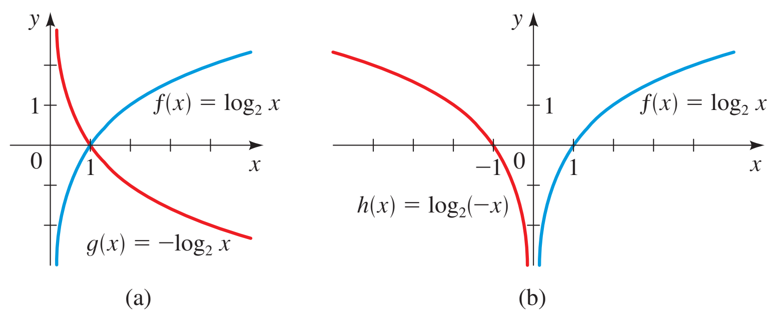


FIGURE 5

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **59**

EXAMPLE 6 | Shifting Graphs of Logarithmic Functions

Find the domain of each function, and sketch the graph.

(a) $g(x) = 2 + \log_5 x$

(b) $h(x) = \log_{10}(x - 3)$

SOLUTION

(a) The graph of g is obtained from the graph of $f(x) = \log_5 x$ (Figure 4) by shifting upward 2 units (see Figure 6). The domain of f is $(0, \infty)$.

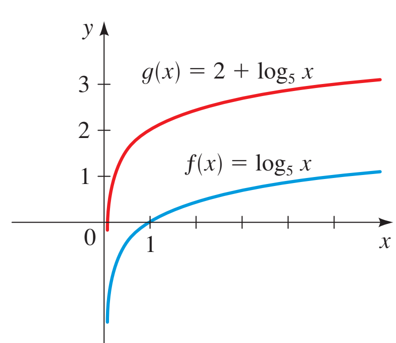


FIGURE 6

(b) The graph of h is obtained from the graph of $f(x) = \log_{10} x$ (Figure 4) by shifting to the right 3 units (see Figure 7). The line $x = 3$ is a vertical asymptote. Since $\log_{10} x$ is defined only when $x > 0$, the domain of $h(x) = \log_{10}(x - 3)$ is

$$\{x \mid x - 3 > 0\} = \{x \mid x > 3\} = (3, \infty)$$

MATHEMATICS IN THE MODERN WORLD

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Law Enforcement

Mathematics aids law enforcement in numerous and surprising ways, from the reconstruction of bullet trajectories to determining the time of death to calculating the probability that a DNA sample is from a particular person. One interesting use is in the search for missing persons. A person who has been missing for several years might look quite different from his or her most recent available photograph. This is particularly true if the missing person is a child. Have you ever wondered what you will look like 5, 10, or 15 years from now?

Researchers have found that different parts of the body grow at different rates. For example, you have no doubt noticed that a baby's head is much larger relative to its body than an adult's. As another example, the ratio of arm length to height is $\frac{1}{3}$ in a child but about $\frac{2}{5}$ in an adult. By collecting data and analyzing the graphs, researchers are able to determine the functions that model growth. As in all growth phenomena, exponential and logarithmic functions play a crucial role. For instance, the formula that relates arm length l to height h is $l = ae^{kh}$ where a and k are constants. By studying various physical characteristics of a person, mathematical biologists model each characteristic by a function that describes how it changes over time. Models of facial characteristics can be programmed into a computer to give a picture of how a person's appearance changes over time. These pictures aid law enforcement agencies in locating missing persons.

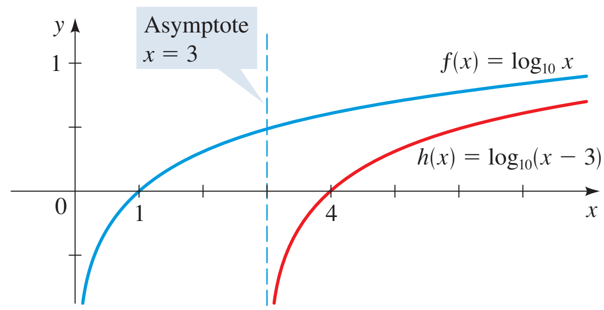
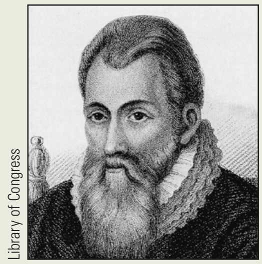


FIGURE 7

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **61** AND **65**



JOHN NAPIER (1550–1617) was a Scottish landowner for whom mathematics was a hobby. We know him today because of his key invention: logarithms, which he published in 1614 under the title *A Description of the Marvelous Rule of Logarithms*. In Napier's time, logarithms were used exclusively for simplifying complicated calculations. For example, to multiply two large numbers, we would write them as powers of 10. The exponents are simply the logarithms of the numbers. For instance,

$$\begin{aligned} 4532 \times 57783 &\approx 10^{3.65629} \times 10^{4.76180} \\ &= 10^{8.41809} \\ &\approx 261{,}872{,}564 \end{aligned}$$

The idea is that multiplying powers of 10 is easy (we simply add their exponents). Napier produced extensive tables giving the logarithms (or exponents) of numbers. Since the advent of calculators and computers, logarithms are no longer used for this purpose. The logarithmic functions, however, have found many applications, some of which are described in this chapter.

Napier wrote on many topics. One of his most colorful works is a book entitled *A Plaine Discovery of the Whole Revelation of Saint John*, in which he predicted that the world would end in the year 1700.

Common Logarithms

We now study logarithms with base 10.

COMMON LOGARITHM

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x = \log_{10} x$$

From the definition of logarithms we can easily find that

$$\log 10 = 1 \qquad \text{and} \qquad \log 100 = 2$$

But how do we find log 50? We need to find the exponent y such that $10^y = 50$. Clearly, 1 is too small and 2 is too large. So

$$1 < \log 50 < 2$$

To get a better approximation, we can experiment to find a power of 10 closer to 50. Fortunately, scientific calculators are equipped with a LOG key that directly gives values of common logarithms.

EXAMPLE 7 | Evaluating Common Logarithms

Use a calculator to find appropriate values of $f(x) = \log x$, and use the values to sketch the graph.

SOLUTION We make a table of values, using a calculator to evaluate the function at those values of x that are not powers of 10. We plot those points and connect them by a smooth curve as in Figure 8.

x	$\log x$
0.01	−2
0.1	−1
0.5	−0.301
1	0
4	0.602
5	0.699
10	1

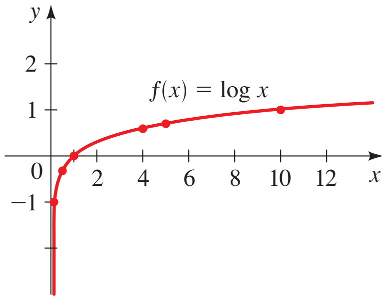


FIGURE 8

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **49**

Human response to sound and light intensity is logarithmic.

Scientists model human response to stimuli (such as sound, light, or pressure) using logarithmic functions. For example, the intensity of a sound must be increased manyfold before we "feel" that the loudness has simply doubled. The psychologist Gustav Fechner formulated the law as

$$S = k \log\left(\frac{I}{I_0}\right)$$

where S is the subjective intensity of the stimulus, I is the physical intensity of the stimulus, I_0 stands for the threshold physical intensity, and k is a constant that is different for each sensory stimulus.

EXAMPLE 8 | Common Logarithms and Sound

We study the decibel scale in more detail in Section 4.6.

The perception of the loudness B (in decibels, dB) of a sound with physical intensity I (in W/m^2) is given by

$$B = 10 \log\left(\frac{I}{I_0}\right)$$

where I_0 is the physical intensity of a barely audible sound. Find the decibel level (loudness) of a sound whose physical intensity I is 100 times that of I_0.

SOLUTION We find the decibel level B by using the fact that $I = 100I_0$.

$$\begin{aligned} B &= 10 \log\left(\frac{I}{I_0}\right) && \text{Definition of } B \\ &= 10 \log\left(\frac{100I_0}{I_0}\right) && I = 100I_0 \\ &= 10 \log 100 && \text{Cancel } I_0 \\ &= 10 \cdot 2 = 20 && \text{Definition of log} \end{aligned}$$

The loudness of the sound is 20 dB.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **95**

Natural Logarithms

The notation ln is an abbreviation for the Latin name *logarithmus naturalis*.

Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of calculus is the number e, which we defined in Section 4.2.

NATURAL LOGARITHM

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

The natural logarithmic function $y = \ln x$ is the inverse function of the natural exponential function $y = e^x$. Both functions are graphed in Figure 9. By the definition of inverse functions we have

$$\boxed{\ln x = y \quad \Leftrightarrow \quad e^y = x}$$

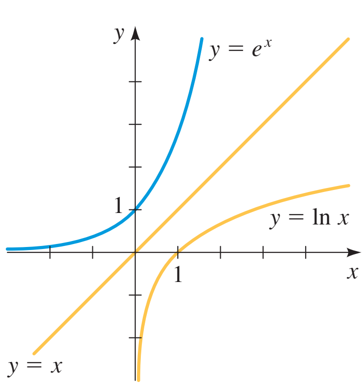


FIGURE 9 Graph of the natural logarithmic function

If we substitute $a = e$ and write "ln" for "$\log_e$" in the properties of logarithms mentioned earlier, we obtain the following properties of natural logarithms.

PROPERTIES OF NATURAL LOGARITHMS

Property	Reason
1. $\ln 1 = 0$	We must raise e to the power 0 to get 1.
2. $\ln e = 1$	We must raise e to the power 1 to get e.
3. $\ln e^x = x$	We must raise e to the power x to get e^x.
4. $e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

Calculators are equipped with an LN key that directly gives the values of natural logarithms.

EXAMPLE 9 | Evaluating the Natural Logarithm Function

(a) $\ln e^8 = 8$ — Definition of natural logarithm

(b) $\ln\left(\dfrac{1}{e^2}\right) = \ln e^{-2} = -2$ — Definition of natural logarithm

(c) $\ln 5 \approx 1.609$ — Use LN key on calculator

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45**

EXAMPLE 10 | Finding the Domain of a Logarithmic Function

Find the domain of the function $f(x) = \ln(4 - x^2)$.

SOLUTION As with any logarithmic function, $\ln x$ is defined when $x > 0$. Thus the domain of f is

$$\begin{aligned}\{x \mid 4 - x^2 > 0\} &= \{x \mid x^2 < 4\} = \{x \mid |x| < 2\}\\ &= \{x \mid -2 < x < 2\} = (-2, 2)\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **71**

EXAMPLE 11 | Drawing the Graph of a Logarithmic Function

Draw the graph of the function $y = x \ln(4 - x^2)$, and use it to find the asymptotes and local maximum and minimum values.

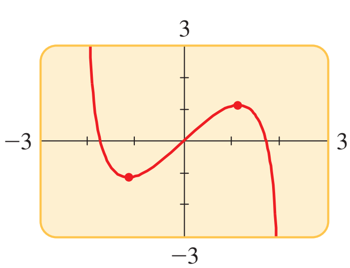


FIGURE 10
$y = x \ln(4 - x^2)$

SOLUTION As in Example 10 the domain of this function is the interval $(-2, 2)$, so we choose the viewing rectangle $[-3, 3]$ by $[-3, 3]$. The graph is shown in Figure 10, and from it we see that the lines $x = -2$ and $x = 2$ are vertical asymptotes.

The function has a local maximum point to the right of $x = 1$ and a local minimum point to the left of $x = -1$. By zooming in and tracing along the graph with the cursor, we find that the local maximum value is approximately 1.13 and this occurs when $x \approx 1.15$. Similarly (or by noticing that the function is odd), we find that the local minimum value is about -1.13, and it occurs when $x \approx -1.15$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **77**

4.3 EXERCISES

CONCEPTS

1. $\log x$ is the exponent to which the base 10 must be raised to get ________. So we can complete the following table for $\log x$.

x	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	$10^{1/2}$
$\log x$								

2. The function $f(x) = \log_9 x$ is the logarithm function with base ________. So $f(9) =$ ________, $f(1) =$ ________, $f(\frac{1}{9}) =$ ________, $f(81) =$ ________, and $f(3) =$ ________.

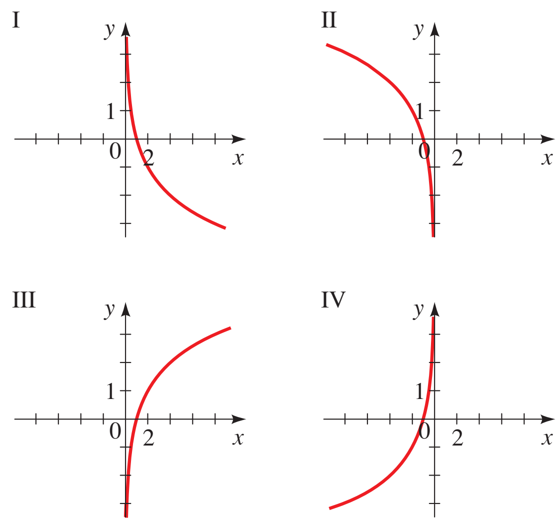
3. (a) $5^3 = 125$, so $\log_{\square} \square = \square$

(b) $\log_5 25 = 2$, so $\square^{\square} = \square$

4. Match the logarithmic function with its graph.

(a) $f(x) = \log_2 x$ **(b)** $f(x) = \log_2(-x)$

(c) $f(x) = -\log_2 x$ **(d)** $f(x) = -\log_2(-x)$



SKILLS

5–6 ■ Complete the table by finding the appropriate logarithmic or exponential form of the equation, as in Example 1.

5.

Logarithmic form	Exponential form
$\log_8 8 = 1$	
$\log_8 64 = 2$	
	$8^{2/3} = 4$
	$8^3 = 512$
$\log_8\left(\frac{1}{8}\right) = -1$	
	$8^{-2} = \frac{1}{64}$

6.

Logarithmic form	Exponential form
	$4^3 = 64$
$\log_4 2 = \frac{1}{2}$	
	$4^{3/2} = 8$
$\log_4\left(\frac{1}{16}\right) = -2$	
$\log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$	
	$4^{-5/2} = \frac{1}{32}$

7–14 ■ Express the equation in exponential form.

7. (a) $\log_5 25 = 2$ **(b)** $\log_5 1 = 0$

8. (a) $\log_{10} 0.1 = -1$ **(b)** $\log_8 512 = 3$

9. (a) $\log_8 2 = \frac{1}{3}$ **(b)** $\log_2\left(\frac{1}{8}\right) = -3$

10. (a) $\log_3 81 = 4$ **(b)** $\log_8 4 = \frac{2}{3}$

11. (a) $\log_3 5 = x$ **(b)** $\log_7(3y) = 2$

12. (a) $\log_6 z = 1$ **(b)** $\log_{10} 3 = 2t$

13. (a) $\ln 5 = 3y$ **(b)** $\ln(t + 1) = -1$

14. (a) $\ln(x + 1) = 2$ **(b)** $\ln(x - 1) = 4$

15–22 ■ Express the equation in logarithmic form.

15. (a) $5^3 = 125$ **(b)** $10^{-4} = 0.0001$

16. (a) $10^3 = 1000$ **(b)** $81^{1/2} = 9$

17. (a) $8^{-1} = \frac{1}{8}$ **(b)** $2^{-3} = \frac{1}{8}$

18. (a) $4^{-3/2} = 0.125$ **(b)** $7^3 = 343$

19. (a) $5^x = 3$ **(b)** $4^5 = z$

20. (a) $2^{3x} = 7$ **(b)** $10^{-0.5x} = 0.01$

21. (a) $e^x = 2$ **(b)** $e^3 = y$

22. (a) $e^{x+1} = 0.5$ **(b)** $e^{0.5x} = t$

23–32 ■ Evaluate the expression.

23. (a) $\log_3 3$ **(b)** $\log_3 1$ **(c)** $\log_3 3^2$

24. (a) $\log_5 5^4$ **(b)** $\log_4 64$ **(c)** $\log_3 9$

25. (a) $\log_6 36$ **(b)** $\log_9 81$ **(c)** $\log_7 7^{10}$

26. (a) $\log_2 32$ **(b)** $\log_8 8^{17}$ **(c)** $\log_6 1$

27. (a) $\log_3\left(\frac{1}{27}\right)$ **(b)** $\log_{10}\sqrt{10}$ **(c)** $\log_5 0.2$

28. (a) $\log_5 125$ **(b)** $\log_{49} 7$ **(c)** $\log_9 \sqrt{3}$

29. (a) $2^{\log_2 37}$ **(b)** $3^{\log_3 8}$ **(c)** $e^{\ln\sqrt{5}}$

30. (a) $e^{\ln \pi}$ **(b)** $10^{\log 5}$ **(c)** $10^{\log 87}$

31. (a) $\log_8 0.25$ **(b)** $\ln e^4$ **(c)** $\ln(1/e)$

32. (a) $\log_4 \sqrt{2}$ **(b)** $\log_4\left(\frac{1}{2}\right)$ **(c)** $\log_4 8$

33–42 ■ Use the definition of the logarithmic function to find x.

33. (a) $\log_2 x = 5$ **(b)** $\log_2 16 = x$

34. (a) $\log_5 x = 4$ **(b)** $\log_{10} 0.1 = x$

35. (a) $\ln x = 3$ **(b)** $\ln e^2 = x$

36. (a) $\ln x = -1$ **(b)** $\ln(1/e) = x$

37. **(a)** $\log_3 243 = x$ **(b)** $\log_3 x = 3$

38. **(a)** $\log_4 2 = x$ **(b)** $\log_4 x = 2$

39. **(a)** $\log_{10} x = 2$ **(b)** $\log_5 x = 2$

40. **(a)** $\log_x 1000 = 3$ **(b)** $\log_x 25 = 2$

41. **(a)** $\log_x 16 = 4$ **(b)** $\log_x 8 = \frac{3}{2}$

42. **(a)** $\log_x 6 = \frac{1}{2}$ **(b)** $\log_x 3 = \frac{1}{3}$

43–46 ■ Use a calculator to evaluate the expression, correct to four decimal places.

43. **(a)** $\log 2$ **(b)** $\log 35.2$ **(c)** $\log\left(\frac{2}{3}\right)$

44. **(a)** $\log 50$ **(b)** $\log \sqrt{2}$ **(c)** $\log(3\sqrt{2})$

45. **(a)** $\ln 5$ **(b)** $\ln 25.3$ **(c)** $\ln(1 + \sqrt{3})$

46. **(a)** $\ln 27$ **(b)** $\ln 7.39$ **(c)** $\ln 54.6$

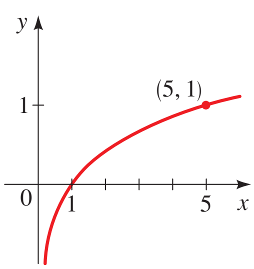
47–50 ■ Sketch the graph of the function by plotting points.

47. $f(x) = \log_3 x$ **48.** $g(x) = \log_4 x$

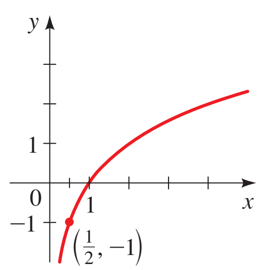
49. $f(x) = 2 \log x$ **50.** $g(x) = 1 + \log x$

51–54 ■ Find the function of the form $y = \log_a x$ whose graph is given.

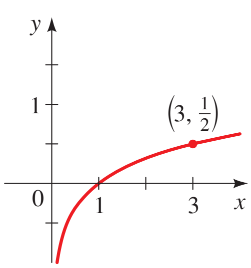
51.



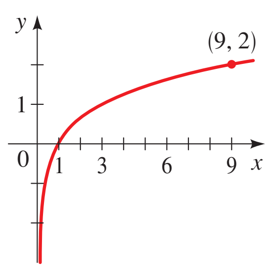
52.



53.



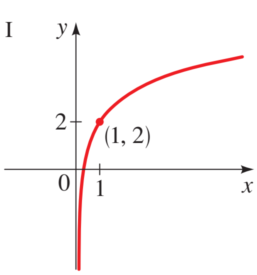
54.



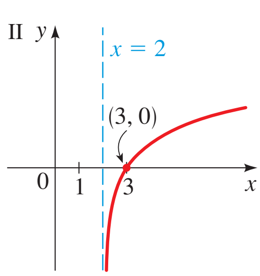
55–56 ■ Match the logarithmic function with one of the graphs labeled I or II.

55. $f(x) = 2 + \ln x$ **56.** $f(x) = \ln(x - 2)$

I



II



57. Draw the graph of $y = 4^x$, then use it to draw the graph of $y = \log_4 x$.

58. Draw the graph of $y = 3^x$, then use it to draw the graph of $y = \log_3 x$.

59–70 ■ Graph the function, not by plotting points, but by starting from the graphs in Figures 4 and 9. State the domain, range, and asymptote.

59. $g(x) = \log_5(-x)$ **60.** $f(x) = -\log_{10} x$

61. $f(x) = \log_2(x - 4)$ **62.** $g(x) = \ln(x + 2)$

63. $h(x) = \ln(x + 5)$ **64.** $g(x) = \log_6(x - 3)$

65. $y = 2 + \log_3 x$ **66.** $y = 1 - \log_{10} x$

67. $y = \log_3(x - 1) - 2$ **68.** $y = 1 + \ln(-x)$

69. $y = |\ln x|$ **70.** $y = \ln|x|$

71–76 ■ Find the domain of the function.

71. $f(x) = \log_{10}(x + 3)$ **72.** $f(x) = \log_5(8 - 2x)$

73. $g(x) = \log_3(x^2 - 1)$ **74.** $g(x) = \ln(x - x^2)$

75. $h(x) = \ln x + \ln(2 - x)$

76. $h(x) = \sqrt{x - 2} - \log_5(10 - x)$

77–82 ■ Draw the graph of the function in a suitable viewing rectangle, and use it to find the domain, the asymptotes, and the local maximum and minimum values.

77. $y = \log_{10}(1 - x^2)$ **78.** $y = \ln(x^2 - x)$

79. $y = x + \ln x$ **80.** $y = x(\ln x)^2$

81. $y = \dfrac{\ln x}{x}$ **82.** $y = x \log_{10}(x + 10)$

83-86 ■ Find the functions $f \circ g$ and $g \circ f$ and their domains.

83. $f(x) = 2^x$, $g(x) = x + 1$

84. $f(x) = 3^x$, $g(x) = x^2 + 1$

85. $f(x) = \log_2 x$, $g(x) = x - 2$

86. $f(x) = \log x$, $g(x) = x^2$

87. Compare the rates of growth of the functions $f(x) = \ln x$ and $g(x) = \sqrt{x}$ by drawing their graphs on a common screen using the viewing rectangle $[-1, 30]$ by $[-1, 6]$.

88. **(a)** By drawing the graphs of the functions

$$f(x) = 1 + \ln(1 + x) \quad \text{and} \quad g(x) = \sqrt{x}$$

in a suitable viewing rectangle, show that even when a logarithmic function starts out higher than a root function, it is ultimately overtaken by the root function.

(b) Find, rounded to two decimal places, the solutions of the equation $\sqrt{x} = 1 + \ln(1 + x)$.

89–90 ■ A family of functions is given. **(a)** Draw graphs of the family for $c = 1, 2, 3,$ and 4. **(b)** How are the graphs in part (a) related?

89. $f(x) = \log(cx)$ **90.** $f(x) = c \log x$

91–92 ■ A function $f(x)$ is given. **(a)** Find the domain of the function f. **(b)** Find the inverse function of f.

91. $f(x) = \log_2(\log_{10} x)$ **92.** $f(x) = \ln(\ln(\ln x))$

93. **(a)** Find the inverse of the function $f(x) = \dfrac{2^x}{1 + 2^x}$.

(b) What is the domain of the inverse function?

APPLICATIONS

94. Absorption of Light A spectrophotometer measures the concentration of a sample dissolved in water by shining a light through it and recording the amount of light that emerges. In other words, if we know the amount of light that is absorbed, we can calculate the concentration of the sample. For a certain substance the concentration (in moles per liter) is found by using the formula

$$C = -2500 \ln\left(\frac{I}{I_0}\right)$$

where I_0 is the intensity of the incident light and I is the intensity of light that emerges. Find the concentration of the substance if the intensity I is 70% of I_0.

95. Carbon Dating The age of an ancient artifact can be determined by the amount of radioactive carbon-14 remaining in it. If D_0 is the original amount of carbon-14 and D is the amount remaining, then the artifact's age A (in years) is given by

$$A = -8267 \ln\left(\frac{D}{D_0}\right)$$

Find the age of an object if the amount D of carbon-14 that remains in the object is 73% of the original amount D_0.

96. Bacteria Colony A certain strain of bacteria divides every three hours. If a colony is started with 50 bacteria, then the time t (in hours) required for the colony to grow to N bacteria is given by

$$t = 3\frac{\log(N/50)}{\log 2}$$

Find the time required for the colony to grow to a million bacteria.

97. Investment The time required to double the amount of an investment at an interest rate r compounded continuously is given by

$$t = \frac{\ln 2}{r}$$

Find the time required to double an investment at 6%, 7%, and 8%.

98. Charging a Battery The rate at which a battery charges is slower the closer the battery is to its maximum charge C_0. The time (in hours) required to charge a fully discharged battery to a charge C is given by

$$t = -k \ln\left(1 - \frac{C}{C_0}\right)$$

where k is a positive constant that depends on the battery. For a certain battery, $k = 0.25$. If this battery is fully discharged, how long will it take to charge to 90% of its maximum charge C_0?

99. Difficulty of a Task The difficulty in "acquiring a target" (such as using your mouse to click on an icon on your computer screen) depends on the distance to the target and the size of the target. According to Fitts's Law, the index of difficulty (ID) is given by

$$\text{ID} = \frac{\log(2A/W)}{\log 2}$$

where W is the width of the target and A is the distance to the center of the target. Compare the difficulty of clicking on an icon that is 5 mm wide to clicking on one that is 10 mm wide. In each case, assume that the mouse is 100 mm from the icon.

DISCOVERY ▪ DISCUSSION ▪ WRITING

100. The Height of the Graph of a Logarithmic Function Suppose that the graph of $y = 2^x$ is drawn on a coordinate plane where the unit of measurement is an inch.

(a) Show that at a distance 2 ft to the right of the origin the height of the graph is about 265 mi.

(b) If the graph of $y = \log_2 x$ is drawn on the same set of axes, how far to the right of the origin do we have to go before the height of the curve reaches 2 ft?

101. The Googolplex A **googol** is 10^{100}, and a **googolplex** is 10^{googol}. Find

$$\log(\log(\text{googol})) \qquad \text{and} \qquad \log(\log(\log(\text{googolplex})))$$

102. Comparing Logarithms Which is larger, $\log_4 17$ or $\log_5 24$? Explain your reasoning.

103. The Number of Digits in an Integer Compare log 1000 to the number of digits in 1000. Do the same for 10,000. How many digits does any number between 1000 and 10,000 have? Between what two values must the common logarithm of such a number lie? Use your observations to explain why the number of digits in any positive integer x is $[\![\log x]\!] + 1$. (The symbol $[\![n]\!]$ is the greatest integer function defined in Section 2.2.) How many digits does the number 2^{100} have?

4.4 Laws of Logarithms

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Use the Laws of Logarithms to evaluate logarithmic expressions ▶ Use the Laws of Logarithms to expand logarithmic expressions ▶ Use the Laws of Logarithms to combine logarithmic expressions ▶ Use the Change of Base Formula

In this section we study properties of logarithms. These properties give logarithmic functions a wide range of applications, as we will see in Section 4.6.

▼ Laws of Logarithms

Since logarithms are exponents, the Laws of Exponents give rise to the Laws of Logarithms.

LAWS OF LOGARITHMS

Let a be a positive number, with $a \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C\log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

PROOF We make use of the property $\log_a a^x = x$ from Section 4.3.

Law 1 Let $\log_a A = u$ and $\log_a B = v$. When written in exponential form, these equations become

$$a^u = A \qquad \text{and} \qquad a^v = B$$

Thus
$$\log_a(AB) = \log_a(a^u a^v) = \log_a(a^{u+v})$$
$$= u + v = \log_a A + \log_a B$$

Law 2 Using Law 1, we have

$$\log_a A = \log_a\left[\left(\frac{A}{B}\right)B\right] = \log_a\left(\frac{A}{B}\right) + \log_a B$$

so
$$\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$$

Law 3 Let $\log_a A = u$. Then $a^u = A$, so

$$\log_a(A^C) = \log_a(a^u)^C = \log_a(a^{uC}) = uC = C\log_a A \qquad \blacksquare$$

EXAMPLE 1 | Using the Laws of Logarithms to Evaluate Expressions

Evaluate each expression.

(a) $\log_4 2 + \log_4 32$

(b) $\log_2 80 - \log_2 5$

(c) $-\frac{1}{3}\log 8$

SOLUTION

(a) $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$ Law 1

$= \log_4 64 = 3$ Because $64 = 4^3$

(b) $\log_2 80 - \log_2 5 = \log_2(\frac{80}{5})$ Law 2

$= \log_2 16 = 4$ Because $16 = 2^4$

(c) $-\frac{1}{3}\log 8 = \log 8^{-1/3}$ Law 3

$= \log(\frac{1}{2})$ Property of negative exponents

≈ -0.301 Calculator

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **7**, **9**, AND **11**

Expanding and Combining Logarithmic Expressions

The Laws of Logarithms allow us to write the logarithm of a product or a quotient as the sum or difference of logarithms. This process, called *expanding* a logarithmic expression, is illustrated in the next example.

EXAMPLE 2 | Expanding Logarithmic Expressions

Use the Laws of Logarithms to expand each expression.

(a) $\log_2(6x)$ **(b)** $\log_5(x^3y^6)$ **(c)** $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right)$

SOLUTION

(a) $\log_2(6x) = \log_2 6 + \log_2 x$ Law 1

(b) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6$ Law 1

$= 3\log_5 x + 6\log_5 y$ Law 3

(c) $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln\sqrt[3]{c}$ Law 2

$= \ln a + \ln b - \ln c^{1/3}$ Law 1

$= \ln a + \ln b - \frac{1}{3}\ln c$ Law 3

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **21**, **33**, AND **35**

The Laws of Logarithms also allow us to reverse the process of expanding that was done in Example 2. That is, we can write sums and differences of logarithms as a single logarithm. This process, called *combining* logarithmic expressions, is illustrated in the next example.

EXAMPLE 3 | Combining Logarithmic Expressions

Combine $3\log x + \frac{1}{2}\log(x + 1)$ into a single logarithm.

SOLUTION

$3\log x + \frac{1}{2}\log(x+1) = \log x^3 + \log(x+1)^{1/2}$ Law 3

$= \log(x^3(x+1)^{1/2})$ Law 1

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **49**

EXAMPLE 4 | Combining Logarithmic Expressions

Combine $3\ln s + \frac{1}{2}\ln t - 4\ln(t^2 + 1)$ into a single logarithm.

SOLUTION

$$\begin{aligned} 3\ln s + \tfrac{1}{2}\ln t - 4\ln(t^2+1) &= \ln s^3 + \ln t^{1/2} - \ln(t^2+1)^4 && \text{Law 3} \\ &= \ln(s^3t^{1/2}) - \ln(t^2+1)^4 && \text{Law 1} \\ &= \ln\left(\frac{s^3\sqrt{t}}{(t^2+1)^4}\right) && \text{Law 2} \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **51** ■

Warning Although the Laws of Logarithms tell us how to compute the logarithm of a product or a quotient, *there is no corresponding rule for the logarithm of a sum or a difference*. For instance,

$$\log_a(x+y) \cancel{=} \log_a x + \log_a y$$

In fact, we know that the right side is equal to $\log_a(xy)$. Also, don't improperly simplify quotients or powers of logarithms. For instance,

$$\frac{\log 6}{\log 2} \cancel{=} \log\left(\frac{6}{2}\right) \qquad \text{and} \qquad (\log_2 x)^3 \cancel{=} 3\log_2 x$$

Logarithmic functions are used to model a variety of situations involving human behavior. One such behavior is how quickly we forget things we have learned. For example, if you learn algebra at a certain performance level (say, 90% on a test) and then don't use algebra for a while, how much will you retain after a week, a month, or a year? Hermann Ebbinghaus (1850–1909) studied this phenomenon and formulated the law described in the next example.

EXAMPLE 5 | The Law of Forgetting

Forgetting what we've learned depends on how long ago we learned it.

If a task is learned at a performance level P_0, then after a time interval t the performance level P satisfies

$$\log P = \log P_0 - c\log(t+1)$$

where c is a constant that depends on the type of task and t is measured in months.

(a) Solve for P.

(b) If your score on a history test is 90, what score would you expect to get on a similar test after two months? After a year? (Assume that $c = 0.2$.)

SOLUTION

(a) We first combine the right-hand side.

$$\begin{aligned} \log P &= \log P_0 - c\log(t+1) && \text{Given equation} \\ \log P &= \log P_0 - \log(t+1)^c && \text{Law 3} \\ \log P &= \log\frac{P_0}{(t+1)^c} && \text{Law 2} \\ P &= \frac{P_0}{(t+1)^c} && \text{Because log is one-to-one} \end{aligned}$$

(b) Here $P_0 = 90$, $c = 0.2$, and t is measured in months.

In two months: $t = 2$ and $P = \dfrac{90}{(2+1)^{0.2}} \approx 72$

In one year: $t = 12$ and $P = \dfrac{90}{(12+1)^{0.2}} \approx 54$

Your expected scores after two months and one year are 72 and 54, respectively.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **71** ■

Change of Base Formula

For some purposes we find it useful to change from logarithms in one base to logarithms in another base. Suppose we are given $\log_a x$ and want to find $\log_b x$. Let

$$y = \log_b x$$

We write this in exponential form and take the logarithm, with base a, of each side.

$$\begin{aligned} b^y &= x && \text{Exponential form} \\ \log_a(b^y) &= \log_a x && \text{Take } \log_a \text{ of each side} \\ y \log_a b &= \log_a x && \text{Law 3} \\ y &= \frac{\log_a x}{\log_a b} && \text{Divide by } \log_a b \end{aligned}$$

This proves the following formula.

We may write the Change of Base Formula as

$$\log_b x = \left(\frac{1}{\log_a b}\right)\log_a x$$

So $\log_b x$ is just a constant multiple of $\log_a x$; the constant is $\dfrac{1}{\log_a b}$.

CHANGE OF BASE FORMULA

$$\log_b x = \frac{\log_a x}{\log_a b}$$

In particular, if we put $x = a$, then $\log_a a = 1$, and this formula becomes

$$\log_b a = \frac{1}{\log_a b}$$

We can now evaluate a logarithm to *any* base by using the Change of Base Formula to express the logarithm in terms of common logarithms or natural logarithms and then using a calculator.

EXAMPLE 6 | Evaluating Logarithms with the Change of Base Formula

Use the Change of Base Formula and common or natural logarithms to evaluate each logarithm, rounded to five decimal places.

(a) $\log_8 5$ **(b)** $\log_9 20$

SOLUTION

(a) We use the Change of Base Formula with $b = 8$ and $a = 10$:

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} \approx 0.77398$$

(b) We use the Change of Base Formula with $b = 9$ and $a = e$:

$$\log_9 20 = \frac{\ln 20}{\ln 9} \approx 1.36342$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **57** AND **59**

EXAMPLE 7 | Using the Change of Base Formula to Graph a Logarithmic Function

Use a graphing calculator to graph $f(x) = \log_6 x$.

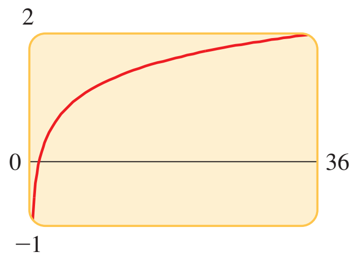


FIGURE 1 $f(x) = \log_6 x = \dfrac{\ln x}{\ln 6}$

SOLUTION Calculators don't have a key for $\log_6$, so we use the Change of Base Formula to write

$$f(x) = \log_6 x = \frac{\ln x}{\ln 6}$$

Since calculators do have an LN key, we can enter this new form of the function and graph it. The graph is shown in Figure 1.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **65** ■

4.4 EXERCISES

CONCEPTS

1. The logarithm of a product of two numbers is the same as the ________ of the logarithms of these numbers. So $\log_5(25 \cdot 125) =$ ________ $+$ ________.

2. The logarithm of a quotient of two numbers is the same as the ________ of the logarithms of these numbers. So $\log_5(\frac{25}{125}) =$ ________ $-$ ________.

3. The logarithm of a number raised to a power is the same as the power ________ the logarithm of the number. So $\log_5(25^{10}) =$ ________ $\cdot$ ________.

4. **(a)** We can expand $\log\left(\dfrac{x^2 y}{z}\right)$ to get ________________.

(b) We can combine $2\log x + \log y - \log z$ to get ________.

5. Most calculators can find logarithms with base ________ and base ________. To find logarithms with different bases, we use the ________________ Formula. To find $\log_7 12$, we write

$$\log_7 12 = \frac{\log \square}{\log \square} = \_\_\_\_\_\_$$

6. *True or false?* We get the same answer if we do the calculation in Exercise 5 using ln in place of log.

SKILLS

7–20 ■ Use the Laws of Logarithms to evaluate the expression.

7. $\log 4 + \log 25$
8. $\log_{12} 9 + \log_{12} 16$
9. $\log_4 192 - \log_4 3$
10. $\log_2 160 - \log_2 5$
11. $-\frac{1}{3}\log 64$
12. $3 \ln 10$
13. $\log_3 \sqrt{27}$
14. $\log \dfrac{1}{\sqrt{1000}}$
15. $\log_2 6 - \log_2 15 + \log_2 20$
16. $\log_3 100 - \log_3 18 - \log_3 50$
17. $\log_4 16^{100}$
18. $\log_2 8^{33}$
19. $\log(\log 10^{10,000})$
20. $\ln(\ln e^{e^{200}})$

21–46 ■ Use the Laws of Logarithms to expand the expression.

21. $\log_2(2x)$
22. $\log_3(5y)$
23. $\log_2(x(x-1))$
24. $\log_4 \dfrac{y}{y+3}$
25. $\log 6^{10}$
26. $\log_6 \sqrt[4]{17}$
27. $\ln \sqrt{z}$
28. $\log_9 \sqrt[4]{t}$
29. $\log_5 \sqrt[3]{x^2+1}$
30. $\log \sqrt{2a^4+1}$
31. $\log_2(xy)^{10}$
32. $\ln \sqrt{ab}$
33. $\log_2(AB^2)$
34. $\log_3(x\sqrt{y})$
35. $\log\left(\dfrac{x^3 y^4}{z^6}\right)$
36. $\log_a\left(\dfrac{x^2}{yz^3}\right)$
37. $\log_2\left(\dfrac{x(x^2+1)}{\sqrt{x^2-1}}\right)$
38. $\log_5 \sqrt{\dfrac{x-1}{x+1}}$
39. $\ln\left(x\sqrt{\dfrac{y}{z}}\right)$
40. $\ln \dfrac{3x^2}{(x+1)^{10}}$
41. $\log \sqrt[4]{x^2+y^2}$
42. $\log\left(\dfrac{x}{\sqrt[3]{1-x}}\right)$
43. $\log \sqrt{\dfrac{x^2+4}{(x^2+1)(x^3-7)^2}}$
44. $\log \sqrt{x\sqrt{y\sqrt{z}}}$
45. $\ln\left(\dfrac{x^3\sqrt{x-1}}{3x+4}\right)$
46. $\log\left(\dfrac{10^x}{x(x^2+1)(x^4+2)}\right)$

47–56 ■ Use the Laws of Logarithms to combine the expression.

47. $\log_3 5 + 5\log_3 2$
48. $\log 12 + \frac{1}{2}\log 7 - \log 2$
49. $\ln 5 + 2\ln x + 3\ln(x^2+5)$
50. $3\log_2 A + 5\log_2 B - 2\log_2 C$
51. $4\log x - \frac{1}{3}\log(x^2+1) + 2\log(x-1)$
52. $\log_5(x^2-1) - \log_5(x-1)$
53. $\ln(a+b) + \ln(a-b) - 2\ln c$
54. $2(\log_5 x + 2\log_5 y - 3\log_5 z)$

55. $\frac{1}{3}\log(x+2)^3 + \frac{1}{2}[\log x^4 - \log(x^2 - x - 6)^2]$

56. $\log_a b + c \log_a d - r \log_a s$

57–64 ■ Use the Change of Base Formula and a calculator to evaluate the logarithm, rounded to six decimal places. Use either natural or common logarithms.

57. $\log_2 5$ **58.** $\log_5 2$

59. $\log_3 16$ **60.** $\log_6 92$

61. $\log_7 2.61$ **62.** $\log_6 532$

63. $\log_4 125$ **64.** $\log_{12} 2.5$

65. Use the Change of Base Formula to show that

$$\log_3 x = \frac{\ln x}{\ln 3}$$

Then use this fact to draw the graph of the function $f(x) = \log_3 x$.

66. Draw graphs of the family of functions $y = \log_a x$ for $a = 2, e,$ 5, and 10 on the same screen, using the viewing rectangle $[0, 5]$ by $[-3, 3]$. How are these graphs related?

67. Use the Change of Base Formula to show that

$$\log e = \frac{1}{\ln 10}$$

68. Simplify: $(\log_2 5)(\log_5 7)$

69. Show that $-\ln(x - \sqrt{x^2 - 1}) = \ln(x + \sqrt{x^2 - 1})$.

APPLICATIONS

70. Forgetting Use the Law of Forgetting (Example 5) to estimate a student's score on a biology test two years after he got a score of 80 on a test covering the same material. Assume that $c = 0.3$ and t is measured in months.

71. Wealth Distribution Vilfredo Pareto (1848–1923) observed that most of the wealth of a country is owned by a few members of the population. **Pareto's Principle** is

$$\log P = \log c - k \log W$$

where W is the wealth level (how much money a person has) and P is the number of people in the population having that much money.

(a) Solve the equation for P.

(b) Assume that $k = 2.1$, $c = 8000$, and that W is measured in millions of dollars. Use part (a) to find the number of people who have \$2 million or more. How many people have \$10 million or more?

72. Biodiversity Some biologists model the number of species S in a fixed area A (such as an island) by the species-area relationship

$$\log S = \log c + k \log A$$

where c and k are positive constants that depend on the type of species and habitat.

(a) Solve the equation for S.

(b) Use part (a) to show that if $k = 3$, then doubling the area increases the number of species eightfold.

73. Magnitude of Stars The magnitude M of a star is a measure of how bright a star appears to the human eye. It is defined by

$$M = -2.5 \log\left(\frac{B}{B_0}\right)$$

where B is the actual brightness of the star and B_0 is a constant.

(a) Expand the right-hand side of the equation.

(b) Use part (a) to show that the brighter a star, the less its magnitude.

(c) Betelgeuse is about 100 times brighter than Albiero. Use part (a) to show that Betelgeuse is 5 magnitudes less bright than Albiero.

DISCOVERY ■ DISCUSSION ■ WRITING

74. True or False? Discuss each equation, and determine whether it is true for all possible values of the variables. (Ignore values of the variables for which any term is undefined.)

(a) $\log\left(\frac{x}{y}\right) = \frac{\log x}{\log y}$

(b) $\log_2(x - y) = \log_2 x - \log_2 y$

(c) $\log_5\left(\frac{a}{b^2}\right) = \log_5 a - 2\log_5 b$

(d) $\log 2^z = z \log 2$

(e) $(\log P)(\log Q) = \log P + \log Q$

(f) $\frac{\log a}{\log b} = \log a - \log b$

(g) $(\log_2 7)^x = x \log_2 7$

(h) $\log_a a^a = a$

(i) $\log(x - y) = \frac{\log x}{\log y}$

(j) $-\ln\left(\frac{1}{A}\right) = \ln A$

75. Find the Error What is wrong with the following argument?

$$\begin{aligned} \log 0.1 &< 2 \log 0.1 \\ &= \log(0.1)^2 \\ &= \log 0.01 \\ \log 0.1 &< \log 0.01 \\ 0.1 &< 0.01 \end{aligned}$$

76. Shifting, Shrinking, and Stretching Graphs of Functions Let $f(x) = x^2$. Show that $f(2x) = 4f(x)$, and explain how this shows that shrinking the graph of f horizontally has the same effect as stretching it vertically. Then use the identities $e^{2+x} = e^2e^x$ and $\ln(2x) = \ln 2 + \ln x$ to show that for $g(x) = e^x$ a horizontal shift is the same as a vertical stretch and for $h(x) = \ln x$ a horizontal shrinking is the same as a vertical shift.

4.5 Exponential and Logarithmic Equations

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Solve exponential equations ▶ Solve logarithmic equations ▶ Solve problems involving compound interest ▶ Calculate annual percentage yield

In this section we solve equations that involve exponential or logarithmic functions. The techniques that we develop here will be used in the next section for solving applied problems.

▼ Exponential Equations

An *exponential equation* is one in which the variable occurs in the exponent. Some exponential equations can be solved by using the fact that exponential functions are one-to-one. This means that

$$a^x = a^y \quad \Rightarrow \quad x = y$$

We use this property in the next example.

EXAMPLE 1 | Exponential Equations

Solve the exponential equation.

(a) $5^x = 125$ **(b)** $5^{2x} = 5^{x+1}$

SOLUTION

(a) We first express 125 as a power of 5 and then use the fact that the exponential function $f(x) = 5^x$ is one-to-one:

$$\begin{aligned} 5^x &= 125 && \text{Given equation} \\ 5^x &= 5^3 && \text{Because } 125 = 5^3 \\ x &= 3 && \text{One-to-one property} \end{aligned}$$

The solution is $x = 3$.

(b) We first use the fact that the function $f(x) = 5^x$ is one-to-one:

$$\begin{aligned} 5^{2x} &= 5^{x+1} && \text{Given equation} \\ 2x &= x + 1 && \text{One-to-one property} \\ x &= 1 && \text{Solve for } x \end{aligned}$$

The solution is $x = 1$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **3** AND **7**

The equations in Example 1 were solved by comparing exponents. This method is not suitable for solving an equation like $5^x = 160$ because 160 is not easily expressed as a power of the base 5. To solve such equations, we take the logarithm of each side and use Law 3 of logarithms to "bring down the exponent." The following guidelines describe the process.

Law 3: $\log_a A^C = C \log_a A$

GUIDELINES FOR SOLVING EXPONENTIAL EQUATIONS

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."
3. Solve for the variable.

EXAMPLE 2 | Solving an Exponential Equation

Consider the exponential equation $3^{x+2} = 7$.

(a) Find the exact solution of the equation expressed in terms of logarithms.

(b) Use a calculator to find an approximation to the solution rounded to six decimal places.

SOLUTION

(a) We take the common logarithm of each side and use Law 3:

$$\begin{aligned} 3^{x+2} &= 7 && \text{Given equation} \\ \log(3^{x+2}) &= \log 7 && \text{Take log of each side} \\ (x+2)\log 3 &= \log 7 && \text{Law 3 (bring down exponent)} \\ x + 2 &= \frac{\log 7}{\log 3} && \text{Divide by log 3} \\ x &= \frac{\log 7}{\log 3} - 2 && \text{Subtract 2} \end{aligned}$$

The exact solution is $x = \dfrac{\log 7}{\log 3} - 2$.

(b) Using a calculator we find the decimal approximation $x \approx -0.228756$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **15**

We could have used natural logarithms instead of common logarithms. In fact, using the same steps, we get

$$x = \frac{\ln 7}{\ln 3} - 2 \approx -0.228756$$

CHECK YOUR ANSWER

Substituting $x = -0.228756$ into the original equation and using a calculator, we get

$$3^{(-0.228756)+2} \approx 7 \quad ✓$$

EXAMPLE 3 | Solving an Exponential Equation

Solve the equation $8e^{2x} = 20$.

SOLUTION We first divide by 8 to isolate the exponential term on one side of the equation:

$$\begin{aligned} 8e^{2x} &= 20 && \text{Given equation} \\ e^{2x} &= \tfrac{20}{8} && \text{Divide by 8} \\ \ln e^{2x} &= \ln 2.5 && \text{Take ln of each side} \\ 2x &= \ln 2.5 && \text{Property of ln} \\ x &= \frac{\ln 2.5}{2} && \text{Divide by 2 (exact solution)} \\ &\approx 0.458 && \text{Calculator (approximate solution)} \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17**

CHECK YOUR ANSWER

Substituting $x = 0.458$ into the original equation and using a calculator, we get

$$8e^{2(0.458)} \approx 20 \quad ✓$$

EXAMPLE 4 | Solving an Exponential Equation Algebraically and Graphically

Solve the equation $e^{3-2x} = 4$ algebraically and graphically.

SOLUTION 1: Algebraic

Since the base of the exponential term is e, we use natural logarithms to solve this equation:

$$
\begin{aligned}
e^{3-2x} &= 4 && \text{Given equation}\\
\ln(e^{3-2x}) &= \ln 4 && \text{Take ln of each side}\\
3 - 2x &= \ln 4 && \text{Property of ln}\\
-2x &= -3 + \ln 4 && \text{Subtract 3}\\
x &= \tfrac{1}{2}(3 - \ln 4) \approx 0.807 && \text{Multiply by } -\tfrac{1}{2}
\end{aligned}
$$

You should check that this answer satisfies the original equation.

SOLUTION 2: Graphical

We graph the equations $y = e^{3-2x}$ and $y = 4$ in the same viewing rectangle as in Figure 1. The solutions occur where the graphs intersect. Zooming in on the point of intersection of the two graphs, we see that $x \approx 0.81$.

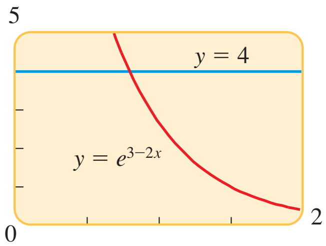


FIGURE 1

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **21**

EXAMPLE 5 | An Exponential Equation of Quadratic Type

Solve the equation $e^{2x} - e^x - 6 = 0$.

SOLUTION To isolate the exponential term, we factor:

$$
\begin{aligned}
e^{2x} - e^x - 6 &= 0 && \text{Given equation}\\
(e^x)^2 - e^x - 6 &= 0 && \text{Law of Exponents}\\
(e^x - 3)(e^x + 2) &= 0 && \text{Factor (a quadratic in } e^x)\\
e^x - 3 = 0 \quad \text{or} \quad e^x + 2 &= 0 && \text{Zero-Product Property}\\
e^x = 3 \qquad\qquad e^x &= -2
\end{aligned}
$$

If we let $w = e^x$, we get the quadratic equation

$$w^2 - w - 6 = 0$$

which factors as

$$(w - 3)(w + 2) = 0$$

The equation $e^x = 3$ leads to $x = \ln 3$. But the equation $e^x = -2$ has no solution because $e^x > 0$ for all x. Thus $x = \ln 3 \approx 1.0986$ is the only solution. You should check that this answer satisfies the original equation.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **37**

EXAMPLE 6 | Solving an Exponential Equation

Solve the equation $3xe^x + x^2e^x = 0$.

SOLUTION First we factor the left side of the equation:

$$
\begin{aligned}
3xe^x + x^2e^x &= 0 && \text{Given equation}\\
x(3 + x)e^x &= 0 && \text{Factor out common factors}\\
x(3 + x) &= 0 && \text{Divide by } e^x \text{ (because } e^x \neq 0)\\
x = 0 \quad \text{or} \quad 3 + x &= 0 && \text{Zero-Product Property}
\end{aligned}
$$

Thus the solutions are $x = 0$ and $x = -3$.

CHECK YOUR ANSWER

$x = 0$:

$3(0)e^0 + 0^2e^0 = 0$ ✓

$x = -3$:

$3(-3)e^{-3} + (-3)^2e^{-3}$

$= -9e^{-3} + 9e^{-3} = 0$ ✓

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41**

Logarithmic Equations

A *logarithmic equation* is one in which a logarithm of the variable occurs. Some logarithmic equations can be solved by using the fact that logarithmic functions are one-to-one. This means that

$$\log_a x = \log_a y \quad \Rightarrow \quad x = y$$

We use this property in the next example.

Radiocarbon Dating is a method that archeologists use to determine the age of ancient objects. The carbon dioxide in the atmosphere always contains a fixed fraction of radioactive carbon, carbon-14 (^{14}C), with a half-life of about 5730 years. Plants absorb carbon dioxide from the atmosphere, which then makes its way to animals through the food chain. Thus, all living creatures contain the same fixed proportions of ^{14}C to nonradioactive ^{12}C as the atmosphere.

After an organism dies, it stops assimilating ^{14}C, and the amount of ^{14}C in it begins to decay exponentially. We can then determine the time that has elapsed since the death of the organism by measuring the amount of ^{14}C left in it.

For example, if a donkey bone contains 73% as much ^{14}C as a living donkey and it died t years ago, then by the formula for radioactive decay (Section 4.6),

$$0.73 = (1.00)e^{-(t \ln 2)/5730}$$

We solve this exponential equation to find $t \approx 2600$, so the bone is about 2600 years old.

EXAMPLE 7 | Solving a Logarithmic Equation

Solve the equation $\log(x^2 + 1) = \log(x - 2) + \log(x + 3)$.

SOLUTION First we combine the logarithms on the right-hand side, and then we use the one-to-one property of logarithms.

$$\begin{aligned}
\log_5(x^2+1) &= \log_5(x-2) + \log_5(x+3) && \text{Given equation}\\
\log_5(x^2+1) &= \log_5[(x-2)(x+3)] && \text{Law 1: } \log_a AB = \log_a A + \log_a B\\
\log_5(x^2+1) &= \log_5(x^2+x-6) && \text{Expand}\\
x^2+1 &= x^2+x-6 && \text{log is one-to-one (or raise 5 to each side)}\\
x &= 7 && \text{Solve for } x
\end{aligned}$$

The solution is $x = 7$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45**

The method of Example 7 is not suitable for solving an equation like $\log_5 x = 13$ because the right-hand side is not expressed as a logarithm (base 5). To solve such equations, we use the following guidelines.

GUIDELINES FOR SOLVING LOGARITHMIC EQUATIONS

1. Isolate the logarithmic term on one side of the equation; you might first need to combine the logarithmic terms.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable.

EXAMPLE 8 | Solving Logarithmic Equations

Solve each equation for x.

(a) $\ln x = 8$ **(b)** $\log_2(25 - x) = 3$

SOLUTION

(a)

$$\begin{aligned}
\ln x &= 8 && \text{Given equation}\\
x &= e^8 && \text{Exponential form}
\end{aligned}$$

Therefore $x = e^8 \approx 2981$.

We can also solve this problem another way:

$$\begin{aligned}
\ln x &= 8 && \text{Given equation}\\
e^{\ln x} &= e^8 && \text{Raise } e \text{ to each side}\\
x &= e^8 && \text{Property of ln}
\end{aligned}$$

(b) The first step is to rewrite the equation in exponential form:

$$\begin{aligned}
\log_2(25-x) &= 3 && \text{Given equation}\\
25 - x &= 2^3 && \text{Exponential form (or raise 2 to each side)}\\
25 - x &= 8\\
x &= 25 - 8 = 17
\end{aligned}$$

CHECK YOUR ANSWER

If $x = 17$, we get

$\log_2(25 - 17) = \log_2 8 = 3$ ✓

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **51** AND **55**

EXAMPLE 9 | Solving a Logarithmic Equation

Solve the equation $4 + 3\log(2x) = 16$.

SOLUTION We first isolate the logarithmic term. This allows us to write the equation in exponential form:

$$\begin{aligned} 4 + 3\log(2x) &= 16 && \text{Given equation} \\ 3\log(2x) &= 12 && \text{Subtract 4} \\ \log(2x) &= 4 && \text{Divide by 3} \\ 2x &= 10^4 && \text{Exponential form (or raise 10 to each side)} \\ x &= 5000 && \text{Divide by 2} \end{aligned}$$

CHECK YOUR ANSWER

If $x = 5000$, we get

$$\begin{aligned} 4 + 3\log 2(5000) &= 4 + 3\log 10{,}000 \\ &= 4 + 3(4) \\ &= 16 \quad \checkmark \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **57** ■

EXAMPLE 10 | Solving a Logarithmic Equation Algebraically and Graphically

Solve the equation $\log(x + 2) + \log(x - 1) = 1$ algebraically and graphically.

SOLUTION 1: Algebraic

We first combine the logarithmic terms, using the Laws of Logarithms:

$$\begin{aligned} \log[(x + 2)(x - 1)] &= 1 && \text{Law 1} \\ (x + 2)(x - 1) &= 10 && \text{Exponential form (or raise 10 to each side)} \\ x^2 + x - 2 &= 10 && \text{Expand left side} \\ x^2 + x - 12 &= 0 && \text{Subtract 10} \\ (x + 4)(x - 3) &= 0 && \text{Factor} \end{aligned}$$

$$x = -4 \quad \text{or} \quad x = 3$$

We check these potential solutions in the original equation and find that $x = -4$ is not a solution (because logarithms of negative numbers are undefined), but $x = 3$ is a solution. (See *Check Your Answers*.)

CHECK YOUR ANSWER

$x = -4$:

$$\begin{aligned} &\log(-4 + 2) + \log(-4 - 1) \\ &\quad = \log(-2) + \log(-5) \\ &\qquad \text{undefined} \quad ✗ \end{aligned}$$

$x = 3$:

$$\begin{aligned} &\log(3 + 2) + \log(3 - 1) \\ &\quad = \log 5 + \log 2 = \log(5 \cdot 2) \\ &\quad = \log 10 = 1 \quad \checkmark \end{aligned}$$

SOLUTION 2: Graphical

We first move all terms to one side of the equation:

$$\log(x + 2) + \log(x - 1) - 1 = 0$$

Then we graph

$$y = \log(x + 2) + \log(x - 1) - 1$$

as in Figure 2. The solutions are the x-intercepts of the graph. Thus the only solution is $x \approx 3$.

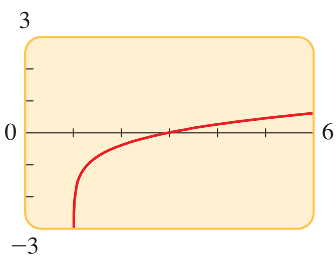


FIGURE 2

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **59** ■

EXAMPLE 11 | Solving a Logarithmic Equation Graphically

In Example 11 it's not possible to isolate x algebraically, so we must solve the equation graphically.

Solve the equation $x^2 = 2\ln(x + 2)$.

SOLUTION We first move all terms to one side of the equation:

$$x^2 - 2\ln(x + 2) = 0$$

Then we graph

$$y = x^2 - 2\ln(x + 2)$$

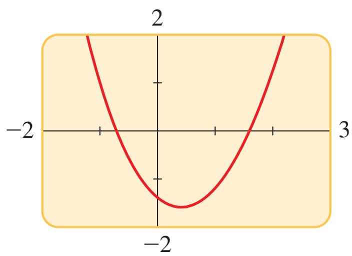


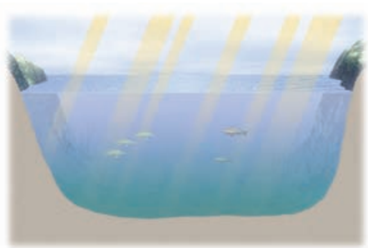
FIGURE 3

as in Figure 3. The solutions are the x-intercepts of the graph. Zooming in on the x-intercepts, we see that there are two solutions:

$$x \approx -0.71 \qquad \text{and} \qquad x \approx 1.60$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **69** ■

Logarithmic equations are used in determining the amount of light that reaches various depths in a lake. (This information helps biologists to determine the types of life a lake can support.) As light passes through water (or other transparent materials such as glass or plastic), some of the light is absorbed. It's easy to see that the murkier the water, the more light is absorbed. The exact relationship between light absorption and the distance light travels in a material is described in the next example.



The intensity of light in a lake diminishes with depth.

EXAMPLE 12 | Transparency of a Lake

If I_0 and I denote the intensity of light before and after going through a material and x is the distance (in feet) the light travels in the material, then according to the **Beer-Lambert Law**,

$$-\frac{1}{k}\ln\left(\frac{I}{I_0}\right) = x$$

where k is a constant depending on the type of material.

(a) Solve the equation for I.

(b) For a certain lake $k = 0.025$, and the light intensity is $I_0 = 14$ lumens (lm). Find the light intensity at a depth of 20 ft.

SOLUTION

(a) We first isolate the logarithmic term:

$$\begin{aligned} -\frac{1}{k}\ln\left(\frac{I}{I_0}\right) &= x && \text{Given equation} \\ \ln\left(\frac{I}{I_0}\right) &= -kx && \text{Multiply by } -k \\ \frac{I}{I_0} &= e^{-kx} && \text{Exponential form} \\ I &= I_0 e^{-kx} && \text{Multiply by } I_0 \end{aligned}$$

(b) We find I using the formula from part (a):

$$\begin{aligned} I &= I_0 e^{-kx} && \text{From part (a)} \\ &= 14e^{(-0.025)(20)} && I_0 = 14,\ k = 0.025,\ x = 20 \\ &\approx 8.49 && \text{Calculator} \end{aligned}$$

The light intensity at a depth of 20 ft is about 8.5 lm.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **95** ■

▼ Compound Interest

Recall the formulas for interest that we found in Section 4.1. If a principal P is invested at an interest rate r for a period of t years, then the amount A of the investment is given by

$$\begin{aligned} A &= P(1 + r) && \text{Simple interest (for one year)} \\ A(t) &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Interest compounded } n \text{ times per year} \\ A(t) &= Pe^{rt} && \text{Interest compounded continuously} \end{aligned}$$

We can use logarithms to determine the time it takes for the principal to increase to a given amount.

EXAMPLE 13 | Finding the Term for an Investment to Double

A sum of \$5000 is invested at an interest rate of 5% per year. Find the time required for the money to double if the interest is compounded according to the following method.

(a) Semiannually **(b)** Continuously

SOLUTION

(a) We use the formula for compound interest with $P = \$5000$, $A(t) = \$10{,}000$, $r = 0.05$, and $n = 2$ and solve the resulting exponential equation for t:

$$
\begin{aligned}
5000\left(1 + \frac{0.05}{2}\right)^{2t} &= 10{,}000 && P\left(1 + \frac{r}{n}\right)^{nt} = A \\
(1.025)^{2t} &= 2 && \text{Divide by 5000} \\
\log 1.025^{2t} &= \log 2 && \text{Take log of each side} \\
2t \log 1.025 &= \log 2 && \text{Law 3 (bring down the exponent)} \\
t &= \frac{\log 2}{2 \log 1.025} && \text{Divide by 2 log 1.025} \\
t &\approx 14.04 && \text{Calculator}
\end{aligned}
$$

The money will double in 14.04 years.

(b) We use the formula for continuously compounded interest with $P = \$5000$, $A(t) = \$10{,}000$, and $r = 0.05$ and solve the resulting exponential equation for t:

$$
\begin{aligned}
5000e^{0.05t} &= 10{,}000 && Pe^{rt} = A \\
e^{0.05t} &= 2 && \text{Divide by 5000} \\
\ln e^{0.05t} &= \ln 2 && \text{Take ln of each side} \\
0.05t &= \ln 2 && \text{Property of ln} \\
t &= \frac{\ln 2}{0.05} && \text{Divide by 0.05} \\
t &\approx 13.86 && \text{Calculator}
\end{aligned}
$$

The money will double in 13.86 years.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **85**

EXAMPLE 14 | Time Required to Grow an Investment

A sum of \$1000 is invested at an interest rate of 4% per year. Find the time required for the amount to grow to \$4000 if interest is compounded continuously.

SOLUTION We use the formula for continuously compounded interest with $P = \$1000$, $A(t) = \$4000$, and $r = 0.04$ and solve the resulting exponential equation for t:

$$
\begin{aligned}
1000e^{0.04t} &= 4000 && Pe^{rt} = A \\
e^{0.04t} &= 4 && \text{Divide by 1000} \\
0.04t &= \ln 4 && \text{Take ln of each side} \\
t &= \frac{\ln 4}{0.04} && \text{Divide by 0.04} \\
t &\approx 34.66 && \text{Calculator}
\end{aligned}
$$

The amount will be \$4000 in about 34 years and 8 months.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **87**

4.5 EXERCISES

CONCEPTS

1. Let's solve the exponential equation $2e^x = 50$.

(a) First, we isolate e^x to get the equivalent equation ________.

(b) Next, we take ln of each side to get the equivalent equation ________________.

(c) Now we use a calculator to find $x =$ ________.

2. Let's solve the logarithmic equation

$$\log 3 + \log(x - 2) = \log x$$

(a) First, we combine the logarithms on the LHS to get the equivalent equation ________________.

(b) Next, we write each side in exponential form to get the equivalent equation ________________.

(c) Now we find $x =$ ________.

SKILLS

3–10 ■ Find the solution of the exponential equation, as in Example 1.

3. $4^{x+1} = 64$ **4.** $e^{x^2} = e^9$

5. $5^{2x-3} = 1$ **6.** $10^{2x-3} = \frac{1}{10}$

7. $3^{2x-8} = 3^{5x+1}$ **8.** $e^{x-2} = e^{3x+2}$

9. $6^{x^2-1} = 6^{1-x^2}$ **10.** $10^{2x^2-3} = 10^{9-x^2}$

11–36 ■ **(a)** Find the exact solution of the exponential equation in terms of logarithms. **(b)** Use a calculator to find an approximation to the solution rounded to six decimal places.

11. $10^x = 25$ **12.** $10^{-x} = 4$

13. $e^{-2x} = 7$ **14.** $e^{3x} = 12$

15. $2^{1-x} = 3$ **16.** $3^{2x-1} = 5$

17. $3e^x = 10$ **18.** $2e^{12x} = 17$

19. $100(1.04)^{2t} = 300$ **20.** $2(1.00625)^{12t} = 8$

21. $e^{1-4x} = 2$ **22.** $e^{3-5x} = 16$

23. $e^{2x+1} = 200$ **24.** $2^{3x} = 34$

25. $8^{0.4x} = 5$ **26.** $\left(\frac{1}{4}\right)^x = 75$

27. $3^{x/14} = 0.1$ **28.** $5^{-x/100} = 2$

29. $4(1 + 10^{5x}) = 9$ **30.** $4 + 3^{5x} = 8$

31. $5^x = 4^{x+1}$ **32.** $10^{1-x} = 6^x$

33. $2^{3x+1} = 3^{x-2}$ **34.** $7^{x/2} = 5^{1-x}$

35. $\dfrac{50}{1 + e^{-x}} = 4$ **36.** $\dfrac{10}{1 + e^{-x}} = 2$

37–44 ■ Solve the equation.

37. $e^{2x} - 3e^x + 2 = 0$ **38.** $e^{2x} - e^x - 6 = 0$

39. $e^{4x} + 4e^{2x} - 21 = 0$ **40.** $e^x - 12e^{-x} - 1 = 0$

41. $x^2 2^x - 2^x = 0$ **42.** $x^2 10^x - x10^x = 2(10^x)$

43. $4x^3 e^{-3x} - 3x^4 e^{-3x} = 0$ **44.** $x^2 e^x + xe^x - e^x = 0$

45–50 ■ Solve the logarithmic equation for x, as in Example 7.

45. $\log x + \log(x - 1) = \log(4x)$

46. $\log_5 x + \log_5(x + 1) = \log_5 20$

47. $2 \log x = \log 2 + \log(3x - 4)$

48. $\ln\left(x - \frac{1}{2}\right) + \ln 2 = 2 \ln x$

49. $\log_2 3 + \log_2 x = \log_2 5 + \log_2(x - 2)$

50. $\log_4(x + 2) + \log_4 3 = \log_4 5 + \log_4(2x - 3)$

51–64 ■ Solve the logarithmic equation for x.

51. $\ln x = 10$ **52.** $\ln(2 + x) = 1$

53. $\log x = -2$ **54.** $\log(x - 4) = 3$

55. $\log(3x + 5) = 2$ **56.** $\log_3(2 - x) = 3$

57. $4 - \log(3 - x) = 3$

58. $\log_2(x^2 - x - 2) = 2$

59. $\log_2 x + \log_2(x - 3) = 2$

60. $\log x + \log(x - 3) = 1$

61. $\log_9(x - 5) + \log_9(x + 3) = 1$

62. $\ln(x - 1) + \ln(x + 2) = 1$

63. $\log_5(x + 1) - \log_5(x - 1) = 2$

64. $\log_3(x + 15) - \log_3(x - 1) = 2$

65. For what value of x is the following true?

$$\log(x + 3) = \log x + \log 3$$

66. For what value of x is it true that $(\log x)^3 = 3 \log x$?

67. Solve for x: $2^{2/\log_5 x} = \frac{1}{16}$

68. Solve for x: $\log_2(\log_3 x) = 4$

69–76 ■ Use a graphing device to find all solutions of the equation, rounded to two decimal places.

69. $\ln x = 3 - x$ **70.** $\log x = x^2 - 2$

71. $x^3 - x = \log(x + 1)$ **72.** $x = \ln(4 - x^2)$

73. $e^x = -x$ **74.** $2^{-x} = x - 1$

75. $4^{-x} = \sqrt{x}$ **76.** $e^{x^2} - 2 = x^3 - x$

77–80 ■ Solve the inequality.

77. $\log(x - 2) + \log(9 - x) < 1$

78. $3 \le \log_2 x \le 4$

79. $2 < 10^x < 5$ **80.** $x^2 e^x - 2e^x < 0$

81–84 ■ Find the inverse function of f.

81. $f(x) = 2^{2x}$ **82.** $f(x) = 3^{x+1}$

83. $f(x) = \log_2(x - 1)$ **84.** $f(x) = \log 3x$

APPLICATIONS

85. Compound Interest A man invests \$5000 in an account that pays 8.5% interest per year, compounded quarterly.
(a) Find the amount after 3 years.
(b) How long will it take for the investment to double?

86. Compound Interest A woman invests \$6500 in an account that pays 6% interest per year, compounded continuously.
(a) What is the amount after 2 years?
(b) How long will it take for the amount to be \$8000?

87. Compound Interest Find the time required for an investment of \$5000 to grow to \$8000 at an interest rate of 7.5% per year, compounded quarterly.

88. Compound Interest Nancy wants to invest \$4000 in saving certificates that bear an interest rate of 9.75% per year, compounded semiannually. How long a time period should she choose to save an amount of \$5000?

89. Doubling an Investment How long will it take for an investment of \$1000 to double in value if the interest rate is 8.5% per year, compounded continuously?

90. Interest Rate A sum of \$1000 was invested for 4 years, and the interest was compounded semiannually. If this sum amounted to \$1435.77 in the given time, what was the interest rate?

91. Radioactive Decay A 15-g sample of radioactive iodine decays in such a way that the mass remaining after t days is given by $m(t) = 15e^{-0.087t}$, where $m(t)$ is measured in grams. After how many days are there only 5 g remaining?

92. Sky Diving The velocity of a sky diver t seconds after jumping is given by $v(t) = 80(1 - e^{-0.2t})$. After how many seconds is the velocity 70 ft/s?

93. Fish Population A small lake is stocked with a certain species of fish. The fish population is modeled by the function

$$P = \frac{10}{1 + 4e^{-0.8t}}$$

where P is the number of fish in thousands and t is measured in years since the lake was stocked.
(a) Find the fish population after 3 years.
(b) After how many years will the fish population reach 5000 fish?

94. Transparency of a Lake Environmental scientists measure the intensity of light at various depths in a lake to find the "transparency" of the water. Certain levels of transparency are required for the biodiversity of the submerged macrophyte population. In a certain lake the intensity of light at depth x is given by

$$I = 10e^{-0.008x}$$

where I is measured in lumens and x in feet.
(a) Find the intensity I at a depth of 30 ft.
(b) At what depth has the light intensity dropped to $I = 5$?

95. Atmospheric Pressure Atmospheric pressure P (in kilopascals, kPa) at altitude h (in kilometers, km) is governed by the formula

$$\ln\left(\frac{P}{P_0}\right) = -\frac{h}{k}$$

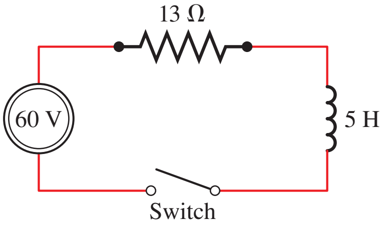
where $k = 7$ and $P_0 = 100$ kPa are constants.
(a) Solve the equation for P.
(b) Use part (a) to find the pressure P at an altitude of 4 km.

96. Cooling an Engine Suppose you're driving your car on a cold winter day (20°F outside) and the engine overheats (at about 220°F). When you park, the engine begins to cool down. The temperature T of the engine t minutes after you park satisfies the equation

$$\ln\left(\frac{T - 20}{200}\right) = -0.11t$$

(a) Solve the equation for T.
(b) Use part (a) to find the temperature of the engine after 20 min ($t = 20$).

97. Electric Circuits An electric circuit contains a battery that produces a voltage of 60 volts (V), a resistor with a resistance of 13 ohms (Ω), and an inductor with an inductance of 5 henrys (H), as shown in the figure. Using calculus, it can be shown that the current $I = I(t)$ (in amperes, A) t seconds after the switch is closed is $I = \frac{60}{13}(1 - e^{-13t/5})$.
(a) Use this equation to express the time t as a function of the current I.
(b) After how many seconds is the current 2 A?



98. Learning Curve A *learning curve* is a graph of a function $P(t)$ that measures the performance of someone learning a skill as a function of the training time t. At first, the rate of learning is rapid. Then, as performance increases and approaches a maximal value M, the rate of learning decreases. It has been found that the function

$$P(t) = M - Ce^{-kt}$$

where k and C are positive constants and $C < M$ is a reasonable model for learning.
(a) Express the learning time t as a function of the performance level P.
(b) For a pole-vaulter in training, the learning curve is given by

$$P(t) = 20 - 14e^{-0.024t}$$

where $P(t)$ is the height he is able to pole-vault after

t months. After how many months of training is he able to vault 12 ft?

(c) Draw a graph of the learning curve in part (b).

DISCOVERY ▪ DISCUSSION ▪ WRITING

99. Estimating a Solution Without actually solving the equation, find two whole numbers between which the solution of $9^x = 20$ must lie. Do the same for $9^x = 100$. Explain how you reached your conclusions.

100. A Surprising Equation Take logarithms to show that the equation

$$x^{1/\log x} = 5$$

has no solution. For what values of k does the equation

$$x^{1/\log x} = k$$

have a solution? What does this tell us about the graph of the function $f(x) = x^{1/\log x}$? Confirm your answer using a graphing device.

101. Disguised Equations Each of these equations can be transformed into an equation of linear or quadratic type by applying the hint. Solve each equation.

(a) $(x - 1)^{\log(x-1)} = 100(x - 1)$ [Take log of each side.]

(b) $\log_2 x + \log_4 x + \log_8 x = 11$ [Change all logs to base 2.]

(c) $4^x - 2^{x+1} = 3$ [Write as a quadratic in 2^x.]

4.6 Modeling with Exponential and Logarithmic Functions

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find exponential models of population growth ▶ Find exponential models of radioactive decay ▶ Solve problems involving compound interest ▶ Find models using Newton's Law of Cooling ▶ Use logarithmic scales (pH, Richter, and decibel scales)

Many processes that occur in nature, such as population growth, radioactive decay, heat diffusion, and numerous others, can be modeled by using exponential functions. Logarithmic functions are used in models for the loudness of sounds, the intensity of earthquakes, and many other phenomena. In this section we study exponential and logarithmic models.

▼ Exponential Growth (Doubling Time)

Suppose we start with a single bacterium, which divides every hour. After one hour we have 2 bacteria, after two hours we have 2^2 or 4 bacteria, after three hours we have 2^3 or 8 bacteria, and so on (see Figure 1). We see that we can model the bacteria population after t hours by $f(t) = 2^t$.

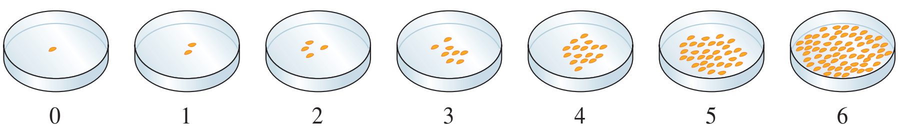


FIGURE 1 Bacteria population

If we start with 10 of these bacteria, then the population is modeled by $f(t) = 10 \cdot 2^t$. A slower-growing strain of bacteria doubles every 3 hours; in this case the population is modeled by $f(t) = 10 \cdot 2^{t/3}$. In general, we have the following.

EXPONENTIAL GROWTH (DOUBLING TIME)

If the intial size of a population is n_0 and the doubling time is a, then the size of the population at time t is

$$n(t) = n_0 2^{t/a}$$

where a and t are measured in the same time units (minutes, hours, days, years, and so on).

EXAMPLE 1 | Bacteria Population

Under ideal conditions a certain bacteria population doubles every three hours. Initially there are 1000 bacteria in a colony.

(a) Find a model for the bacteria population after t hours.

(b) How many bacteria are in the colony after 15 hours?

(c) After how many hours will the bacteria count reach 100,000?

SOLUTION

(a) The population at time t is modeled by

$$n(t) = 1000 \cdot 2^{t/3}$$

where t is measured in hours.

(b) After 15 hours the number of bacteria is

$$n(15) = 1000 \cdot 2^{15/3} = 32{,}000$$

(c) We set $n(t) = 100{,}000$ in the model that we found in part (a) and solve the resulting exponential equation for t:

$$\begin{aligned} 100{,}000 &= 1000 \cdot 2^{t/3} && n(t) = 1000 \cdot 2^{t/3} \\ 100 &= 2^{t/3} && \text{Divide by 1000} \\ \log 100 &= \log 2^{t/3} && \text{Take log of each side} \\ 2 &= \frac{t}{3} \log 2 && \text{Properties of log} \\ t &= \frac{6}{\log 2} \approx 19.93 && \text{Solve for } t \end{aligned}$$

The bacteria level reaches 100,000 in about 20 hours.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **1**

EXAMPLE 2 | Rabbit Population

A certain breed of rabbit was introduced onto a small island 8 months ago. The current rabbit population on the island is estimated to be 4100 and doubling every 3 months.

(a) What was the initial size of the rabbit population?

(b) Estimate the population one year after the rabbits were introduced to the island.

(c) Sketch a graph of the rabbit population.

SOLUTION

(a) The doubling time is $a = 3$, so the population at time t is

$$n(t) = n_0 2^{t/3} \qquad \text{Model}$$

where n_0 is the initial population. Since the population is 4100 when t is 8 months, we have

$$n(8) = n_0 2^{8/3} \qquad \text{From model}$$

$$4100 = n_0 2^{8/3} \qquad \text{Because } n(8) = 4100$$

$$n_0 = \frac{4100}{2^{8/3}} \qquad \text{Divide by } 2^{8/3} \text{ and switch sides}$$

$$n_0 \approx 645 \qquad \text{Calculator}$$

Thus we estimate that 645 rabbits were introduced onto the island.

(b) From part (a) we know that the initial population is $n_0 = 645$, so we can model the population after t months by

$$n(t) = 645 \cdot 2^{t/3} \qquad \text{Model}$$

After one year $t = 12$, so

$$n(12) = 645 \cdot 2^{12/3} \approx 10{,}320$$

So after one year there would be about 10,000 rabbits.

(c) We first note that the domain is $t \geq 0$. The graph is shown in Figure 2.

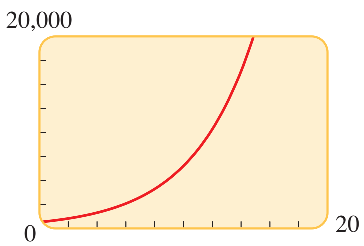


FIGURE 2 $n(t) = 645 \cdot 2^{t/3}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **3**

Exponential Growth (Relative Growth Rate)

We have used an exponential function with base 2 to model population growth (in terms of the doubling time). We could also model the same population with an exponential function with base 3 (in terms of the tripling time). In fact, we can find an exponential model with any base. If we use the base e, we get the following model of a population in terms of the **relative growth rate** r: the rate of population growth expressed as a proportion of the population at any time. For instance, if $r = 0.02$, then at any time t the growth rate is 2% of the population at time t.

EXPONENTIAL GROWTH (RELATIVE GROWTH RATE)

A population that experiences **exponential growth** increases according to the model

$$n(t) = n_0 e^{rt}$$

where $n(t)$ = population at time t

n_0 = initial size of the population

r = relative rate of growth (expressed as a proportion of the population)

t = time

Notice that the formula for population growth is the same as that for continuously compounded interest. In fact, the same principle is at work in both cases: The growth of a population (or an investment) per time period is proportional to the size of the population (or the amount of the investment). A population of 1,000,000 will increase more in one year than a population of 1000; in exactly the same way, an investment of $1,000,000 will increase more in one year than an investment of $1000.

In the following examples we assume that the populations grow exponentially.

EXAMPLE 3 | Predicting the Size of a Population

The initial bacterium count in a culture is 500. A biologist later makes a sample count of bacteria in the culture and finds that the relative rate of growth is 40% per hour.

(a) Find a function that models the number of bacteria after t hours.

(b) What is the estimated count after 10 hours?

(c) After how many hours will the bacteria count reach 80,000?

(d) Sketch the graph of the function $n(t)$.

SOLUTION

(a) We use the exponential growth model with $n_0 = 500$ and $r = 0.4$ to get

$$n(t) = 500e^{0.4t}$$

where t is measured in hours.

(b) Using the function in part (a), we find that the bacterium count after 10 hours is

$$n(10) = 500e^{0.4(10)} = 500e^4 \approx 27{,}300$$

(c) We set $n(t) = 80{,}000$ and solve the resulting exponential equation for t:

$$\begin{aligned} 80{,}000 &= 500 \cdot e^{0.4t} && n(t) = 500 \cdot e^{0.4t} \\ 160 &= e^{0.4t} && \text{Divide by 500} \\ \ln 160 &= 0.4t && \text{Take ln of each side} \\ t &= \frac{\ln 160}{0.4} \approx 12.68 && \text{Solve for } t \end{aligned}$$

The bacteria level reaches 80,000 in about 12.7 hours.

(d) The graph is shown in Figure 3.

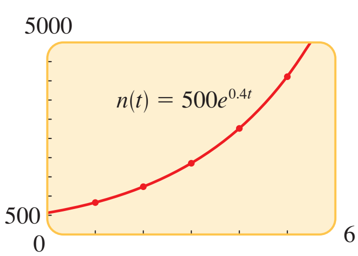


FIGURE 3

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5** ■

The relative growth of world population has been declining over the past few decades—from 2% in 1995 to 1.3% in 2006.

EXAMPLE 4 | Comparing Different Rates of Population Growth

In 2000 the population of the world was 6.1 billion, and the relative rate of growth was 1.4% per year. It is claimed that a rate of 1.0% per year would make a significant difference in the total population in just a few decades. Test this claim by estimating the population of the world in the year 2050 using a relative rate of growth of (a) 1.4% per year and (b) 1.0% per year.

Graph the population functions for the next 100 years for the two relative growth rates in the same viewing rectangle.

> **Standing Room Only**
> The population of the world was about 6.1 billion in 2000 and was increasing at 1.4% per year. Assuming that each person occupies an average of 4 ft^2 of the surface of the earth, the exponential model for population growth projects that by the year 2801 there will be standing room only! (The total land surface area of the world is about 1.8×10^{15} ft^2.)

SOLUTION

(a) By the exponential growth model we have

$$n(t) = 6.1e^{0.014t}$$

where $n(t)$ is measured in billions and t is measured in years since 2000. Because the year 2050 is 50 years after 2000, we find

$$n(50) = 6.1e^{0.014(50)} = 6.1e^{0.7} \approx 12.3$$

The estimated population in the year 2050 is about 12.3 billion.

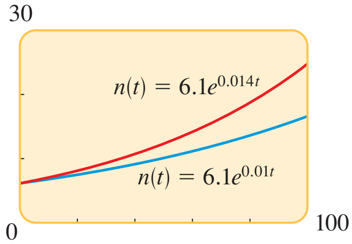


FIGURE 4

(b) We use the function

$$n(t) = 6.1e^{0.010t}$$

and find

$$n(50) = 6.1e^{0.010(50)} = 6.1e^{0.50} \approx 10.1$$

The estimated population in the year 2050 is about 10.1 billion.

The graphs in Figure 4 show that a small change in the relative rate of growth will, over time, make a large difference in population size.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **7**

EXAMPLE 5 | Expressing a Model in Terms of e

A culture starts with 10,000 bacteria, and the number doubles every 40 minutes.

(a) Find a function $n(t) = n_0 2^{t/a}$ that models the number of bacteria after t minutes.

(b) Find a function $n(t) = n_0 e^{rt}$ that models the number of bacteria after t minutes.

(c) Sketch a graph of the number of bacteria at time t.

SOLUTION

(a) The initial population is $n_0 = 10{,}000$. The doubling time is $a = 40$ min $= 2/3$ h. Since $1/a = 3/2 = 1.5$, the model is

$$n(t) = 10{,}000 \cdot 2^{1.5t}$$

(b) The initial population is $n_0 = 10{,}000$. We need to find the relative growth rate r. Since there are 20,000 bacteria when $t = 2/3$ h, we have

$$
\begin{aligned}
20{,}000 &= 10{,}000e^{r(2/3)} && n(t) = 10{,}000e^{rt} \\
2 &= e^{r(2/3)} && \text{Divide by 10,000} \\
\ln 2 &= \ln e^{r(2/3)} && \text{Take ln of each side} \\
\ln 2 &= r(2/3) && \text{Property of ln} \\
r &= \frac{3 \ln 2}{2} \approx 1.0397 && \text{Solve for } r
\end{aligned}
$$

Now that we know the relative growth rate r, we can find the model:

$$n(t) = 10{,}000e^{1.0397t}$$

(c) We can graph the model in part (a) or the one in part (b). The graphs are identical. See Figure 5.

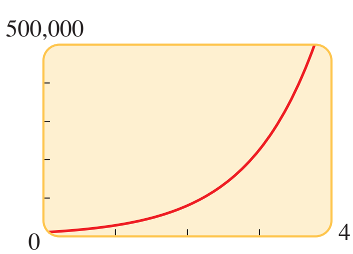


FIGURE 5
Graphs of $y = 10{,}000 \cdot 2^{1.5t}$ and $y = 10{,}000e^{1.0397t}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **9**

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. The rate of decay is proportional to the mass of the substance. This is analogous to population growth except that the mass *decreases*. Physicists express the rate of decay in terms of **half-life**. For example, the half-life of radium-226 is 1600 years, so a 100-g sample decays to 50 g (or $\frac{1}{2} \times 100$ g) in 1600 years, then to 25 g (or $\frac{1}{2} \times \frac{1}{2} \times 100$ g) in 3200 years, and so on. In

The half-lives of **radioactive elements** vary from very long to very short. Here are some examples.

Element	Half-life
Thorium-232	14.5 billion years
Uranium-235	4.5 billion years
Thorium-230	80,000 years
Plutonium-239	24,360 years
Carbon-14	5,730 years
Radium-226	1,600 years
Cesium-137	30 years
Strontium-90	28 years
Polonium-210	140 days
Thorium-234	25 days
Iodine-135	8 days
Radon-222	3.8 days
Lead-211	3.6 minutes
Krypton-91	10 seconds

general, for a radioactive substance with mass m_0 and half-life h, the amount remaining at time t is modeled by

$$\boxed{m(t) = m_0 2^{-t/h}}$$

where h and t are measured in the same time units (minutes, hours, days, years, and so on).

To express this model in the form $m(t) = m_0 e^{rt}$, we need to find the relative decay rate r. Since h is the half-life, we have

$$m(t) = m_0 e^{-rt} \qquad \text{Model}$$

$$\frac{m_0}{2} = m_0 e^{-rh} \qquad h \text{ is the half-life}$$

$$\frac{1}{2} = e^{-rh} \qquad \text{Divide by } m_0$$

$$\ln\frac{1}{2} = -rh \qquad \text{Take ln of each side}$$

$$r = \frac{\ln 2}{h} \qquad \text{Solve for } r$$

This last equation allows us to find the rate r from the half-life h.

RADIOACTIVE DECAY MODEL

If m_0 is the initial mass of a radioactive substance with half-life h, then the mass remaining at time t is modeled by the function

$$m(t) = m_0 e^{-rt}$$

where $r = \dfrac{\ln 2}{h}$.

EXAMPLE 6 | Radioactive Decay

Polonium-210 (^{210}Po) has a half-life of 140 days. Suppose a sample of this substance has a mass of 300 mg.

(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t days.

(b) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t days.

(c) Find the mass remaining after one year.

(d) How long will it take for the sample to decay to a mass of 200 mg?

(e) Draw a graph of the sample mass as a function of time.

SOLUTION

(a) We have $m_0 = 300$ and $h = 140$, so the amount remaining after t days is

$$m(t) = 300 \cdot 2^{-t/140}$$

(b) We have $m_0 = 300$ and $r = \ln 2/140 \approx -0.00495$, so the amount remaining after t days is

$$m(t) = 300 \cdot e^{-0.00495t}$$

In parts (c) and (d) we can also use the model found in part (a). Check that the result is the same using either model.

(c) We use the function we found in part (a) with $t = 365$ (one year):

$$m(365) = 300e^{-0.00495(365)} \approx 49.256$$

Thus approximately 49 mg of ^{210}Po remains after one year.

Radioactive Waste

Harmful radioactive isotopes are produced whenever a nuclear reaction occurs, whether as the result of an atomic bomb test, a nuclear accident such as the one at Chernobyl in 1986, or the uneventful production of electricity at a nuclear power plant.

One radioactive material that is produced in atomic bombs is the isotope strontium-90 (^{90}Sr), with a half-life of 28 years. This is deposited like calcium in human bone tissue, where it can cause leukemia and other cancers. However, in the decades since atmospheric testing of nuclear weapons was halted, ^{90}Sr levels in the environment have fallen to a level that no longer poses a threat to health.

Nuclear power plants produce radioactive plutonium-239 (^{239}Pu), which has a half-life of 24,360 years. Because of its long half-life, ^{239}Pu could pose a threat to the environment for thousands of years. So great care must be taken to dispose of it properly. The difficulty of ensuring the safety of the disposed radioactive waste is one reason that nuclear power plants remain controversial.

(d) We use the function that we found in part (b) with $m(t) = 200$ and solve the resulting exponential equation for t:

$$
\begin{aligned}
300e^{-0.00495t} &= 200 && m(t) = m_0e^{-rt} \\
e^{-0.00495t} &= \tfrac{2}{3} && \text{Divided by 300} \\
\ln e^{-0.00495t} &= \ln \tfrac{2}{3} && \text{Take ln of each side} \\
-0.00495t &= \ln \tfrac{2}{3} && \text{Property of ln} \\
t &= -\frac{\ln \frac{2}{3}}{0.00495} && \text{Solve for } t \\
t &\approx 81.9 && \text{Calculator}
\end{aligned}
$$

The time required for the sample to decay to 200 mg is about 82 days.

(e) We can graph the model in part (a) or the one in part (b). The graphs are identical. See Figure 6.

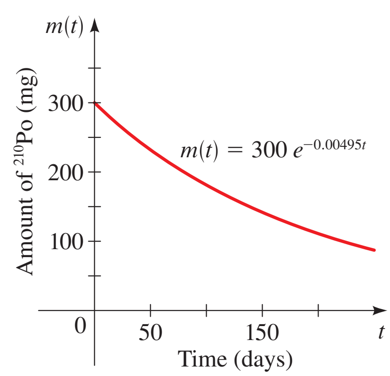


FIGURE 6

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17** ■

▼ Newton's Law of Cooling

Newton's Law of Cooling states that the rate at which an object cools is proportional to the temperature difference between the object and its surroundings, provided that the temperature difference is not too large. By using calculus, the following model can be deduced from this law.

NEWTON'S LAW OF COOLING

If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_s, then the temperature of the object at time t is modeled by the function

$$T(t) = T_s + D_0e^{-kt}$$

where k is a positive constant that depends on the type of object.

EXAMPLE 7 | Newton's Law of Cooling

A cup of coffee has a temperature of 200°F and is placed in a room that has a temperature of 70°F. After 10 min the temperature of the coffee is 150°F.

(a) Find a function that models the temperature of the coffee at time t.

(b) Find the temperature of the coffee after 15 min.

(c) After how long will the coffee have cooled to 100°F?

(d) Illustrate by drawing a graph of the temperature function.

SOLUTION

(a) The temperature of the room is $T_s = 70°\text{F}$, and the initial temperature difference is

$$D_0 = 200 - 70 = 130°\text{F}$$

So by Newton's Law of Cooling, the temperature after t minutes is modeled by the function

$$T(t) = 70 + 130e^{-kt}$$

We need to find the constant k associated with this cup of coffee. To do this, we use the fact that when $t = 10$, the temperature is $T(10) = 150$. So we have

$$\begin{aligned} 70 + 130e^{-10k} &= 150 && T_s + D_0e^{-kt} = T(t) \\ 130e^{-10k} &= 80 && \text{Subtract 70} \\ e^{-10k} &= \tfrac{8}{13} && \text{Divide by 130} \\ -10k &= \ln \tfrac{8}{13} && \text{Take ln of each side} \\ k &= -\tfrac{1}{10} \ln \tfrac{8}{13} && \text{Solve for } k \\ k &\approx 0.04855 && \text{Calculator} \end{aligned}$$

Substituting this value of k into the expression for $T(t)$, we get

$$T(t) = 70 + 130e^{-0.04855t}$$

(b) We use the function that we found in part (a) with $t = 15$.

$$T(15) = 70 + 130e^{-0.04855(15)} \approx 133°\text{F}$$

(c) We use the function that we found in part (a) with $T(t) = 100$ and solve the resulting exponential equation for t:

$$\begin{aligned} 70 + 130e^{-0.04855t} &= 100 && T_s + D_0e^{-kt} = T(t) \\ 130e^{-0.04855t} &= 30 && \text{Subtract 70} \\ e^{-0.04855t} &= \tfrac{3}{13} && \text{Divide by 130} \\ -0.04855t &= \ln \tfrac{3}{13} && \text{Take ln of each side} \\ t &= \frac{\ln \frac{3}{13}}{-0.04855} && \text{Solve for } t \\ t &\approx 30.2 && \text{Calculator} \end{aligned}$$

The coffee will have cooled to 100°F after about half an hour.

(d) The graph of the temperature function is sketched in Figure 7. Notice that the line $t = 70$ is a horizontal asymptote. (Why?)

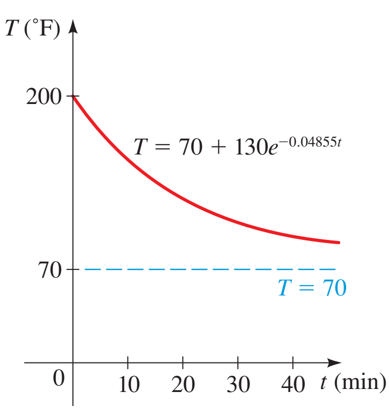


FIGURE 7 Temperature of coffee after t minutes

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **25**

Logarithmic Scales

When a physical quantity varies over a very large range, it is often convenient to take its logarithm in order to work with more manageable numbers. On a **logarithmic scale**, num-

Animal	W (kg)	$\log W$
Ant	0.000003	−5.5
Elephant	4000	3.6
Whale	170,000	5.2

bers are represented by their logarithms. For example, the table in the margin gives the weights W of some animals (in kilograms) and their logarithms ($\log W$).

The weights (W) vary enormously, but on a logarithmic scale, the weights are represented by more manageable numbers ($\log W$). Figure 8 shows that it is difficult to compare the weights W graphically but easy to compare them on a logarithmic scale.

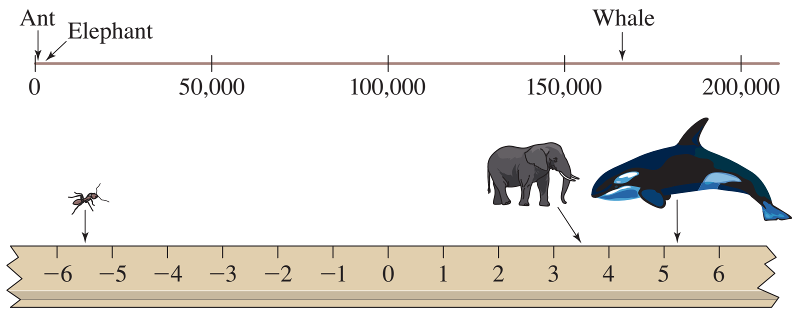


FIGURE 8 Weight graphed on the real line (top) and on a logarithmic scale (bottom)

We discuss three commonly used logarithmic scales: the pH scale, which measures acidity; the Richter scale, which measures the intensity of earthquakes; and the decibel scale, which measures the loudness of sounds. Other quantities that are measured on logarithmic scales are light intensity, information capacity, and radiation.

pH for Some Common Substances

Substance	pH
Milk of magnesia	10.5
Seawater	8.0–8.4
Human blood	7.3–7.5
Crackers	7.0–8.5
Hominy	6.9–7.9
Cow's milk	6.4–6.8
Spinach	5.1–5.7
Tomatoes	4.1–4.4
Oranges	3.0–4.0
Apples	2.9–3.3
Limes	1.3–2.0
Battery acid	1.0

The pH Scale Chemists measured the acidity of a solution by giving its hydrogen ion concentration until Søren Peter Lauritz Sørensen, in 1909, proposed a more convenient measure. He defined

$$\boxed{\mathrm{pH} = -\log[\mathrm{H}^+]}$$

where $[\mathrm{H}^+]$ is the concentration of hydrogen ions measured in moles per liter (M). He did this to avoid very small numbers and negative exponents. For instance,

$$\text{if} \qquad [\mathrm{H}^+] = 10^{-4}\,\mathrm{M}, \qquad \text{then} \qquad \mathrm{pH} = -\log_{10}(10^{-4}) = -(-4) = 4$$

Solutions with a pH of 7 are defined as *neutral*, those with pH < 7 are *acidic*, and those with pH > 7 are *basic*. Notice that when the pH increases by one unit, $[\mathrm{H}^+]$ decreases by a factor of 10.

EXAMPLE 8 | pH Scale and Hydrogen Ion Concentration

(a) The hydrogen ion concentration of a sample of human blood was measured to be $[\mathrm{H}^+] = 3.16 \times 10^{-8}$ M. Find the pH, and classify the blood as acidic or basic.

(b) The most acidic rainfall ever measured occurred in Scotland in 1974; its pH was 2.4. Find the hydrogen ion concentration.

SOLUTION

(a) A calculator gives

$$\mathrm{pH} = -\log[\mathrm{H}^+] = -\log(3.16 \times 10^{-8}) \approx 7.5$$

Since this is greater than 7, the blood is basic.

(b) To find the hydrogen ion concentration, we need to solve for $[\mathrm{H}^+]$ in the logarithmic equation

$$\log[\mathrm{H}^+] = -\mathrm{pH}$$

So we write it in exponential form:

$$[\mathrm{H}^+] = 10^{-\mathrm{pH}}$$

In this case pH $= 2.4$, so

$$[\mathrm{H}^+] = 10^{-2.4} \approx 4.0 \times 10^{-3}\,\mathrm{M}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **29** AND **31**

Largest Earthquakes

Location	Date	Magnitude
Chile	1960	9.5
Alaska	1964	9.2
Japan	2011	9.1
Sumatra	2004	9.1
Alaska	1957	9.1
Kamchatka	1952	9.0
Chile	2010	8.8
Ecuador	1906	8.8
Alaska	1965	8.7
Sumatra	2005	8.7
Tibet	1950	8.6
Kamchatka	1923	8.5
Indonesia	1938	8.5

The Richter Scale In 1935 the American geologist Charles Richter (1900–1984) defined the magnitude M of an earthquake to be

$$\boxed{M = \log \frac{I}{S}}$$

where I is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake) and S is the intensity of a "standard" earthquake (whose amplitude is 1 micron $= 10^{-4}$ cm). The magnitude of a standard earthquake is

$$M = \log \frac{S}{S} = \log 1 = 0$$

Richter studied many earthquakes that occurred between 1900 and 1950. The largest had magnitude 8.9 on the Richter scale, and the smallest had magnitude 0. This corresponds to a ratio of intensities of 800,000,000, so the Richter scale provides more manageable numbers to work with. For instance, an earthquake of magnitude 6 is ten times stronger than an earthquake of magnitude 5.

EXAMPLE 9 | Magnitude of Earthquakes

The 1906 earthquake in San Francisco had an estimated magnitude of 8.3 on the Richter scale. In the same year a powerful earthquake occurred on the Colombia-Ecuador border that was four times as intense. What was the magnitude of the Colombia-Ecuador earthquake on the Richter scale?

SOLUTION If I is the intensity of the San Francisco earthquake, then from the definition of magnitude we have

$$M = \log \frac{I}{S} = 8.3$$

The intensity of the Colombia-Ecuador earthquake was $4I$, so its magnitude was

$$M = \log \frac{4I}{S} = \log 4 + \log \frac{I}{S} = \log 4 + 8.3 \approx 8.9$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **35** ■

EXAMPLE 10 | Intensity of Earthquakes

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The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. How many times more intense was the 1906 earthquake (see Example 9) than the 1989 event?

SOLUTION If I_1 and I_2 are the intensities of the 1906 and 1989 earthquakes, then we are required to find I_1/I_2. To relate this to the definition of magnitude, we divide the numerator and denominator by S:

$$\log \frac{I_1}{I_2} = \log \frac{I_1/S}{I_2/S} \qquad \text{Divide numerator and denominator by } S$$

$$= \log \frac{I_1}{S} - \log \frac{I_2}{S} \qquad \text{Law 2 of logarithms}$$

$$= 8.3 - 7.1 = 1.2 \qquad \text{Definition of earthquake magnitude}$$

Therefore

$$\frac{I_1}{I_2} = 10^{\log(I_1/I_2)} = 10^{1.2} \approx 16$$

The 1906 earthquake was about 16 times as intense as the 1989 earthquake.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **37** ■

The Decibel Scale The ear is sensitive to an extremely wide range of sound intensities. We take as a reference intensity $I_0 = 10^{-12}\ \mathrm{W/m^2}$ (watts per square meter) at a frequency of 1000 hertz, which measures a sound that is just barely audible (the threshold of hearing). The psychological sensation of loudness varies with the logarithm of the intensity (the Weber-Fechner Law), so the **intensity level** B, measured in decibels (dB), is defined as

$$\boxed{B = 10 \log \frac{I}{I_0}}$$

The intensity level of the barely audible reference sound is

$$B = 10 \log \frac{I_0}{I_0} = 10 \log 1 = 0 \text{ dB}$$

The **intensity levels of sounds** that we can hear vary from very loud to very soft. Here are some examples of the decibel levels of commonly heard sounds.

Source of sound	B (dB)
Jet takeoff	140
Jackhammer	130
Rock concert	120
Subway	100
Heavy traffic	80
Ordinary traffic	70
Normal conversation	50
Whisper	30
Rustling leaves	10–20
Threshold of hearing	0

EXAMPLE 11 | Sound Intensity of a Jet Takeoff

Find the decibel intensity level of a jet engine during takeoff if the intensity was measured at $100\ \mathrm{W/m^2}$.

SOLUTION From the definition of intensity level we see that

$$B = 10 \log \frac{I}{I_0} = 10 \log \frac{10^2}{10^{-12}} = 10 \log 10^{14} = 140 \text{ dB}$$

Thus the intensity level is 140 dB. ■

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **39**

The table in the margin lists decibel intensity levels for some common sounds ranging from the threshold of human hearing to the jet takeoff of Example 11. The threshold of pain is about 120 dB.

4.6 EXERCISES

APPLICATIONS

1–16 ■ These exercises use the population growth model.

1. **Bacteria Culture** A certain culture of the bacterium *Streptococcus A* initially has 10 bacteria and is observed to double every 1.5 hours.
   (a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the number of bacteria in the culture after t hours.
   (b) Estimate the number of bacteria after 35 hours.
   (c) After how many hours will the bacteria count reach 10,000?

Streptococcus A
(12,000 × magnification)

2. **Bacteria Culture** A certain culture of the bacterium *Rhodobacter sphaeroides* initially has 25 bacteria and is observed to double every 5 hours.
   (a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the number of bacteria in the culture after t hours.
   (b) Estimate the number of bacteria after 18 hours.
   (c) After how many hours will the bacteria count reach 1 million?

3. **Squirrel Population** A grey squirrel population was introduced in a certain county of Great Britain 30 years ago. Biologists observe that the population doubles every 6 years, and now the population is 100,000.
   (a) What was the initial size of the squirrel population?
   (b) Estimate the squirrel population 10 years from now.
   (c) Sketch a graph of the squirrel population.

4. **Bird Population** A certain species of bird was introduced in a certain county 25 years ago. Biologists observe that the population doubles every 10 years, and now the population is 13,000.
   (a) What was the initial size of the bird population?
   (b) Estimate the bird population 5 years from now.
   (c) Sketch a graph of the bird population.

5. **Fox Population** The fox population in a certain region has a relative growth rate of 8% per year. It is estimated that the population in 2005 was 18,000.
   (a) Find a function $n(t) = n_0e^{rt}$ that models the population t years after 2005.
   (b) Use the function from part (a) to estimate the fox population in the year 2013.
   (c) After how many years will the fox population reach 25,000?
   (d) Sketch a graph of the fox population function for the years 2005–2013.

6. **Fish Population** The population of a certain species of fish has a relative growth rate of 1.2% per year. It is estimated that the population in 2000 was 12 million.
   (a) Find an exponential model $n(t) = n_0e^{rt}$ for the population t years after 2000.
   (b) Estimate the fish population in the year 2005.
   (c) After how many years will the fish population reach 14 million?
   (d) Sketch a graph of the fish population.

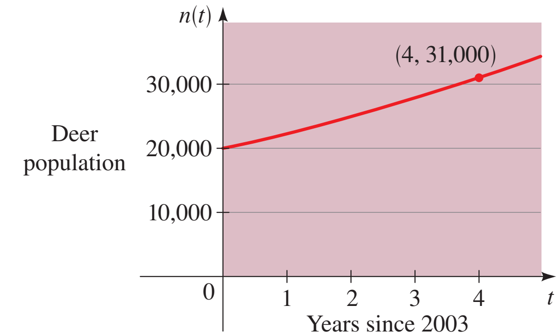
7. **Population of a Country** The population of a country has a relative growth rate of 3% per year. The government is trying to reduce the growth rate to 2%. The population in 1995 was approximately 110 million. Find the projected population for the year 2020 for the following conditions.
   (a) The relative growth rate remains at 3% per year.
   (b) The relative growth rate is reduced to 2% per year.

8. **Bacteria Culture** It is observed that a certain bacteria culture has a relative growth rate of 12% per hour, but in the presence of an antibiotic the relative growth rate is reduced to 5% per hour. The initial number of bacteria in the culture is 22. Find the projected population after 24 hours for the following conditions.
   (a) No antibiotic is present, so the relative growth rate is 12%.
   (b) An antibiotic is present in the culture, so the relative growth rate is reduced to 5%.

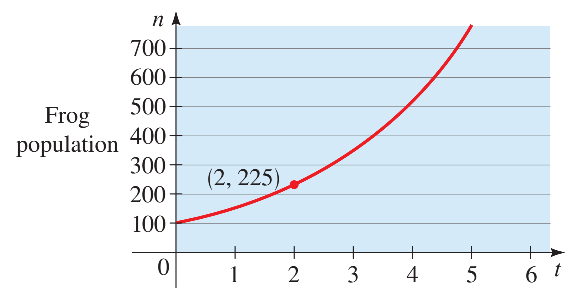
9. **Population of a City** The population of a certain city was 112,000 in 2006, and the observed doubling time for the population is 18 years.
   (a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the population t years after 2006.
   (b) Find an exponential model $n(t) = n_0e^{rt}$ for the population t years after 2006.
   (c) Sketch a graph of the population at time t.
   (d) Estimate how long it takes the population to reach 500,000.

10. **Bat Population** The bat population in a certain Midwestern county was 350,000 in 2009, and the observed doubling time for the population is 25 years.
   (a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the population t years after 2006.
   (b) Find an exponential model $n(t) = n_0e^{rt}$ for the population t years after 2006.
   (c) Sketch a graph of the population at time t.
   (d) Estimate how long it takes the population to reach 2 million.

11. **Deer Population** The graph shows the deer population in a Pennsylvania county between 2003 and 2007. Assume that the population grows exponentially.
   (a) What was the deer population in 2003?
   (b) Find a function that models the deer population t years after 2003.
   (c) What is the projected deer population in 2011?
   (d) Estimate how long it takes the population to reach 100,000.



12. **Frog Population** Some bullfrogs were introduced into a small pond. The graph shows the bullfrog population for the next few years. Assume that the population grows exponentially.
   (a) What was the initial bullfrog population?
   (b) Find a function that models the bullfrog population t years since the bullfrogs were put into the pond.
   (c) What is the projected bullfrog population after 15 years?
   (d) Estimate how long it takes the population to reach 75,000.



13. **Bacteria Culture** A culture starts with 8600 bacteria. After one hour the count is 10,000.
   (a) Find a function that models the number of bacteria $n(t)$ after t hours.
   (b) Find the number of bacteria after 2 hours.
   (c) After how many hours will the number of bacteria double?

14. **Bacteria Culture** The count in a culture of bacteria was 400 after 2 hours and 25,600 after 6 hours.
   (a) What is the relative rate of growth of the bacteria population? Express your answer as a percentage.
   (b) What was the initial size of the culture?
   (c) Find a function that models the number of bacteria $n(t)$ after t hours.
   (d) Find the number of bacteria after 4.5 hours.
   (e) After how many hours will the number of bacteria reach 50,000?

15. **Population of California** The population of California was 29.76 million in 1990 and 33.87 million in 2000. Assume that the population grows exponentially.
   (a) Find a function that models the population t years after 1990.
   (b) Find the time required for the population to double.
   (c) Use the function from part (a) to predict the population of California in the year 2010. Look up California's actual population in 2010, and compare.

16. World Population The population of the world was 5.7 billion in 1995, and the observed relative growth rate was 2% per year.
(a) Estimate how long it takes the population to double.
(b) Estimate how long it takes the population to triple.

17–24 ■ These exercises use the radioactive decay model.

17. Radioactive Radium The half-life of radium-226 is 1600 years. Suppose we have a 22-mg sample.
(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t years.
(b) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t years.
(c) How much of the sample will remain after 4000 years?
(d) After how many years will only 18 mg of the sample remain?

18. Radioactive Cesium The half-life of cesium-137 is 30 years. Suppose we have a 10-g sample.
(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t years.
(b) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t years.
(c) How much of the sample will remain after 80 years?
(d) After how many years will only 2 g of the sample remain?

19. Radioactive Strontium The half-life of strontium-90 is 28 years. How long will it take a 50-mg sample to decay to a mass of 32 mg?

20. Radioactive Radium Radium-221 has a half-life of 30 s. How long will it take for 95% of a sample to decay?

21. Finding Half-life If 250 mg of a radioactive element decays to 200 mg in 48 hours, find the half-life of the element.

22. Radioactive Radon After 3 days a sample of radon-222 has decayed to 58% of its original amount.
(a) What is the half-life of radon-222?
(b) How long will it take the sample to decay to 20% of its original amount?

23. Carbon-14 Dating A wooden artifact from an ancient tomb contains 65% of the carbon-14 that is present in living trees. How long ago was the artifact made? (The half-life of carbon-14 is 5730 years.)

24. Carbon-14 Dating The burial cloth of an Egyptian mummy is estimated to contain 59% of the carbon-14 it contained originally. How long ago was the mummy buried? (The half-life of carbon-14 is 5730 years.)

25–28 ■ These exercises use Newton's Law of Cooling.

25. Cooling Soup A hot bowl of soup is served at a dinner party. It starts to cool according to Newton's Law of Cooling, so its temperature at time t is given by

$$T(t) = 65 + 145e^{-0.05t}$$

where t is measured in minutes and T is measured in °F.
(a) What is the initial temperature of the soup?
(b) What is the temperature after 10 min?
(c) After how long will the temperature be 100°F?

26. Time of Death Newton's Law of Cooling is used in homicide investigations to determine the time of death. The normal body temperature is 98.6°F. Immediately following death, the body begins to cool. It has been determined experimentally that the constant in Newton's Law of Cooling is approximately $k = 0.1947$, assuming that time is measured in hours. Suppose that the temperature of the surroundings is 60°F.
(a) Find a function $T(t)$ that models the temperature t hours after death.
(b) If the temperature of the body is now 72°F, how long ago was the time of death?

27. Cooling Turkey A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.
(a) If the temperature of the turkey is 150°F after half an hour, what is its temperature after 45 min?
(b) After how many hours will the turkey cool to 100°F?

28. Boiling Water A kettle full of water is brought to a boil in a room with temperature 20°C. After 15 min the temperature of the water has decreased from 100°C to 75°C. Find the temperature after another 10 min. Illustrate by graphing the temperature function.

29–42 ■ These exercises deal with logarithmic scales.

29. Finding pH The hydrogen ion concentration of a sample of each substance is given. Calculate the pH of the substance.
(a) Lemon juice: $[H^+] = 5.0 \times 10^{-3}$ M
(b) Tomato juice: $[H^+] = 3.2 \times 10^{-4}$ M
(c) Seawater: $[H^+] = 5.0 \times 10^{-9}$ M

30. Finding pH An unknown substance has a hydrogen ion concentration of $[H^+] = 3.1 \times 10^{-8}$ M. Find the pH and classify the substance as acidic or basic.

31. Ion Concentration The pH reading of a sample of each substance is given. Calculate the hydrogen ion concentration of the substance.
(a) Vinegar: pH = 3.0
(b) Milk: pH = 6.5

32. Ion Concentration The pH reading of a glass of liquid is given. Find the hydrogen ion concentration of the liquid.
(a) Beer: pH = 4.6
(b) Water: pH = 7.3

33. Finding pH The hydrogen ion concentrations in cheeses range from 4.0×10^{-7} M to 1.6×10^{-5} M. Find the corresponding range of pH readings.

34. Ion Concentration in Wine The pH readings for wines vary from 2.8 to 3.8. Find the corresponding range of hydrogen ion concentrations.

35. Earthquake Magnitudes If one earthquake is 20 times as intense as another, how much larger is its magnitude on the Richter scale?

36. Earthquake Magnitudes The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time in Japan an earthquake with magnitude 4.9 caused only minor damage. How many times more intense was the San Francisco earthquake than the Japan earthquake?

37. Earthquake Magnitudes The Japan earthquake of 2011 had a magnitude of 9.1 on the Richter scale. How many times more intense was this than the 1906 San Francisco earthquake? (See Exercise 36.)

38. Earthquake Magnitudes The Northridge, California, earthquake of 1994 had a magnitude of 6.8 on the Richter scale. A year later, a 7.2-magnitude earthquake struck Kobe, Japan. How many times more intense was the Kobe earthquake than the Northridge earthquake?

39. Traffic Noise The intensity of the sound of traffic at a busy intersection was measured at 2.0×10^{-5} W/m^2. Find the intensity level in decibels.

40. Subway Noise The intensity of the sound of a subway train was measured at 98 dB. Find the intensity in W/m^2.

41. Comparing Decibel Levels The noise from a power mower was measured at 106 dB. The noise level at a rock concert was measured at 120 dB. Find the ratio of the intensity of the rock music to that of the power mower.

42. Inverse Square Law for Sound A law of physics states that the intensity of sound is inversely proportional to the square of the distance d from the source: $I = k/d^2$.

(a) Use this model and the equation

$$B = 10 \log \frac{I}{I_0}$$

(described in this section) to show that the decibel levels B_1 and B_2 at distances d_1 and d_2 from a sound source are related by the equation

$$B_2 = B_1 + 20 \log \frac{d_1}{d_2}$$

(b) The intensity level at a rock concert is 120 dB at a distance 2 m from the speakers. Find the intensity level at a distance of 10 m.

CHAPTER 4 | REVIEW

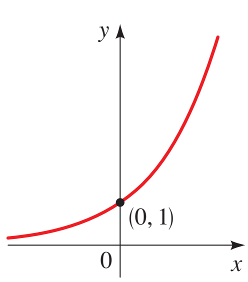
PROPERTIES AND FORMULAS

Exponential Functions (pp. 346–348)

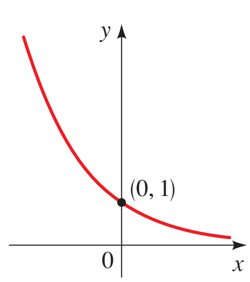
The **exponential function** f with base a (where $a > 0$, $a \neq 1$) is defined for all real numbers x by

$$f(x) = a^x$$

The domain of f is $\mathbb{R}$, and the range of f is $(0, \infty)$ The graph of f has one of the following shapes, depending on the value of a:



$f(x) = a^x$ for $a > 1$



$f(x) = a^x$ for $0 < a < 1$

The Natural Exponential Function (p. 354)

The **natural exponential function** is the exponential function with base e:

$$f(x) = e^x$$

The number e is defined to be the number that the expression $(1 + 1/n)^n$ approaches as $n \to \infty$. An approximate value for the irrational number e is

$$e \approx 2.7182818284590\ldots$$

Compound Interest (pp. 350, 356)

If a principal P is invested in an account paying an annual interest rate r, compounded n times a year, then after t years the **amount** $A(t)$ in the account is

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

If the interest is compounded **continuously**, then the amount is

$$A(t) = Pe^{rt}$$

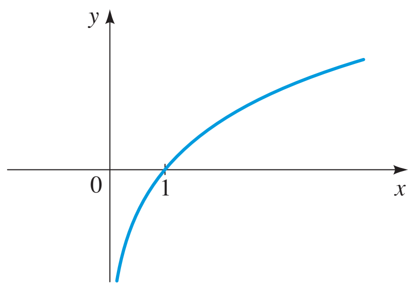
Logarithmic Functions (pp. 359–361)

The **logarithmic function** $\log_a$ with base a (where $a > 0$, $a \neq 1$) is defined for $x > 0$ by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

So $\log_a x$ is the exponent to which the base a must be raised to give y.

The domain of $\log_a$ is $(0, \infty)$, and the range is $\mathbb{R}$. For $a > 1$, the graph of the function $\log_a$ has the following shape:



$y = \log_a x$, $a > 1$

Common and Natural Logarithms (pp. 363–365)

The logarithm function with base 10 is called the **common logarithm** and is denoted **log**. So

$$\log x = \log_{10} x$$

The logarithm function with base e is called the **natural logarithm** and is denoted **ln**. So

$$\ln x = \log_e x$$

Properties of Logarithms (pp. 360, 365)

1. $\log_a 1 = 0$ **2.** $\log_a a = 1$

3. $\log_a a^x = x$ **4.** $a^{\log_a x} = x$

Laws of Logarithms (p. 369)

Let a be a logarithm base ($a > 0$, $a \neq 1$), and let A, B, and C be any real numbers or algebraic expressions that represent real numbers, with $A > 0$ and $B > 0$. Then:

1. $\log_a(AB) = \log_a A + \log_a B$

2. $\log_a(A/B) = \log_a A - \log_a B$

3. $\log_a(A^C) = C \log_a A$

Change of Base Formula (p. 372)

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Guidelines for Solving Exponential Equations (p. 376)

1. Isolate the exponential term on one side of the equation.

2. Take the logarithm of each side, and use the Laws of Logarithms to "bring down the exponent."

3. Solve for the variable.

Guidelines for Solving Logarithmic Equations (p. 378)

1. Isolate the logarithmic term(s) on one side of the equation, and use the Laws of Logarithms to combine logarithmic terms if necessary.

2. Rewrite the equation in exponential form.

3. Solve for the variable.

Exponential Growth Model (p. 386)

A population experiences **exponential growth** if it can be modeled by the exponential function

$$n(t) = n_0 e^{rt}$$

where $n(t)$ is the population at time t, n_0 is the initial population (at time $t = 0$), and r is the relative growth rate (expressed as a proportion of the population).

Radioactive Decay Model (p. 389)

If a **radioactive substance** with half-life h has initial mass m_0, then at time t the mass $m(t)$ of the substance that remains is modeled by the exponential function

$$m(t) = m_0 e^{-rt}$$

where $r = \dfrac{\ln 2}{h}$.

Newton's Law of Cooling (p. 390)

If an object has an initial temperature that is D_0 degrees warmer than the surrounding temperature T_s, then at time t the temperature $T(t)$ of the object is modeled by the function

$$T(t) = T_s + D_0 e^{-kt}$$

where the constant $k > 0$ depends on the size and type of the object.

Logarithmic Scales (pp. 391–394)

The **pH scale** measures the acidity of a solution:

$$\text{pH} = -\log[\text{H}^+]$$

The **Richter scale** measures the intensity of earthquakes:

$$M = \log \frac{I}{S}$$

The **decibel scale** measures the intensity of sound:

$$B = 10 \log \frac{I}{I_0}$$

LEARNING OBJECTIVES SUMMARY

Section	After completing this chapter, you should be able to . . .	Review Exercises
4.1	▪ Evaluate exponential functions	1–2
	▪ Graph exponential functions	5–8, 75–76, 82, 83
	▪ Calculate compound interest	91–92
4.2	▪ Evaluate the natural exponential function	3–4
	▪ Graph the natural exponential function	9–10
	▪ Calculate continuously compounded interest	91–92

4.3	▪ Evaluate logarithmic functions	29–44
	▪ Graph logarithmic functions	11–16, 17–20, 77–78, 81, 84
	▪ Change between logarithmic and exponential forms of an expression	21–28
	▪ Use basic properties of logarithms	29–37
	▪ Use common and natural logarithms	23–24, 27–28, 31–33, 38–39, 44
4.4	▪ Use the Laws of Logarithms to evaluate logarithmic expressions	38–44
	▪ Use the Laws of Logarithms to expand logarithmic expressions	45–50
	▪ Use the Laws of Logarithms to combine logarithmic expressions	51–56
	▪ Use the Change of Base Formula	85–88
4.5	▪ Solve exponential equations	57–64, 71–76, 80
	▪ Solve logarithmic equations	65–70, 79
	▪ Solve problems involving compound interest	92–94
	▪ Calculate annual percentage yield	95–96
4.6	▪ Find exponential models of population growth	97–98, 103
	▪ Find exponential models of radioactive decay	99–102
	▪ Find models using Newton's Law of Cooling	104
	▪ Use logarithmic scales (pH, Richter, and decibel scales)	105–108

EXERCISES

1–4 ▪ Use a calculator to find the indicated values of the exponential function, rounded to three decimal places.

1. $f(x) = 5^x$; $f(-1.5), f(\sqrt{2}), f(2.5)$

2. $f(x) = 3 \cdot 2^x$; $f(-2.2), f(\sqrt{7}), f(5.5)$

3. $g(x) = 4e^{x-2}$; $g(-0.7), g(1), g(\pi)$

4. $g(x) = \frac{7}{4}e^{x+1}$; $g(-2), g(\sqrt{3}), g(3.6)$

5–16 ▪ Sketch the graph of the function. State the domain, range, and asymptote.

5. $f(x) = 3^{x-2}$ **6.** $f(x) = 2^{-x+1}$

7. $g(x) = 3 + 2^x$ **8.** $g(x) = 5^{-x} - 5$

9. $F(x) = e^{x-1} + 1$ **10.** $G(x) = -e^{x+1} - 2$

11. $f(x) = \log_3(x - 1)$ **12.** $g(x) = \log(-x)$

13. $f(x) = 2 - \log_2 x$ **14.** $f(x) = 3 + \log_5(x + 4)$

15. $g(x) = 2 \ln x$ **16.** $g(x) = \ln(x^2)$

17–20 ▪ Find the domain of the function.

17. $f(x) = 10^{x^2} + \log(1 - 2x)$

18. $g(x) = \log(2 + x - x^2)$

19. $h(x) = \ln(x^2 - 4)$

20. $k(x) = \ln|x|$

21–24 ▪ Write the equation in exponential form.

21. $\log_2 1024 = 10$ **22.** $\log_6 37 = x$

23. $\log x = y$ **24.** $\ln c = 17$

25–28 ▪ Write the equation in logarithmic form.

25. $2^6 = 64$ **26.** $49^{-1/2} = \frac{1}{7}$

27. $10^x = 74$ **28.** $e^k = m$

29–44 ▪ Evaluate the expression without using a calculator.

29. $\log_2 128$ **30.** $\log_8 1$

31. $10^{\log 45}$ **32.** $\log 0.000001$

33. $\ln(e^6)$ **34.** $\log_4 8$

35. $\log_3\left(\frac{1}{27}\right)$ **36.** $2^{\log_2 13}$

37. $\log_5 \sqrt{5}$ **38.** $e^{2\ln 7}$

39. $\log 25 + \log 4$ **40.** $\log_3 \sqrt{243}$

41. $\log_2 16^{23}$ **42.** $\log_5 250 - \log_5 2$

43. $\log_8 6 - \log_8 3 + \log_8 2$ **44.** $\log\log 10^{100}$

45–50 ▪ Expand the logarithmic expression.

45. $\log(AB^2C^3)$ **46.** $\log_2(x\sqrt{x^2 + 1})$

47. $\ln\sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$ **48.** $\log\left(\dfrac{4x^3}{y^2(x - 1)^5}\right)$

49. $\log_5\left(\dfrac{x^2(1 - 5x)^{3/2}}{\sqrt{x^3 - x}}\right)$ **50.** $\ln\left(\dfrac{\sqrt[3]{x^4 + 12}}{(x + 16)\sqrt{x - 3}}\right)$

51–56 ▪ Combine into a single logarithm.

51. $\log 6 + 4 \log 2$

52. $\log x + \log(x^2y) + 3 \log y$

53. $\frac{3}{2}\log_2(x - y) - 2\log_2(x^2 + y^2)$

54. $\log_5 2 + \log_5(x+1) - \frac{1}{3}\log_5(3x+7)$

55. $\log(x-2) + \log(x+2) - \frac{1}{2}\log(x^2+4)$

56. $\frac{1}{2}[\ln(x-4) + 5\ln(x^2+4x)]$

57–70 ■ Solve the equation. Find the exact solution if possible; otherwise, use a calculator to approximate to two decimals.

57. $3^{2x-7} = 27$

58. $5^{4-x} = \frac{1}{125}$

59. $2^{3x-5} = 7$

60. $10^{6-3x} = 18$

61. $4^{1-x} = 3^{2x+5}$

62. $e^{3x/4} = 10$

63. $x^2e^{2x} + 2xe^{2x} = 8e^{2x}$

64. $3^{2x} - 3^x - 6 = 0$

65. $\log x + \log(x+1) = \log 12$

66. $\ln(x-2) + \ln 3 = \ln(5x-7)$

67. $\log_2(1-x) = 4$

68. $\ln(2x-3) + 1 = 0$

69. $\log_3(x-8) + \log_3 x = 2$

70. $\log_8(x+5) - \log_8(x-2) = 1$

71–74 ■ Use a calculator to find the solution of the equation, rounded to six decimal places.

71. $5^{-2x/3} = 0.63$

72. $2^{3x-5} = 7$

73. $5^{2x+1} = 3^{4x-1}$

74. $e^{-15k} = 10{,}000$

75–78 ■ Draw a graph of the function and use it to determine the asymptotes and the local maximum and minimum values.

75. $y = e^{x/(x+2)}$

76. $y = 10^x - 5^x$

77. $y = \log(x^3 - x)$

78. $y = 2x^2 - \ln x$

79–80 ■ Find the solutions of the equation, rounded to two decimal places.

79. $3\log x = 6 - 2x$

80. $4 - x^2 = e^{-2x}$

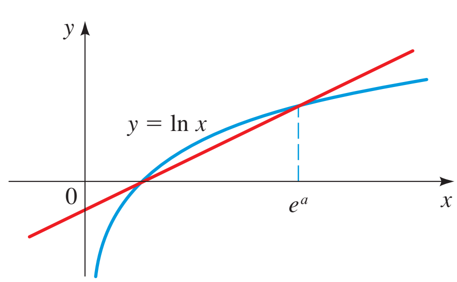
81–82 ■ Solve the inequality graphically.

81. $\ln x > x - 2$

82. $e^x < 4x^2$

83. Use a graph of $f(x) = e^x - 3e^{-x} - 4x$ to find, approximately, the intervals on which f is increasing and on which f is decreasing.

84. Find an equation of the line shown in the figure.



85–88 ■ Use the Change of Base Formula to evaluate the logarithm, rounded to six decimal places.

85. $\log_4 15$

86. $\log_7(\frac{3}{4})$

87. $\log_9 0.28$

88. $\log_{100} 250$

89. Which is larger, $\log_4 258$ or $\log_5 620$?

90. Find the inverse of the function $f(x) = 2^{3^x}$, and state its domain and range.

91. If \$12,000 is invested at an interest rate of 10% per year, find the amount of the investment at the end of 3 years for each compounding method.
(a) Semiannually **(b)** Monthly
(c) Daily **(d)** Continuously

92. A sum of \$5000 is invested at an interest rate of $8\frac{1}{2}$% per year, compounded semiannually.
(a) Find the amount of the investment after $1\frac{1}{2}$ years.
(b) After what period of time will the investment amount to \$7000?
(c) If interest were compounded continously instead of semiannually, how long would it take for the amount to grow to \$7000?

93. A money market account pays 5.2% annual interest, compounded daily. If \$100,000 is invested in this account, how long will it take for the account to accumulate \$10,000 in interest?

94. A retirement savings plan pays 4.5% interest, compounded continuously. How long will it take for an investment in this plan to double?

95–96 ■ Determine the annual percentage yield (APY) for the given nominal annual interest rate and compounding frequency.

95. 4.25%; daily

96. 3.2%; monthly

97. The stray-cat population in a small town grows exponentially. In 1999 the town had 30 stray cats, and the relative growth rate was 15% per year.
(a) Find a function that models the stray-cat population $n(t)$ after t years.
(b) Find the projected population after 4 years.
(c) Find the number of years required for the stray-cat population to reach 500.

98. A culture contains 10,000 bacteria initially. After an hour the bacteria count is 25,000.
(a) Find the doubling period.
(b) Find the number of bacteria after 3 hours.

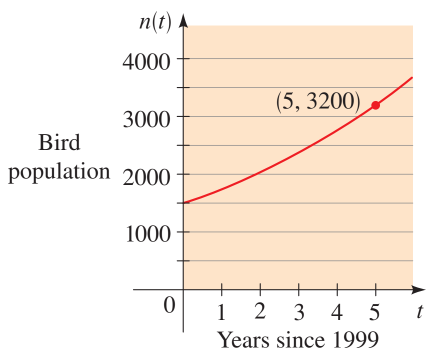
99. Uranium-234 has a half-life of 2.7×10^5 years.
(a) Find the amount remaining from a 10-mg sample after a thousand years.
(b) How long will it take this sample to decompose until its mass is 7 mg?

100. A sample of bismuth-210 decayed to 33% of its original mass after 8 days.
(a) Find the half-life of this element.
(b) Find the mass remaining after 12 days.

101. The half-life of radium-226 is 1590 years.
(a) If a sample has a mass of 150 mg, find a function that models the mass that remains after t years.
(b) Find the mass that will remain after 1000 years.
(c) After how many years will only 50 mg remain?

102. The half-life of palladium-100 is 4 days. After 20 days a sample has been reduced to a mass of 0.375 g.
(a) What was the initial mass of the sample?
(b) Find a function that models the mass remaining after t days.
(c) What is the mass after 3 days?
(d) After how many days will only 0.15 g remain?

103. The graph shows the population of a rare species of bird, where t represents years since 1999 and $n(t)$ is measured in thousands.
(a) Find a function that models the bird population at time t in the form $n(t) = n_0 e^{rt}$.
(b) What is the bird population expected to be in the year 2010?



104. A car engine runs at a temperature of 190°F. When the engine is turned off, it cools according to Newton's Law of Cooling with constant $k = 0.0341$, where the time is measured in minutes. Find the time needed for the engine to cool to 90°F if the surrounding temperature is 60°F.

105. The hydrogen ion concentration of fresh egg whites was measured as

$$[H^+] = 1.3 \times 10^{-8}\text{ M}$$

Find the pH, and classify the substance as acidic or basic.

106. The pH of lime juice is 1.9. Find the hydrogen ion concentration.

107. If one earthquake has magnitude 6.5 on the Richter scale, what is the magnitude of another quake that is 35 times as intense?

108. The drilling of a jackhammer was measured at 132 dB. The sound of whispering was measured at 28 dB. Find the ratio of the intensity of the drilling to that of the whispering.

CHAPTER 4 TEST

1. Sketch the graph of each function, and state its domain, range, and asymptote. Show the x- and y-intercepts on the graph.
   (a) $f(x) = 2^{-x} + 4$ (b) $g(x) = \log_3(x + 3)$
2. Find the domain of the function.
   (a) $f(t) = \ln(2t - 3)$ (b) $g(x) = \log(x^2 - 1)$
3. (a) Write the equation $6^{2x} = 25$ in logarithmic form.
   (b) Write the equation $\ln A = 3$ in exponential form.
4. Find the exact value of each expression.
   (a) $10^{\log 36}$ (b) $\ln e^3$
   (c) $\log_3 \sqrt{27}$ (d) $\log_2 80 - \log_2 10$
   (e) $\log_8 4$ (f) $\log_6 4 + \log_6 9$
5. Use the Laws of Logarithms to expand the expression:
   (a) $\log\left(\dfrac{xy^3}{z^2}\right)$ (b) $\ln\sqrt{\dfrac{x}{y}}$ (c) $\log\sqrt[3]{\dfrac{x + 2}{x^4(x^2 + 4)}}$
6. Use the Laws of Logarithms to combine the expression into a single logarithm.
   (a) $\log a + 2\log b$ (b) $\ln(x^2 - 25) - \ln(x + 5)$ (c) $\log_2 3 - 3\log_2 x + \frac{1}{2}\log_2(x + 1)$
7. Find the solution of the exponential equation, rounded to two decimal places.
   (a) $3^{4x} = 3^{100}$ (b) $e^{3x-2} = e^{x^2}$
   (c) $5^{x/10} + 1 = 7$ (d) $10^{x+3} = 6^{2x}$
8. Solve the logarithmic equation for x.
   (a) $\log(2x) = 3$ (b) $\log(x + 1) + \log 2 = \log(5x)$
   (c) $5\ln(3 - x) = 4$ (d) $\log_2(x + 2) + \log_2(x - 1) = 2$
9. Use the Change of Base Formula to evaluate $\log_{12} 27$.
10. The initial size of a culture of bacteria is 1000. After one hour the bacteria count is 8000.
    (a) Find a function $n(t) = n_0e^{rt}$ that models the population after t hours.
    (b) Find the population after 1.5 hours.
    (c) After how many hours will the number of bacteria reach 15,000?
    (d) Sketch the graph of the population function.
11. Suppose that \$12,000 is invested in a savings account paying 5.6% interest per year.
    (a) Write the formula for the amount in the account after t years if interest is compounded monthly.
    (b) Find the amount in the account after 3 years if interest is compounded daily.
    (c) How long will it take for the amount in the account to grow to \$20,000 if interest is compounded continuously?
12. The half-life of krypton-91 (^{91}Kr) is 10 seconds. At time $t = 0$ a heavy canister contains 3 g of this radioactive gas.
    (a) Find a function $m(t) = m_0 2^{-t/h}$ that models the amount of ^{91}Kr remaining in the canister after t seconds.
    (b) Find a function $m(t) = m_0 e^{-rt}$ that models the amount of ^{91}Kr remaining in the canister after t seconds.
    (c) How much ^{91}Kr remains after one minute?
    (d) After how long will the amount of ^{91}Kr remaining be reduced to 1 μg (1 microgram, or 10^{-6} g)?
13. An earthquake measuring 6.4 on the Richter scale struck Japan in July 2007, causing extensive damage. Earlier that year, a minor earthquake measuring 3.1 on the Richter scale was felt in parts of Pennsylvania. How many times more intense was the Japanese earthquake than the Pennsylvania earthquake?

Fitting Exponential and Power Curves to Data

In a previous *Focus on Modeling* (page 340) we learned that the shape of a scatter plot helps us to choose the type of curve to use in modeling data. The first plot in Figure 1 strongly suggests that a line be fitted through it, and the second one points to a cubic polynomial. For the third plot it is tempting to fit a second-degree polynomial. But what if an exponential curve fits better? How do we decide this? In this section we learn how to fit exponential and power curves to data and how to decide which type of curve fits the data better. We also learn that for scatter plots like those in the last two plots in Figure 1, the data can be modeled by logarithmic or logistic functions.

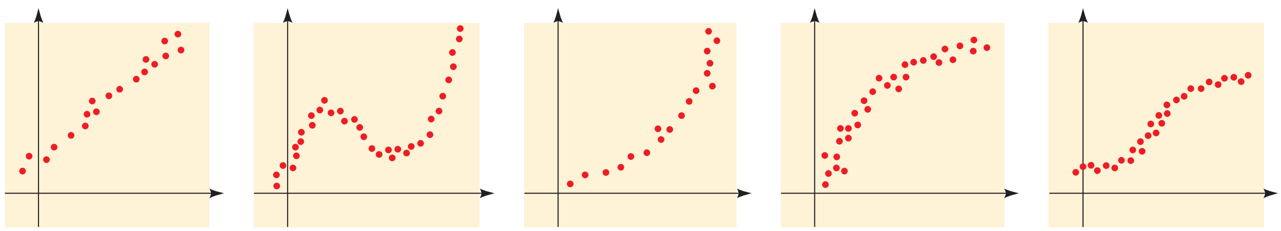


FIGURE 1

▼ Modeling with Exponential Functions

If a scatter plot shows that the data increase rapidly, we might want to model the data using an *exponential model*, that is, a function of the form

$$\boxed{f(x) = Ce^{kx}}$$

where C and k are constants. In the first example we model world population by an exponential model. Recall from Section 4.6 that population tends to increase exponentially.

TABLE 1
World population

Year (t)	World population (P in millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2520
1960	3020
1970	3700
1980	4450
1990	5300
2000	6060

EXAMPLE 1 | An Exponential Model for World Population

Table 1 gives the population of the world in the 20th century.

(a) Draw a scatter plot, and note that a linear model is not appropriate.

(b) Find an exponential function that models population growth.

(c) Draw a graph of the function that you found together with the scatter plot. How well does the model fit the data?

(d) Use the model that you found to predict world population in the year 2020.

SOLUTION

(a) The scatter plot is shown in Figure 2. The plotted points do not appear to lie along a straight line, so a linear model is not appropriate.

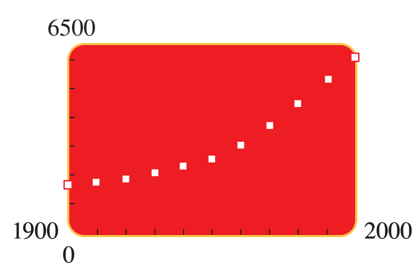


FIGURE 2 Scatter plot of world population

The population of the world increases exponentially.

(b) Using a graphing calculator and the `ExpReg` command (see Figure 3(a)), we get the exponential model

$$P(t) = (0.0082543) \cdot (1.0137186)^t$$

This is a model of the form $y = Cb^t$. To convert this to the form $y = Ce^{kt}$, we use the properties of exponentials and logarithms as follows:

$$\begin{aligned} 1.0137186^t &= e^{\ln 1.0137186^t} && A = e^{\ln A} \\ &= e^{t \ln 1.0137186} && \ln A^B = B \ln A \\ &= e^{0.013625t} && \ln 1.0137186 \approx 0.013625 \end{aligned}$$

Thus we can write the model as

$$P(t) = 0.0082543e^{0.013625t}$$

(c) From the graph in Figure 3(b) we see that the model appears to fit the data fairly well. The period of relatively slow population growth is explained by the depression of the 1930s and the two world wars.

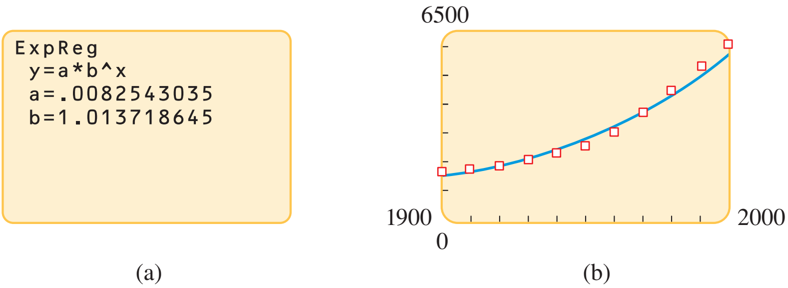


FIGURE 3 Exponential model for world population

(d) The model predicts that the world population in 2020 will be

$$\begin{aligned} P(2020) &= 0.0082543e^{(0.013625)(2020)} \\ &\approx 7{,}405{,}400{,}000 \end{aligned}$$

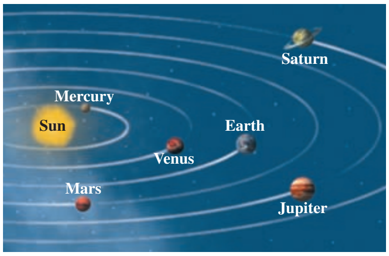
Modeling with Power Functions

If the scatter plot of the data we are studying resembles the graph of $y = ax^2$, $y = ax^{1.32}$, or some other power function, then we seek a *power model*, that is, a function of the form

$$\boxed{f(x) = ax^n}$$

where a is a positive constant and n is any real number.

In the next example we seek a power model for some astronomical data. In astronomy, distance in the solar system is often measured in astronomical units. An *astronomical unit* (AU) is the mean distance from the earth to the sun. The *period* of a planet is the time it takes the planet to make a complete revolution around the sun (measured in earth years). In this example we derive the remarkable relationship, first discovered by Johannes Kepler (see page 550), between the mean distance of a planet from the sun and its period.



EXAMPLE 2 | A Power Model for Planetary Periods

Table 2 gives the mean distance d of each planet from the sun in astronomical units and its period T in years.

TABLE 2
Distances and periods of the planets

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784
Pluto	39.507	248.350

(a) Sketch a scatter plot. Is a linear model appropriate?

(b) Find a power function that models the data.

(c) Draw a graph of the function you found and the scatter plot on the same graph. How well does the model fit the data?

(d) Use the model that you found to calculate the period of an asteroid whose mean distance from the sun is 5 AU.

SOLUTION

(a) The scatter plot shown in Figure 4 indicates that the plotted points do not lie along a straight line, so a linear model is not appropriate.

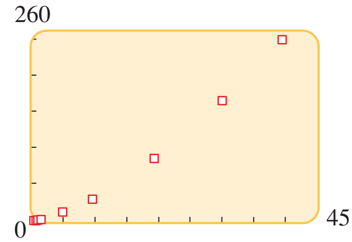


FIGURE 4 Scatter plot of planetary data

(b) Using a graphing calculator and the `PwrReg` command (see Figure 5(a)), we get the power model

$$T = 1.000396d^{1.49966}$$

If we round both the coefficient and the exponent to three significant figures, we can write the model as

$$T = d^{1.5}$$

This is the relationship discovered by Kepler (see page 550). Sir Isaac Newton (page 613) later used his Law of Gravity to derive this relationship theoretically, thereby providing strong scientific evidence that the Law of Gravity must be true.

(c) The graph is shown in Figure 5(b). The model appears to fit the data very well.

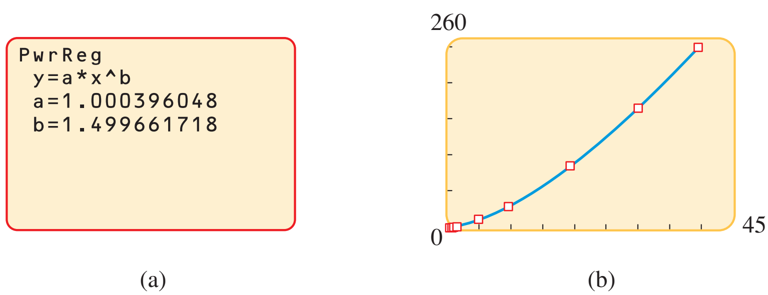


FIGURE 5 Power model for planetary data

(d) In this case $d = 5$ AU, so our model gives

$$T = 1.00039 \cdot 5^{1.49966} \approx 11.22$$

The period of the asteroid is about 11.2 years.

▼ Linearizing Data

We have used the shape of a scatter plot to decide which type of model to use: linear, exponential, or power. This works well if the data points lie on a straight line. But it's difficult to distinguish a scatter plot that is exponential from one that requires a power model. So to help decide which model to use, we can *linearize* the data, that is, apply a function that "straightens" the scatter plot. The inverse of the linearizing function is then an appropriate model. We now describe how to linearize data that can be modeled by exponential or power functions.

▶ Linearizing Exponential Data

If we suspect that the data points (x, y) lie on an exponential curve $y = Ce^{kx}$, then the points

$$(x, \ln y)$$

should lie on a straight line. We can see this from the following calculations:

$$\begin{aligned} \ln y &= \ln Ce^{kx} && \text{Assume that } y = Ce^{kx} \text{ and take ln} \\ &= \ln e^{kx} + \ln C && \text{Property of ln} \\ &= kx + \ln C && \text{Property of ln} \end{aligned}$$

To see that $\ln y$ is a linear function of x, let $Y = \ln y$ and $A = \ln C$; then

$$Y = kx + A$$

We apply this technique to the world population data (t, P) to obtain the points $(t, \ln P)$ in Table 3. The scatter plot of $(t, \ln P)$ in Figure 6, called a **semi-log plot**, shows that the linearized data lie approximately on a straight line, so an exponential model should be appropriate.

TABLE 3
World population data

t	Population P (in millions)	$\ln P$
1900	1650	21.224
1910	1750	21.283
1920	1860	21.344
1930	2070	21.451
1940	2300	21.556
1950	2520	21.648
1960	3020	21.829
1970	3700	22.032
1980	4450	22.216
1990	5300	22.391
2000	6060	22.525

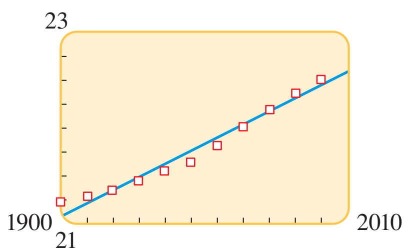


FIGURE 6 Semi-log plot of data in Table 3

▶ Linearizing Power Data

If we suspect that the data points (x, y) lie on a power curve $y = ax^n$, then the points

$$(\ln x, \ln y)$$

should be on a straight line. We can see this from the following calculations:

$$\begin{aligned} \ln y &= \ln ax^n && \text{Assume that } y = ax^n \text{ and take ln} \\ &= \ln a + \ln x^n && \text{Property of ln} \\ &= \ln a + n \ln x && \text{Property of ln} \end{aligned}$$

To see that $\ln y$ is a linear function of $\ln x$, let $Y = \ln y$, $X = \ln x$, and $A = \ln a$; then

$$Y = nX + A$$

We apply this technique to the planetary data (d, T) in Table 2 to obtain the points $(\ln d, \ln T)$ in Table 4. The scatter plot of $(\ln d, \ln T)$ in Figure 7, called a **log-log plot**, shows that the data lie on a straight line, so a power model seems appropriate.

TABLE 4
Log-log table

$\ln d$	$\ln T$
−0.94933	−1.4230
−0.32435	−0.48613
0	0
0.42068	0.6318
1.6492	2.4733
2.2556	3.3829
2.9544	4.4309
3.4041	5.1046
3.6765	5.5148

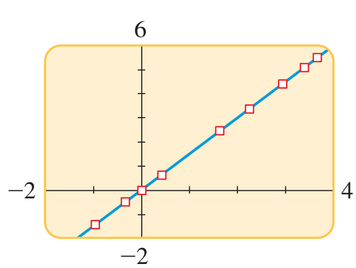


FIGURE 7 Log-log plot of data in Table 4

▼ An Exponential or Power Model?

Suppose that a scatter plot of the data points (x, y) shows a rapid increase. Should we use an exponential function or a power function to model the data? To help us decide, we draw two scatter plots: one for the points $(x, \ln y)$ and the other for the points $(\ln x, \ln y)$. If the first scatter plot appears to lie along a line, then an exponential model is appropriate. If the second plot appears to lie along a line, then a power model is appropriate.

EXAMPLE 3 | An Exponential or Power Model?

Data points (x, y) are shown in Table 5.

(a) Draw a scatter plot of the data.

(b) Draw scatter plots of $(x, \ln y)$ and $(\ln x, \ln y)$.

(c) Is an exponential function or a power function appropriate for modeling this data?

(d) Find an appropriate function to model the data.

TABLE 5

x	y
1	2
2	6
3	14
4	22
5	34
6	46
7	64
8	80
9	102
10	130

SOLUTION

(a) The scatter plot of the data is shown in Figure 8.

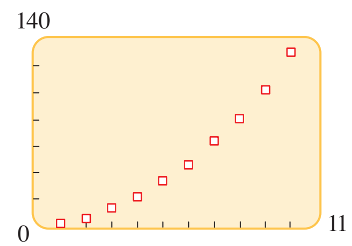


FIGURE 8

(b) We use the values from Table 6 to graph the scatter plots in Figures 9 and 10.

TABLE 6

x	$\ln x$	$\ln y$
1	0	0.7
2	0.7	1.8
3	1.1	2.6
4	1.4	3.1
5	1.6	3.5
6	1.8	3.8
7	1.9	4.2
8	2.1	4.4
9	2.2	4.6
10	2.3	4.9

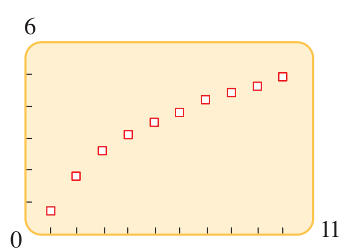


FIGURE 9 Semi-log plot

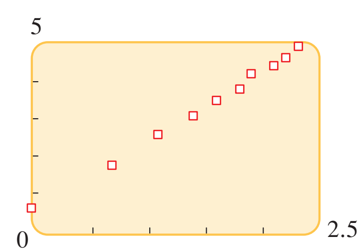


FIGURE 10 Log-log plot

(c) The scatter plot of $(x, \ln y)$ in Figure 9 does not appear to be linear, so an exponential model is not appropriate. On the other hand, the scatter plot of $(\ln x, \ln y)$ in Figure 10 is very nearly linear, so a power model is appropriate.

(d) Using the `PwrReg` command on a graphing calculator, we find that the power function that best fits the data point is

$$y = 1.85x^{1.82}$$

The graph of this function and the original data points are shown in Figure 11. ■

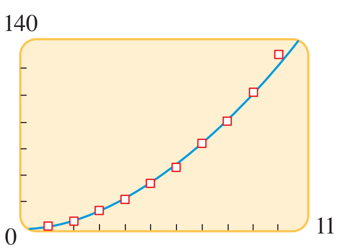


FIGURE 11

Before graphing calculators and statistical software became common, exponential and power models for data were often constructed by first finding a linear model for the linearized data. Then the model for the actual data was found by taking exponentials. For instance, if we find that $\ln y = A \ln x + B$, then by taking exponentials we get the model $y = e^B \cdot e^{A \ln x}$, or $y = Cx^A$ (where $C = e^B$). Special graphing paper called "log paper" or "log-log paper" was used to facilitate this process.

Modeling with Logistic Functions

A logistic growth model is a function of the form

$$f(t) = \frac{c}{1 + ae^{-bt}}$$

where a, b, and c are positive constants. Logistic functions are used to model populations where the growth is constrained by available resources. (See Exercises 25–28 of Section 4.2.)

EXAMPLE 4 | Stocking a Pond with Catfish

Much of the fish that is sold in supermarkets today is raised on commercial fish farms, not caught in the wild. A pond on one such farm is initially stocked with 1000 catfish, and the fish population is then sampled at 15-week intervals to estimate its size. The population data are given in Table 7.

TABLE 7

Week	Catfish
0	1000
15	1500
30	3300
45	4400
60	6100
75	6900
90	7100
105	7800
120	7900

(a) Find an appropriate model for the data.

(b) Make a scatter plot of the data and graph the model that you found in part (a) on the scatter plot.

(c) How does the model predict that the fish population will change with time?

SOLUTION

(a) Since the catfish population is restricted by its habitat (the pond), a logistic model is appropriate. Using the `Logistic` command on a calculator (see Figure 12(a)), we find the following model for the catfish population $P(t)$:

$$P(t) = \frac{7925}{1 + 7.7e^{-0.052t}}$$

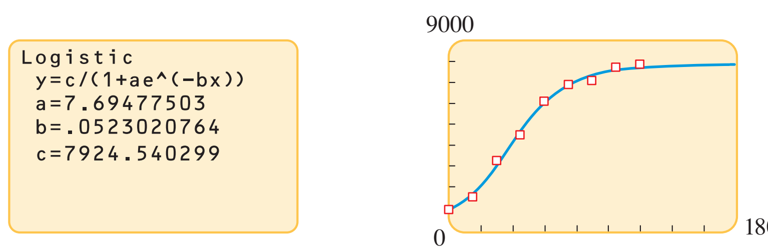


FIGURE 12 (a) (b) Catfish population $y = P(t)$

(b) The scatter plot and the logistic curve are shown in Figure 12(b).

(c) From the graph of P in Figure 12(b) we see that the catfish population increases rapidly until about $t = 80$ weeks. Then growth slows down, and at about $t = 120$ weeks the population levels off and remains more or less constant at slightly over 7900. ■

The behavior that is exhibited by the catfish population in Example 4 is typical of logistic growth. After a rapid growth phase, the population approaches a constant level called the **carrying capacity** of the environment. This occurs because as $t \to \infty$, we have $e^{-bt} \to 0$ (see Section 4.2), and so

$$P(t) = \frac{c}{1 + ae^{-bt}} \longrightarrow \frac{c}{1 + 0} = c$$

Thus the carrying capacity is c.

PROBLEMS

1. U.S. Population The U.S. Constitution requires a census every 10 years. The census data for 1790–2010 are given in the table.

(a) Make a scatter plot of the data.

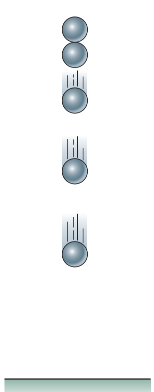
(b) Use a calculator to find an exponential model for the data.

(c) Use your model to predict the population at the 2020 census.

(d) Use your model to estimate the population in 1995.

(e) Compare your answers from parts (c) and (d) to the values in the table. Do you think an exponential model is appropriate for these data?

Year	Population (in millions)	Year	Population (in millions)	Year	Population (in millions)
1790	3.9	1870	38.6	1950	151.3
1800	5.3	1880	50.2	1960	179.3
1810	7.2	1890	63.0	1970	203.3
1820	9.6	1900	76.2	1980	226.5
1830	12.9	1910	92.2	1990	248.7
1840	17.1	1920	106.0	2000	281.4
1850	23.2	1930	123.2	2010	308.7
1860	31.4	1940	132.2		



Time (s)	Distance (m)
0.1	0.048
0.2	0.197
0.3	0.441
0.4	0.882
0.5	1.227
0.6	1.765
0.7	2.401
0.8	3.136
0.9	3.969
1.0	4.902

2. A Falling Ball In a physics experiment a lead ball is dropped from a height of 5 m. The students record the distance the ball has fallen every one-tenth of a second. (This can be done by using a camera and a strobe light.) Their data are shown in the margin.

(a) Make a scatter plot of the data.

(b) Use a calculator to find a power model.

(c) Use your model to predict how far a dropped ball would fall in 3 s.

3. Health-Care Expenditures The U.S. health-care expenditures for 1970–2008 are given in the table below, and a scatter plot of the data is shown in the figure.

(a) Does the scatter plot shown suggest an exponential model?

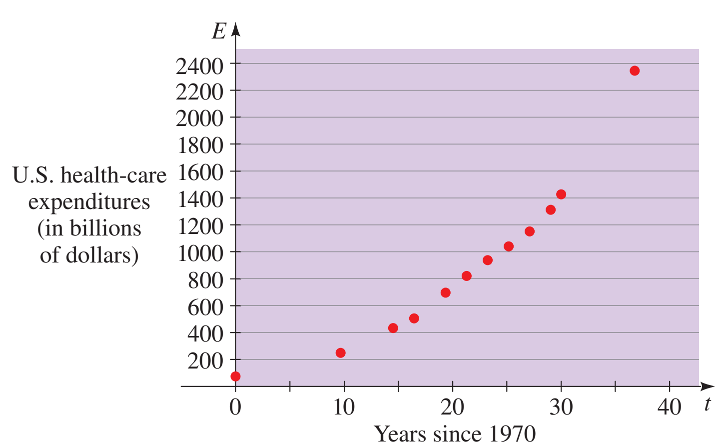
(b) Make a table of the values $(t, \ln E)$ and a scatter plot, where t is the number of years since 1970 and E is health-care expenditures in billions of dollars. Does the scatter plot appear to be linear?

(c) Find the regression line for the data in part (b).

(d) Use the results of part (c) to find an exponential model for the growth of health-care expenditures.

(e) Use your model to predict the total health-care expenditures in 2015.

Year	Health-care expenditures (in billions of dollars)
1970	74.3
1980	251.1
1985	434.5
1987	506.2
1990	696.6
1992	820.3
1994	937.2
1996	1039.4
1998	1150.0
2000	1310.0
2001	1424.5
2008	2339.5



Time (h)	Amount of ^{131}I (g)
0	4.80
8	4.66
16	4.51
24	4.39
32	4.29
40	4.14
48	4.04

4. Half-Life of Radioactive Iodine A student is trying to determine the half-life of radioactive iodine-131. He measures the amount of iodine-131 in a sample solution every 8 hours. His data are shown in the table in the margin.

(a) Make a scatter plot of the data.

(b) Use a calculator to find an exponential model.

(c) Use your model to find the half-life of iodine-131.

5. The Beer-Lambert Law As sunlight passes through the waters of lakes and oceans, the light is absorbed, and the deeper it penetrates, the more its intensity diminishes. The light intensity I at depth x is given by the Beer-Lambert Law:

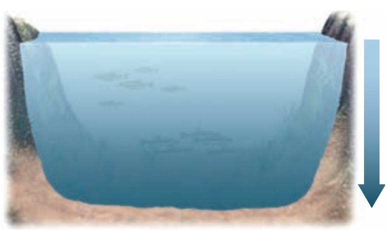
$$I = I_0 e^{-kx}$$

where I_0 is the light intensity at the surface and k is a constant that depends on the murkiness of the water (see page 380). A biologist uses a photometer to investigate light penetration in a northern lake, obtaining the data in the table.

(a) Use a graphing calculator to find an exponential function of the form given by the Beer-Lambert Law to model these data. What is the light intensity I_0 at the surface on this day, and what is the "murkiness" constant k for this lake? [*Hint:* If your calculator gives you a function of the form $I = ab^x$, convert this to the form you want using the identities $b^x = e^{\ln(b^x)} = e^{x \ln b}$. See Example 1(b).]

(b) Make a scatter plot of the data, and graph the function that you found in part (a) on your scatter plot.

(c) If the light intensity drops below 0.15 lumen (lm), a certain species of algae can't survive because photosynthesis is impossible. Use your model from part (a) to determine the depth below which there is insufficient light to support this algae.



Light intensity decreases exponentially with depth.

Depth (ft)	Light intensity (lm)	Depth (ft)	Light intensity (lm)
5	13.0	25	1.8
10	7.6	30	1.1
15	4.5	35	0.5
20	2.7	40	0.3

6. Experimenting with "Forgetting" Curves Every one of us is all too familiar with the phenomenon of forgetting. Facts that we clearly understood at the time we first learned them sometimes fade from our memory by the time the final exam rolls around. Psychologists have proposed several ways to model this process. One such model is Ebbinghaus' Law of Forgetting, described on page 371. Other models use exponential or logarithmic functions. To develop her own model, a psychologist performs an experiment on a group of volunteers by asking them to memorize a list of 100 related words. She then tests how many of these words they can recall after various periods of time. The average results for the group are shown in the table.

(a) Use a graphing calculator to find a *power* function of the form $y = at^b$ that models the average number of words y that the volunteers remember after t hours. Then find an *exponential* function of the form $y = ab^t$ to model the data.

(b) Make a scatter plot of the data, and graph both the functions that you found in part (a) on your scatter plot.

(c) Which of the two functions seems to provide the better model?

Time	Words recalled
15 min	64.3
1 h	45.1
8 h	37.3
1 day	32.8
2 days	26.9
3 days	25.6
5 days	22.9

The number of different bat species in a cave is related to the size of the cave by a power function.

7. Modeling the Species-Area Relation The table gives the areas of several caves in central Mexico and the number of bat species that live in each cave.*

(a) Find a power function that models the data.

(b) Draw a graph of the function you found in part (a) and a scatter plot of the data on the same graph. Does the model fit the data well?

(c) The cave called El Sapo near Puebla, Mexico, has a surface area of $A = 205\ \text{m}^2$. Use the model to estimate the number of bat species you would expect to find in that cave.

Cave	Area (m^2)	Number of species
La Escondida	18	1
El Escorpion	19	1
El Tigre	58	1
Mision Imposible	60	2
San Martin	128	5
El Arenal	187	4
La Ciudad	344	6
Virgen	511	7

8. Auto Exhaust Emissions A study by the U.S. Office of Science and Technology in 1972 estimated the cost of reducing automobile emissions by certain percentages. Find an exponential model that captures the "diminishing returns" trend of these data shown in the table below.

Reduction in emissions (%)	Cost per car ($)
50	45
55	55
60	62
65	70
70	80
75	90
80	100
85	200
90	375
95	600

9. Exponential or Power Model? Data points (x, y) are shown in the table.

(a) Draw a scatter plot of the data.

(b) Draw scatter plots of $(x, \ln y)$ and $(\ln x, \ln y)$.

(c) Which is more appropriate for modeling this data: an exponential function or a power function?

(d) Find an appropriate function to model the data.

x	y
2	0.08
4	0.12
6	0.18
8	0.25
10	0.36
12	0.52
14	0.73
16	1.06

*A. K. Brunet and R. A. Medallin, "The Species-Area Relationship in Bat Assemblages of Tropical Caves." *Journal of Mammalogy, 82*(4):1114–1122, 2001.

x	y
10	29
20	82
30	151
40	235
50	330
60	430
70	546
80	669
90	797

10. Exponential or Power Model? Data points (x, y) are shown in the table in the margin.

(a) Draw a scatter plot of the data.

(b) Draw scatter plots of $(x, \ln y)$ and $(\ln x, \ln y)$.

(c) Which is more appropriate for modeling this data: an exponential function or a power function?

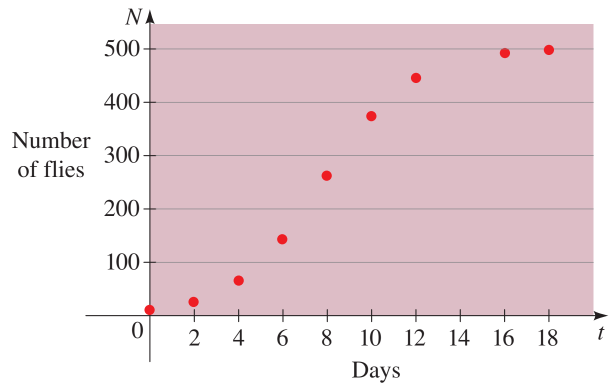
(d) Find an appropriate function to model the data.

Time (days)	Number of flies
0	10
2	25
4	66
6	144
8	262
10	374
12	446
16	492
18	498

11. Logistic Population Growth The table and scatter plot give the population of black flies in a closed laboratory container over an 18-day period.

(a) Use the `Logistic` command on your calculator to find a logistic model for these data.

(b) Use the model to estimate the time when there were 400 flies in the container.



12. Logarithmic Models A **logarithmic model** is a function of the form

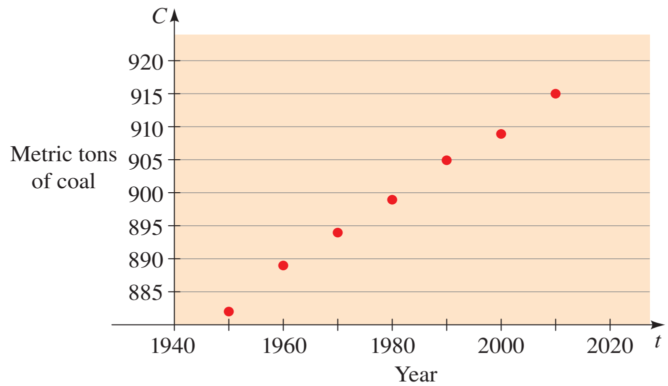
$$y = a + b \ln x$$

Many relationships between variables in the real world can be modeled by this type of function. The table and the scatter plot show the coal production (in metric tons) from a small mine in northern British Columbia.

(a) Use the `LnReg` command on your calculator to find a logarithmic model for these production figures.

(b) Use the model to predict coal production from this mine in 2020.

Year	Metric tons of coal
1950	882
1960	889
1970	894
1980	899
1990	905
2000	909
2010	915



CUMULATIVE REVIEW TEST CHAPTERS 2, 3, and 4

1. Let $f(x) = x^2 - 4x$ and $g(x) = \sqrt{x + 4}$. Find each of the following:

(a) The domain of f

(b) The domain of g

(c) $f(-2), f(0), f(4), g(0), g(8), g(-6)$

(d) $f(x + 2), g(x + 2), f(2 + h)$

(e) The average rate of change of g between $x = 5$ and $x = 21$

(f) $f \circ g, g \circ f, f(g(12)), g(f(12))$

(g) The inverse of g

2. Let $f(x) = \begin{cases} 4 & \text{if } x \le 2 \\ x - 3 & \text{if } x > 2 \end{cases}$

(a) Evaluate $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

(b) Sketch the graph of f.

3. Let f be the quadratic function $f(x) = -2x^2 + 8x + 5$.

(a) Express f in standard form.

(b) Find the maximum or minimum value of f.

(c) Sketch the graph of f.

(d) Find the interval on which f is increasing and the interval on which f is decreasing.

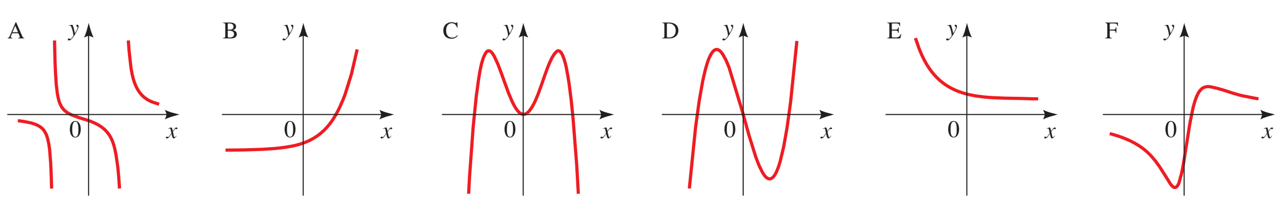
(e) How is the graph of $g(x) = -2x^2 + 8x + 10$ obtained from the graph of f?

(f) How is the graph of $h(x) = -2(x + 3)^2 + 8(x + 3) + 5$ obtained from the graph of f?

4. Without using a graphing calculator, match each of the following functions to the graphs below. Give reasons for your choices.

$$f(x) = x^3 - 8x \qquad g(x) = -x^4 + 8x^2 \qquad r(x) = \frac{2x + 3}{x^2 - 9}$$

$$s(x) = \frac{2x - 3}{x^2 + 9} \qquad h(x) = 2^x - 5 \qquad k(x) = 2^{-x} + 3$$



5. Let $P(x) = 2x^3 - 11x^2 + 10x + 8$.

(a) List all possible rational zeros of P.

(b) Determine which of the numbers you listed in part (a) actually are zeros of P.

(c) Factor P completely.

(d) Sketch a graph of P.

6. Let $Q(x) = x^5 - 3x^4 + 3x^3 + x^2 - 4x + 2$.

(a) Find all zeros of Q, real and complex, and state their multiplicities.

(b) Factor Q completely.

(c) Factor Q into linear and irreducible quadratic factors.

7. Let $r(x) = \dfrac{3x^2 + 6x}{x^2 - x - 2}$. Find the x- and y-intercepts and the horizontal and vertical asymptotes. Then sketch the graph of r.

8. A survey finds that the average starting salary for young people in their first full-time job is proportional to the square of the number of years of education they have completed. College graduates with 16 years of education have an average starting salary of \$48,000.

(a) Write an equation that expresses the relationship between years of education x and average starting salary S.

(b) What is the average starting salary for a person who drops out of high school after completing the tenth grade?

(c) A person with a master's degree has an average starting salary of \$60,750. How many years of education does this represent?

9. Sketch graphs of the following functions on the same coordinate plane.

(a) $f(x) = 2 - e^x$ **(b)** $g(x) = \ln(x + 1)$

10. **(a)** Find the exact value of $\log_3 16 - 2\log_3 36$.

(b) Use the Laws of Logarithms to expand the expression

$$\log\left(\frac{x^5\sqrt{x - 1}}{2x - 3}\right)$$

11. Solve the equations.

(a) $\log_2 x + \log_2(x - 2) = 3$

(b) $2e^{3x} - 11e^{2x} + 10e^x + 8 = 0$ [*Hint:* Compare to the polynomial in Problem 5.]

12. A sum of \$25,000 is deposited into an account paying 5.4% interest per year, compounded daily.

(a) What will the amount in the account be after 3 years?

(b) When will the account have grown to \$35,000?

(c) How long will it take for the initial deposit to double?

13. After a shipwreck, 120 rats manage to swim from the wreckage to a deserted island. The rat population on the island grows exponentially, and after 15 months there are 280 rats on the island.

(a) Find a function that models the population t months after the arrival of the rats.

(b) What will the population be 3 years after the shipwreck?

(c) When will the population reach 2000?

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CHAPTER 5

Systems of Equations and Inequalities

Equations Working Together In the preceding chapters we learned that a real-world situation can often be modeled by an equation. But many real-world situations involve too many variables to be modeled by a single equation. For example, weather depends on the relationships among many variables, including temperature, wind speed, air pressure, and humidity. So to model (and forecast) the weather, scientists use many equations, each having many variables. These equations *work together* to model the weather. Such collections of equations are called *systems of equations.* In this chapter we learn how to solve systems of equations that consist of several equations in several variables. Airlines use systems of equations with hundreds of variables to establish consistent flight schedules, and telecommunications companies use them to find efficient routings for telephone calls. In both of these situations the problem is to allocate limited resources in an optimal way. In *Focus on Modeling* at the end of the chapter we use systems of inequalities to model and solve these types of problems.

5.1 Systems of Linear Equations in Two Variables

LEARNING OBJECTIVES After completing this section, you will be able to:

Solve a system in two variables using the Substitution Method, the Elimination Method, and the Graphical Method ▶ Determine the number of solutions of a linear system in two variables ▶ Model with linear systems

GET READY Prepare for this section by reviewing how to solve linear equations in Section P.8 and how to graph lines in Section 1.3.

▼ Systems of Linear Equations and Their Solutions

A linear equation in two variables is an equation of the form

$$ax + by = c$$

The graph of a linear equation is a line (see Section 1.3).

A **system of equations** is a set of equations that involve the same variables. A **system of linear equations** (or a **linear system**) is a system of equations in which each equation is linear. A **solution** of a system is an assignment of values for the variables that makes *each* equation in the system true. To **solve** a system means to find all solutions of the system.

Here is an example of a system of linear equations in two variables:

$$\begin{cases} 2x - y = 5 & \text{Equation 1} \\ x + 4y = 7 & \text{Equation 2} \end{cases}$$

We can check that $x = 3$ and $y = 1$ is a solution of this system.

Equation 1	**Equation 2**
$2x - y = 5$	$x + 4y = 7$
$2(3) - 1 = 5$ ✓	$3 + 4(1) = 7$ ✓

The solution can also be written as the ordered pair $(3, 1)$. Note that the graphs of Equations 1 and 2 are lines (see Figure 1). Since the solution $(3, 1)$ satisfies each equation, the point $(3, 1)$ lies on each line. So it is the point of intersection of the two lines.

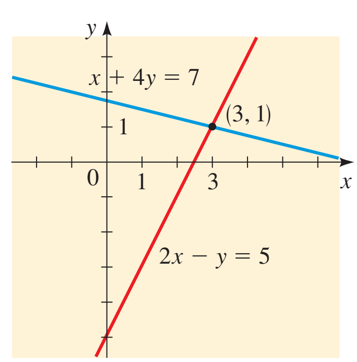

FIGURE 1

▼ Substitution Method

To solve a system using the **substitution method**, we start with one equation in the system and solve for one variable in terms of the other variable. The following box describes the procedure.

> **SUBSTITUTION METHOD**
>
> 1. **Solve for One Variable.** Choose one equation, and solve for one variable in terms of the other variable.
> 2. **Substitute.** Substitute the expression you found in Step 1 into the other equation to get an equation in one variable, then solve for that variable.
> 3. **Back-Substitute.** Substitute the value you found in Step 2 back into the expression found in Step 1 to solve for the remaining variable.

EXAMPLE 1 | Substitution Method

Find all solutions of the system.

$$\begin{cases} 2x + y = 1 & \text{Equation 1} \\ 3x + 4y = 14 & \text{Equation 2} \end{cases}$$

SOLUTION Solve for one variable. We solve for y in the first equation:

$$y = 1 - 2x \qquad \text{Solve for } y \text{ in Equation 1}$$

Substitute. Now we substitute for y in the second equation and solve for x:

$$\begin{aligned} 3x + 4(1 - 2x) &= 14 && \text{Substitute } y = 1 - 2x \text{ into Equation 2} \\ 3x + 4 - 8x &= 14 && \text{Expand} \\ -5x + 4 &= 14 && \text{Simplify} \\ -5x &= 10 && \text{Subtract 4} \\ x &= -2 && \text{Solve for } x \end{aligned}$$

Back-substitute. Next we back-substitute $x = -2$ into the equation $y = 1 - 2x$:

$$y = 1 - 2(-2) = 5 \qquad \text{Back-substitute}$$

Thus $x = -2$ and $y = 5$, so the solution is the ordered pair $(-2, 5)$. Figure 2 shows that the graphs of the two equations intersect at the point $(-2, 5)$.

CHECK YOUR ANSWER

$x = -2, y = 5$:

$$\begin{cases} 2(-2) + 5 = 1 \\ 3(-2) + 4(5) = 14 \end{cases} \checkmark$$

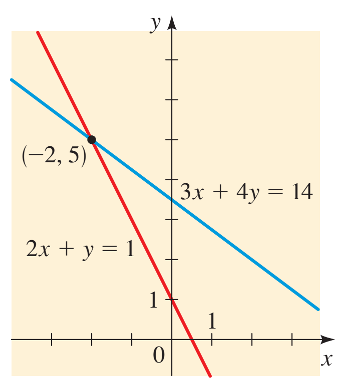

FIGURE 2

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5** ■

Elimination Method

To solve a system using the **elimination method**, we try to combine the equations using sums or differences so as to eliminate one of the variables.

ELIMINATION METHOD

1. **Adjust the Coefficients.** Multiply one or more of the equations by appropriate numbers so that the coefficient of one variable in one equation is the negative of its coefficient in the other equation.
2. **Add the Equations.** Add the two equations to eliminate one variable, then solve for the remaining variable.
3. **Back-Substitute.** Substitute the value that you found in Step 2 back into one of the original equations, and solve for the remaining variable.

EXAMPLE 2 | Elimination Method

Find all solutions of the system.

$$\begin{cases} 3x + 2y = 14 & \text{Equation 1} \\ 2x - 4y = 4 & \text{Equation 2} \end{cases}$$

SOLUTION Adjust the coefficients. Multiply Equation 2 by $\frac{1}{2}$ so that the coefficients of the y-terms are negatives of each other: $x - 2y = 2$.

Add the equations. We now add the equations to eliminate y:

$$\begin{array}{rll} \begin{cases} 3x + 2y = 14 \\ x - 2y = 2 \end{cases} & & \text{System} \\ \hline 4x \qquad = 16 & & \text{Add} \\ x = 4 & & \text{Solve for } x \end{array}$$

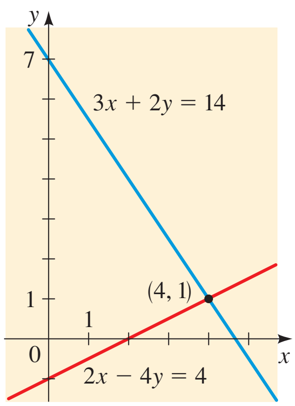

FIGURE 3

Back-substitute. Now we back-substitute $x = 4$ into one of the original equations and solve for y. Let's choose the second equation because it looks simpler.

$$\begin{aligned} x - 2y &= 2 && \text{Equation 2} \\ 4 - 2y &= 2 && \text{Back-substitute } x = 4 \text{ into Equation 2} \\ -2y &= -2 && \text{Subtract 4} \\ y &= 1 && \text{Solve for } y \end{aligned}$$

The solution is $(4, 1)$. Figure 3 shows that the graphs of the equations in the system intersect at the point $(4, 1)$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **9**

Graphical Method

In the **graphical method** we use a graphing device to solve the system of equations.

GRAPHICAL METHOD

1. **Graph Each Equation.** Express each equation in a form suitable for the graphing calculator by solving for y as a function of x. Graph the equations on the same screen.
2. **Find the Intersection Points.** The solutions are the x- and y-coordinates of the points of intersection.

MATHEMATICS IN THE MODERN WORLD

Weather Prediction

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Modern meteorologists do much more than predict tomorrow's weather. They research long-term weather patterns, depletion of the ozone layer, global warming, and other effects of human activity on the weather. But daily weather prediction is still a major part of meteorology; its value is measured by the innumerable human lives that are saved each year through accurate prediction of hurricanes, blizzards, and other catastrophic weather phenomena. Early in the 20th century mathematicians proposed to model weather with equations that used hundreds of variables. However, it was impossible to predict future weather with this model because it took several days to solve the equations—too late for predicting tomorrow's weather. Currently, new mathematical models combined with high-speed computer simulations have vastly improved weather prediction. As a result, many human as well as economic disasters have been averted. Mathematicians at the National Oceanographic and Atmospheric Administration (NOAA) are continually researching better methods of weather prediction.

See Appendix B, *Graphing with a Graphing Calculator*, for guidelines on using a graphing calculator. See Appendix C, *Using the TI-83/84 Graphing Calculator*, for specific graphing instructions.

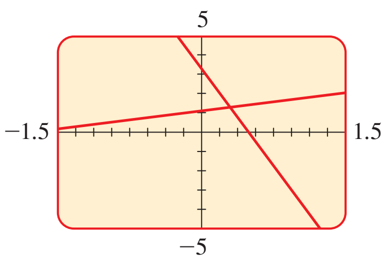


FIGURE 4

EXAMPLE 3 | Graphical Method

Find all solutions of the system.

$$\begin{cases} 1.35x - 2.13y = -2.36 \\ 2.16x + 0.32y = \quad 1.06 \end{cases}$$

SOLUTION To graph each equation we solve for y in terms of x, and we get the equivalent system

$$\begin{cases} y = \quad 0.63x + 1.11 \\ y = -6.75x + 3.31 \end{cases}$$

where we have rounded the coefficients to two decimals. Figure 4 shows that the two lines intersect. Zooming in, we see that the solution is approximately $(0.30, 1.30)$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **13** AND **49**

The Number of Solutions of a Linear System in Two Variables

The graph of a linear system in two variables is a pair of lines, so to solve the system graphically, we must find the intersection point(s) of the lines. Two lines may intersect in a single point, they may be parallel, or they may coincide, as shown in Figure 5. So there are three possible outcomes in solving such a system.

NUMBER OF SOLUTIONS OF A LINEAR SYSTEM IN TWO VARIABLES

For a system of linear equations in two variables, exactly one of the following is true. (See Figure 5.)

1. The system has exactly one solution.
2. The system has no solution.
3. The system has infinitely many solutions.

A system that has no solution is said to be **inconsistent**. A system with infinitely many solutions is called **dependent**.

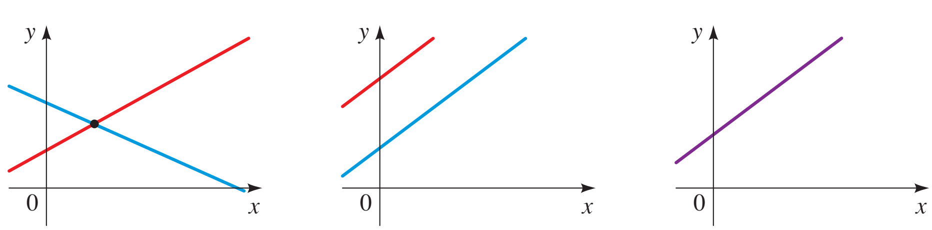


FIGURE 5

(a) Lines intersect at a single point. The system has one solution.

(b) Lines are parallel and do not intersect. The system has no solution.

(c) Lines coincide—equations are for the same line. The system has infinitely many solutions.

EXAMPLE 4 | A Linear System with One Solution

Solve the system and graph the lines.

$$\begin{cases} 3x - \quad y = 0 & \text{Equation 1} \\ 5x + 2y = 22 & \text{Equation 2} \end{cases}$$

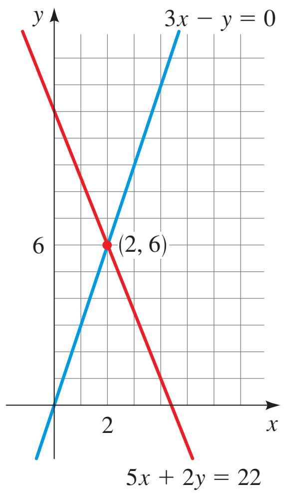


FIGURE 6

CHECK YOUR ANSWER

$x = 2, y = 6$:

$$\begin{cases} 3(2) - (6) = 0 \\ 5(2) + 2(6) = 22 \end{cases} \checkmark$$

SOLUTION We eliminate y from the equations and solve for x.

$$\begin{cases} 6x - 2y = 0 & 2 \times \text{Equation 1} \\ 5x + 2y = 22 \end{cases}$$

$$11x = 22 \qquad \text{Add}$$

$$x = 2 \qquad \text{Solve for } x$$

Now we back-substitute into the first equation and solve for y:

$$6(2) - 2y = 0 \qquad \text{Back-substitute } x = 2$$

$$-2y = -12 \qquad \text{Subtract } 6 \times 2 = 12$$

$$y = 6 \qquad \text{Solve for } y$$

The solution of the system is the ordered pair (2, 6), that is,

$$x = 2, \qquad y = 6$$

The graph in Figure 6 shows that the lines in the system intersect at the point (2, 6).

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **23**

EXAMPLE 5 | A Linear System with No Solution

Solve the system.

$$\begin{cases} 8x - 2y = 5 & \text{Equation 1} \\ -12x + 3y = 7 & \text{Equation 2} \end{cases}$$

SOLUTION This time we try to find a suitable combination of the two equations to eliminate the variable y. Multiplying the first equation by 3 and the second equation by 2 gives

$$\begin{cases} 24x - 6y = 15 & 3 \times \text{Equation 1} \\ -24x + 6y = 14 & 2 \times \text{Equation 2} \end{cases}$$

$$0 = 29 \qquad \text{Add}$$

Adding the two equations eliminates *both x and y* in this case, and we end up with $0 = 29$, which is obviously false. No matter what values we assign to x and y, we cannot make this statement true, so the system has *no solution*. Figure 7 shows that the lines in the system are parallel and do not intersect. The system is inconsistent.

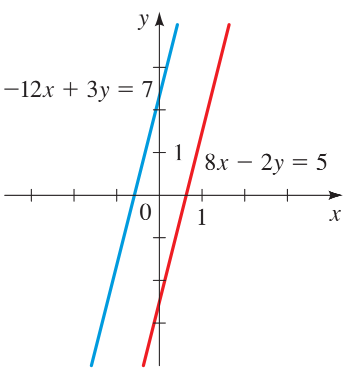


FIGURE 7

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **35**

EXAMPLE 6 | A Linear System with Infinitely Many Solutions

Solve the system.

$$\begin{cases} 3x - 6y = 12 & \text{Equation 1} \\ 4x - 8y = 16 & \text{Equation 2} \end{cases}$$

SOLUTION We multiply the first equation by 4 and the second by 3 to prepare for subtracting the equations to eliminate x. The new equations are

$$\begin{cases} 12x - 24y = 48 & 4 \times \text{Equation 1} \\ 12x - 24y = 48 & 3 \times \text{Equation 2} \end{cases}$$

We see that the two equations in the original system are simply different ways of expressing the equation of one single line. The coordinates of any point on this line give a

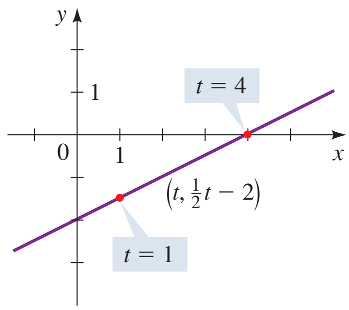


FIGURE 8

solution of the system. Writing the equation in slope-intercept form, we have $y = \frac{1}{2}x - 2$. So if we let t represent any real number, we can write the solution as

$$x = t$$

$$y = \tfrac{1}{2}t - 2$$

We can also write the solution in ordered-pair form as

$$\left(t, \tfrac{1}{2}t - 2\right)$$

where t is any real number. The system has infinitely many solutions (see Figure 8).

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **37**

In Example 3, to get specific solutions we have to assign values to t. For instance, if $t = 1$, we get the solution $\left(1, -\frac{3}{2}\right)$. If $t = 4$, we get the solution $(4, 0)$. For every value of t we get a different solution. (See Figure 8.)

Modeling with Linear Systems

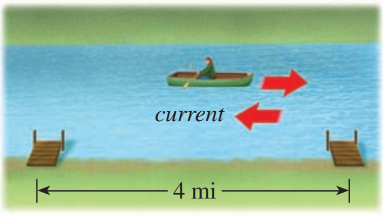
Frequently, when we use equations to solve problems in the sciences or in other areas, we obtain systems like the ones we've been considering. When modeling with systems of equations, we use the following guidelines, which are similar to those in Section 1.5.

GUIDELINES FOR MODELING WITH SYSTEMS OF EQUATIONS

1. **Identify the Variables.** Identify the quantities that the problem asks you to find. These are usually determined by a careful reading of the question posed at the end of the problem. Introduce notation for the variables (call them x and y or some other letters).
2. **Express All Unknown Quantities in Terms of the Variables.** Read the problem again, and express all the quantities mentioned in the problem in terms of the variables you defined in Step 1.
3. **Set Up a System of Equations.** Find the crucial facts in the problem that give the relationships between the expressions you found in Step 2. Set up a system of equations (or a model) that expresses these relationships.
4. **Solve the System and Interpret the Results.** Solve the system you found in Step 3, check your solutions, and state your final answer as a sentence that answers the question posed in the problem.

The next two examples illustrate how to model with systems of equations.

EXAMPLE 7 | A Distance-Speed-Time Problem



A woman rows a boat upstream from one point on a river to another point 4 mi away in $1\frac{1}{2}$ hours. The return trip, traveling with the current, takes only 45 min. How fast does she row relative to the water, and at what speed is the current flowing?

SOLUTION Identify the variables. We are asked to find the rowing speed and the speed of the current, so we let

$$x = \text{rowing speed (mi/h)}$$

$$y = \text{current speed (mi/h)}$$

Express unknown quantities in terms of the variable. The woman's speed when she rows upstream is her rowing speed minus the speed of the current; her speed downstream is her rowing speed plus the speed of the current. Now we translate this information into the language of algebra.

In Words	In Algebra
Rowing speed	x
Current speed	y
Speed upstream	$x - y$
Speed downstream	$x + y$

Set up a system of equations. The distance upstream and downstream is 4 mi, so using the fact that speed $\times$ time $=$ distance for both legs of the trip, we get

speed upstream $\times$ time upstream $=$ distance traveled

speed downstream $\times$ time downstream $=$ distance traveled

In algebraic notation this translates into the following equations:

$$(x - y)\tfrac{3}{2} = 4 \qquad \text{Equation 1}$$

$$(x + y)\tfrac{3}{4} = 4 \qquad \text{Equation 2}$$

(The times have been converted to hours, since we are expressing the speeds in miles per *hour*.)

Solve the system. We multiply the equations by 2 and 4, respectively, to clear the denominators.

$$\begin{aligned}
&\begin{cases} 3x - 3y = 8 \\ 3x + 3y = 16 \end{cases} && \begin{aligned} &2 \times \text{Equation 1} \\ &4 \times \text{Equation 2} \end{aligned} \\
&6x \qquad\quad = 24 && \text{Add} \\
&x \qquad\quad\;\; = 4 && \text{Solve for } x
\end{aligned}$$

Back-substituting this value of x into the first equation (the second works just as well) and solving for y gives

$$\begin{aligned}
3(4) - 3y &= 8 && \text{Back-substitute } x = 4 \\
-3y &= 8 - 12 && \text{Subtract 12} \\
y &= \tfrac{4}{3} && \text{Solve for } y
\end{aligned}$$

The woman rows at 4 mi/h, and the current flows at $1\frac{1}{3}$ mi/h.

CHECK YOUR ANSWER

Speed upstream is

$$\frac{\text{distance}}{\text{time}} = \frac{4 \text{ mi}}{1\frac{1}{2}\text{ h}} = 2\tfrac{2}{3}\text{ mi/h}$$

and this should equal

rowing speed $-$ current flow

$$= 4 \text{ mi/h} - \tfrac{4}{3}\text{ mi/h} = 2\tfrac{2}{3}\text{ mi/h} \quad ✓$$

Speed downstream is

$$\frac{\text{distance}}{\text{time}} = \frac{4 \text{ mi}}{\frac{3}{4}\text{ h}} = 5\tfrac{1}{3}\text{ mi/h}$$

and this should equal

rowing speed $+$ current flow

$$= 4 \text{ mi/h} + \tfrac{4}{3}\text{ mi/h} = 5\tfrac{1}{3}\text{ mi/h} \quad ✓$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **63**

EXAMPLE 8 | A Mixture Problem

A vintner fortifies wine that contains 10% alcohol by adding a 70% alcohol solution to it. The resulting mixture has an alcoholic strength of 16% and fills 1000 one-liter bottles. How many liters (L) of the wine and of the alcohol solution does the vintner use?

SOLUTION Identify the variables. Since we are asked for the amounts of wine and alcohol, we let

$$x = \text{amount of wine used (L)}$$

$$y = \text{amount of alcohol solution used (L)}$$

Express all unknown quantities in terms of the variable. From the fact that the wine contains 10% alcohol and the solution contains 70% alcohol, we get the following.

In Words	In Algebra
Amount of wine used (L)	x
Amount of alcohol solution used (L)	y
Amount of alcohol in wine (L)	$0.10x$
Amount of alcohol in solution (L)	$0.70y$

Set up a system of equations. The volume of the mixture must be the total of the two volumes the vintner is adding together, so

$$x + y = 1000$$

Also, the amount of alcohol in the mixture must be the total of the alcohol contributed by the wine and by the alcohol solution, that is,

$$0.10x + 0.70y = (0.16)1000$$

$$0.10x + 0.70y = 160 \qquad \text{Simplify}$$

$$x + 7y = 1600 \qquad \text{Multiply by 10 to clear decimals}$$

Thus we get the system

$$\begin{cases} x + y = 1000 & \text{Equation 1} \\ x + 7y = 1600 & \text{Equation 2} \end{cases}$$

Solve the system. Subtracting the first equation from the second eliminates the variable x, and we get

$$6y = 600 \qquad \text{Subtract Equation 1 from Equation 2}$$

$$y = 100 \qquad \text{Solve for } y$$

We now back-substitute $y = 100$ into the first equation and solve for x:

$$x + 100 = 1000 \qquad \text{Back-substitute } y = 100$$

$$x = 900 \qquad \text{Solve for } x$$

The vintner uses 900 L of wine and 100 L of the alcohol solution.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **65** ■

5.1 EXERCISES

CONCEPTS

1. The system of equations

$$\begin{cases} 2x + 3y = 7 \\ 5x - y = 9 \end{cases}$$

is a system of two equations in the two variables ________ and ________. To determine whether $(5, -1)$ is a solution of this system, we check whether $x = 5$ and $y = -1$ satisfy each ________ in the system. Which of the following are solutions of this system?

$$(5, -1), \quad (-1, 3), \quad (2, 1)$$

2. A system of equations in two variables can be solved by the ____________ method, the ____________ method, or the ____________ method.

3. A system of two linear equations in two variables can have one solution, ________ solution, or ________ ________ solutions.

4. The following is a system of two linear equations in two variables.

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

The graph of the first equation is the same as the graph of the second equation, so the system has ________ ________ solutions. We express these solutions by writing

$x = t$
$y =$ ________

where t is any real number. Some of the solutions of this system are $(1, \_\_)$, $(-3, \_\_)$, and $(5, \_\_)$.

SKILLS

5–8 ■ Use the substitution method to find all solutions of the system of equations.

5. $\begin{cases} x - y = 1 \\ 4x + 3y = 18 \end{cases}$ **6.** $\begin{cases} 3x + y = 1 \\ 5x + 2y = 1 \end{cases}$

7. $\begin{cases} x - y = 2 \\ 2x + 3y = 9 \end{cases}$ **8.** $\begin{cases} 2x + y = 7 \\ x + 2y = 2 \end{cases}$

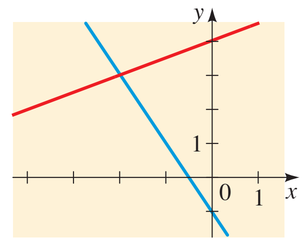
9–12 ■ Use the elimination method to find all solutions of the system of equations.

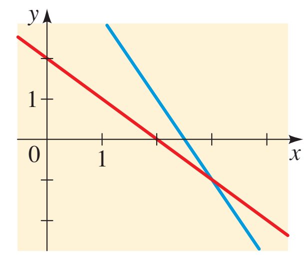
9. $\begin{cases} 3x + 4y = 10 \\ x - 4y = -2 \end{cases}$ **10.** $\begin{cases} 2x + 5y = 15 \\ 4x + y = 21 \end{cases}$

11. $\begin{cases} x + 2y = 5 \\ 2x + 3y = 8 \end{cases}$ **12.** $\begin{cases} 4x - 3y = 11 \\ 8x + 4y = 12 \end{cases}$

13–14 ■ Two equations and their graphs are given. Find the intersection point(s) of the graphs by solving the system.

13. $\begin{cases} 2x + y = -1 \\ x - 2y = -8 \end{cases}$ **14.** $\begin{cases} x + y = 2 \\ 2x + y = 5 \end{cases}$





15–20 ■ Graph each linear system, either by hand or using a graphing device. Use the graph to determine whether the system has one solution, no solution, or infinitely many solutions. If there is exactly one solution, use the graph to find it.

15. $\begin{cases} x - y = 4 \\ 2x + y = 2 \end{cases}$ **16.** $\begin{cases} 2x - y = 4 \\ 3x + y = 6 \end{cases}$

17. $\begin{cases} 2x - 3y = 12 \\ -x + \frac{3}{2}y = 4 \end{cases}$ **18.** $\begin{cases} 2x + 6y = 0 \\ -3x - 9y = 18 \end{cases}$

19. $\begin{cases} -x + \frac{1}{2}y = -5 \\ 2x - y = 10 \end{cases}$ **20.** $\begin{cases} 12x + 15y = -18 \\ 2x + \frac{5}{2}y = -3 \end{cases}$

21–48 ■ Solve the system, or show that it has no solution. If the system has infinitely many solutions, express them in the ordered-pair form given in Example 6.

21. $\begin{cases} x + y = 4 \\ -x + y = 0 \end{cases}$ **22.** $\begin{cases} x - y = 3 \\ x + 3y = 7 \end{cases}$

23. $\begin{cases} 2x - 3y = 9 \\ 4x + 3y = 9 \end{cases}$ **24.** $\begin{cases} 3x + 2y = 0 \\ -x - 2y = 8 \end{cases}$

25. $\begin{cases} x + 3y = 5 \\ 2x - y = 3 \end{cases}$ **26.** $\begin{cases} x + y = 7 \\ 2x - 3y = -1 \end{cases}$

27. $\begin{cases} -x + y = 2 \\ 4x - 3y = -3 \end{cases}$ **28.** $\begin{cases} 4x - 3y = 28 \\ 9x - y = -6 \end{cases}$

29. $\begin{cases} x + 2y = 7 \\ 5x - y = 2 \end{cases}$ **30.** $\begin{cases} -4x + 12y = 0 \\ 12x + 4y = 160 \end{cases}$

31. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 2 \\ \frac{1}{5}x - \frac{2}{3}y = 8 \end{cases}$ **32.** $\begin{cases} 0.2x - 0.2y = -1.8 \\ -0.3x + 0.5y = 3.3 \end{cases}$

33. $\begin{cases} 3x + 2y = 8 \\ x - 2y = 0 \end{cases}$ **34.** $\begin{cases} 4x + 2y = 16 \\ x - 5y = 70 \end{cases}$

35. $\begin{cases} x + 4y = 8 \\ 3x + 12y = 2 \end{cases}$ **36.** $\begin{cases} -3x + 5y = 2 \\ 9x - 15y = 6 \end{cases}$

37. $\begin{cases} 2x - 6y = 10 \\ -3x + 9y = -15 \end{cases}$ **38.** $\begin{cases} 2x - 3y = -8 \\ 14x - 21y = 3 \end{cases}$

39. $\begin{cases} 6x + 4y = 12 \\ 9x + 6y = 18 \end{cases}$

40. $\begin{cases} 25x - 75y = 100 \\ -10x + 30y = -40 \end{cases}$

41. $\begin{cases} 8s - 3t = -3 \\ 5s - 2t = -1 \end{cases}$

42. $\begin{cases} u - 30v = -5 \\ -3u + 80v = 5 \end{cases}$

43. $\begin{cases} \frac{1}{2}x + \frac{3}{5}y = 3 \\ \frac{5}{3}x + 2y = 10 \end{cases}$

44. $\begin{cases} \frac{3}{2}x - \frac{1}{3}y = \frac{1}{2} \\ 2x - \frac{1}{2}y = -\frac{1}{2} \end{cases}$

45. $\begin{cases} 0.4x + 1.2y = 14 \\ 12x - 5y = 10 \end{cases}$

46. $\begin{cases} 26x - 10y = -4 \\ -0.6x + 1.2y = 3 \end{cases}$

47. $\begin{cases} \frac{1}{3}x - \frac{1}{4}y = 2 \\ -8x + 6y = 10 \end{cases}$

48. $\begin{cases} -\frac{1}{10}x + \frac{1}{2}y = 4 \\ 2x - 10y = -80 \end{cases}$

49–52 ■ Use a graphing device to graph both lines in the same viewing rectangle. (Note that you must solve for y in terms of x before graphing if you are using a graphing calculator.) Solve the system rounded to two decimal places, either by zooming in and using TRACE or by using `Intersect`.

49. $\begin{cases} 0.21x + 3.17y = 9.51 \\ 2.35x - 1.17y = 5.89 \end{cases}$

50. $\begin{cases} 18.72x - 14.91y = 12.33 \\ 6.21x - 12.92y = 17.82 \end{cases}$

51. $\begin{cases} 2371x - 6552y = 13{,}591 \\ 9815x + 992y = 618{,}555 \end{cases}$

52. $\begin{cases} -435x + 912y = 0 \\ 132x + 455y = 994 \end{cases}$

53–56 ■ Find x and y in terms of a and b.

53. $\begin{cases} x + y = 0 \\ x + ay = 1 \end{cases} \quad (a \neq 1)$

54. $\begin{cases} ax + by = 0 \\ x + y = 1 \end{cases} \quad (a \neq b)$

55. $\begin{cases} ax + by = 1 \\ bx + ay = 1 \end{cases} \quad (a^2 - b^2 \neq 0)$

56. $\begin{cases} ax + by = 0 \\ a^2x + b^2y = 1 \end{cases} \quad (a \neq 0, b \neq 0, a \neq b)$

APPLICATIONS

57. Number Problem Find two numbers whose sum is 34 and whose difference is 10.

58. Number Problem The sum of two numbers is twice their difference. The larger number is 6 more than twice the smaller. Find the numbers.

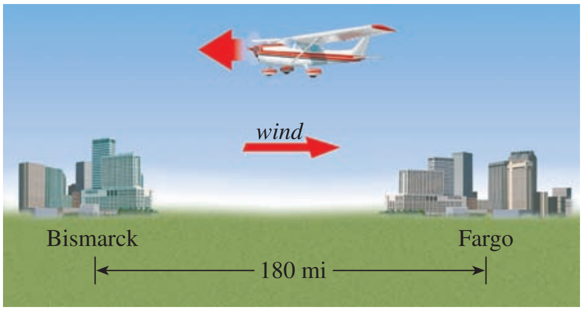
59. Value of Coins A man has 14 coins in his pocket, all of which are dimes and quarters. If the total value of his change is \$2.75, how many dimes and how many quarters does he have?

60. Admission Fees The admission fee at an amusement park is \$1.50 for children and \$4.00 for adults. On a certain day, 2200 people entered the park, and the admission fees that were collected totaled \$5050. How many children and how many adults were admitted?

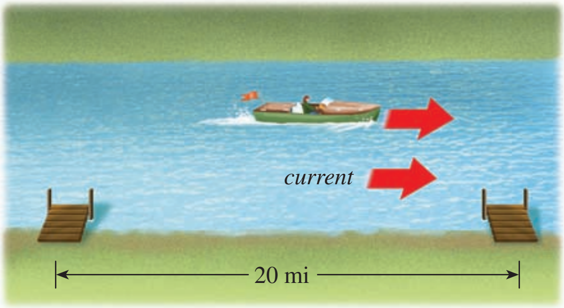
61. Gas Station A gas station sells regular gas for \$2.20 per gallon and premium gas for \$3.00 a gallon. At the end of a business day 280 gallons of gas were sold, and receipts totaled \$680. How many gallons of each type of gas were sold?

62. Fruit Stand A fruit stand sells two varieties of strawberries: standard and deluxe. A box of standard strawberries sells for \$7, and a box of deluxe strawberries sells for \$10. In one day the stand sells 135 boxes of strawberries for a total of \$1110. How many boxes of each type were sold?

63. Airplane Speed A man flies a small airplane from Fargo to Bismarck, North Dakota—a distance of 180 mi. Because he is flying into a head wind, the trip takes him 2 hours. On the way back, the wind is still blowing at the same speed, so the return trip takes only 1 h 12 min. What is his speed in still air, and how fast is the wind blowing?



64. Boat Speed A boat on a river travels downstream between two points, 20 mi apart, in one hour. The return trip against the current takes $2\frac{1}{2}$ hours. What is the boat's speed, and how fast does the current in the river flow?



65. Nutrition A researcher performs an experiment to test a hypothesis that involves the nutrients niacin and retinol. She feeds one group of laboratory rats a daily diet of precisely 32 units of niacin and 22,000 units of retinol. She uses two types of commercial pellet foods. Food A contains 0.12 unit of niacin and 100 units of retinol per gram. Food B contains 0.20 unit of niacin and 50 units of retinol per gram. How many grams of each food does she feed this group of rats each day?

66. Coffee Blends A customer in a coffee shop purchases a blend of two coffees: Kenyan, costing \$3.50 a pound, and Sri Lankan, costing \$5.60 a pound. He buys 3 lb of the blend, which costs him \$11.55. How many pounds of each kind went into the mixture?

67. Mixture Problem A chemist has two large containers of sulfuric acid solution, with different concentrations of acid in each container. Blending 300 mL of the first solution and 600 mL of the second gives a mixture that is 15% acid, whereas blending 100 mL of the first with 500 mL of the second gives a $12\frac{1}{2}\%$ acid mixture. What are the concentrations of sulfuric acid in the original containers?

68. Mixture Problem A biologist has two brine solutions, one containing 5% salt and another containing 20% salt. How many milliliters of each solution should she mix to obtain 1 L of a solution that contains 14% salt?

69. Investments A woman invests a total of \$20,000 in two accounts, one paying 5% and the other paying 8% simple interest per year. Her annual interest is \$1180. How much did she invest at each rate?

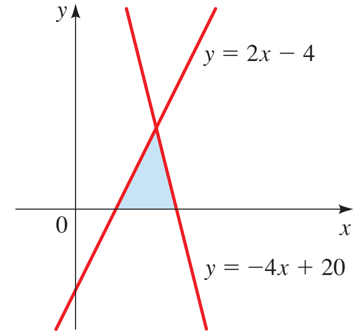
70. Investments A man invests his savings in two accounts, one paying 6% and the other paying 10% simple interest per year. He puts twice as much in the lower-yielding account because it is less risky. His annual interest is \$3520. How much did he invest at each rate?

71. Distance, Speed, and Time John and Mary leave their house at the same time and drive in opposite directions. John drives at 60 mi/h and travels 35 mi farther than Mary, who drives at 40 mi/h. Mary's trip takes 15 min longer than John's. For what length of time does each of them drive?

72. Aerobic Exercise A woman keeps fit by bicycling and running every day. On Monday she spends $\frac{1}{2}$ hour at each activity, covering a total of $12\frac{1}{2}$ mi. On Tuesday she runs for 12 min and cycles for 45 min, covering a total of 16 mi. Assuming that her running and cycling speeds don't change from day to day, find these speeds.

73. Number Problem The sum of the digits of a two-digit number is 7. When the digits are reversed, the number is increased by 27. Find the number.

74. Area of a Triangle Find the area of the triangle that lies in the first quadrant (with its base on the x-axis) and that is bounded by the lines $y = 2x - 4$ and $y = -4x + 20$.



DISCOVERY ■ DISCUSSION ■ WRITING

75. The Least Squares Line The *least squares* line or *regression* line is the line that best fits a set of points in the plane. We studied this line in the *Focus on Modeling* that follows Chapter 1 (see page 162). By using calculus, it can be shown that the line that best fits the n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is the line $y = ax + b$, where the coefficients a and b satisfy the following pair of linear equations. (The notation $\sum_{k=1}^{n} x_k$ stands for the sum of all the x's. See Section 8.1 for a complete description of sigma (Σ) notation.)

$$\left(\sum_{k=1}^{n} x_k\right)a + nb = \sum_{k=1}^{n} y_k$$

$$\left(\sum_{k=1}^{n} x_k^2\right)a + \left(\sum_{k=1}^{n} x_k\right)b = \sum_{k=1}^{n} x_k y_k$$

Use these equations to find the least squares line for the following data points.

$$(1, 3), \quad (2, 5), \quad (3, 6), \quad (5, 6), \quad (7, 9)$$

Sketch the points and your line to confirm that the line fits these points well. If your calculator computes regression lines, see whether it gives you the same line as the formulas.

5.2 Systems of Linear Equations in Several Variables

LEARNING OBJECTIVES After completing this section, you will be able to:

Solve a linear system in several variables ▶ Determine the number of solutions of a linear system in several variables ▶ Model with linear systems

A **linear equation in *n* variables** is an equation that can be put in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

where $a_1, a_2, \ldots, a_n$ and c are real numbers, and $x_1, x_2, \ldots, x_n$ are the variables. If we have only three or four variables, we generally use x, y, z, and w instead of x_1, x_2, x_3, and x_4. Such equations are called *linear* because if we have just two variables, the equation is $a_1x + a_2y = c$, which is the equation of a line. Here are some examples of equations in three variables that illustrate the difference between linear and nonlinear equations.

Linear equations	Nonlinear equations	
$6x_1 - 3x_2 + \sqrt{5}x_3 = 10$	$x^2 + 3y - \sqrt{z} = 5$	Not linear because it contains the square and the square root of a variable
$x + y + z = 2w - \frac{1}{2}$	$x_1x_2 + 6x_3 = -6$	Not linear because it contains a product of variables

In this section we study systems of linear equations in three or more variables.

Solving a Linear System

The following are two examples of systems of linear equations in three variables. The second system is in **triangular form**; that is, the variable x doesn't appear in the second equation, and the variables x and y do not appear in the third equation.

A system of linear equations

$$\begin{cases} x - 2y - z = 1 \\ -x + 3y + 3z = 4 \\ 2x - 3y + z = 10 \end{cases}$$

A system in triangular form

$$\begin{cases} x - 2y - z = 1 \\ y + 2z = 5 \\ z = 3 \end{cases}$$

It's easy to solve a system that is in triangular form by using back-substitution. So our goal in this section is to start with a system of linear equations and change it to a system in triangular form that has the same solutions as the original system. We begin by showing how to use back-substitution to solve a system that is already in triangular form.

EXAMPLE 1 | Solving a Triangular System Using Back-Substitution

Solve the system using back-substitution,

$$\begin{cases} x - 2y - z = 1 & \text{Equation 1} \\ y + 2z = 5 & \text{Equation 2} \\ z = 3 & \text{Equation 3} \end{cases}$$

SOLUTION From the last equation we know that $z = 3$. We back-substitute this into the second equation and solve for y:

$$y + 2(3) = 5 \qquad \text{Back-substitute } z = 3 \text{ into Equation 2}$$

$$y = -1 \qquad \text{Solve for } y$$

Then we back-substitute $y = -1$ and $z = 3$ into the first equation and solve for x:

$$x - 2(-1) - (3) = 1 \qquad \text{Back-substitute } y = -1 \text{ and } z = 3 \text{ into Equation 1}$$

$$x = 2 \qquad \text{Solve for } x$$

The solution of the system is $x = 2$, $y = -1$, $z = 3$. We can also write the solution as the ordered triple $(2, -1, 3)$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **7**

To change a system of linear equations to an **equivalent system** (that is, a system with the same solutions as the original system), we use the elimination method. This means that we can use the following operations.

OPERATIONS THAT LEAD TO AN EQUIVALENT SYSTEM

1. Add a nonzero multiple of one equation to another.
2. Multiply an equation by a nonzero constant.
3. Interchange the positions of two equations.

To solve a linear system, we use these operations to change the system to an equivalent triangular system. Then we use back-substitution as in Example 1. This process is called **Gaussian elimination**.

EXAMPLE 2 | Solving a System of Three Equations in Three Variables

Solve the system using Gaussian elimination.

$$\begin{cases} x - 2y + 3z = 1 & \text{Equation 1} \\ x + 2y - z = 13 & \text{Equation 2} \\ 3x + 2y - 5z = 3 & \text{Equation 3} \end{cases}$$

SOLUTION We need to change this to a triangular system, so we begin by eliminating the x-term from the second equation.

$$\begin{array}{rll} x + 2y - z = 13 & & \text{Equation 2} \\ x - 2y + 3z = 1 & & \text{Equation 1} \\ \hline 4y - 4z = 12 & & \text{Equation 2} + (-1) \times \text{Equation 1} = \text{new Equation 2} \end{array}$$

This gives us a new, equivalent system that is one step closer to triangular form.

$$\begin{cases} x - 2y + 3z = 1 & \text{Equation 1} \\ 4y - 4z = 12 & \text{Equation 2} \\ 3x + 2y - 5z = 3 & \text{Equation 3} \end{cases}$$

Now we eliminate the x-term from the third equation:

$$\begin{array}{r} 3x + 2y - 5z = 3 \\ -3x + 6y - 9z = -3 \\ \hline 8y - 14z = 0 \end{array}$$

$$\begin{cases} x - 2y + 3z = 1 \\ 4y - 4z = 12 \\ 8y - 14z = 0 & \text{Equation 3} + (-3) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

Then we eliminate the y-term from the third equation:

$$\begin{array}{r} 8y - 14z = 0 \\ -8y + 8z = -24 \\ \hline -6z = -24 \end{array}$$

$$\begin{cases} x - 2y + 3z = 1 \\ 4y - 4z = 12 \\ -6z = -24 & \text{Equation 3} + (-2) \times \text{Equation 2} = \text{new Equation 3} \end{cases}$$

The system is now in triangular form, but it will be easier to work with if we divide the second and third equations by the common factors of each term.

$$\begin{cases} x - 2y + 3z = 1 \\ y - z = 3 & \frac{1}{4} \times \text{Equation 2} = \text{new Equation 2} \\ z = 4 & -\frac{1}{6} \times \text{Equation 3} = \text{new Equation 3} \end{cases}$$

Now we use back-substitution to solve the system. From the third equation we get $z = 4$. We back-substitute this into the second equation and solve for y:

$$\begin{array}{rl} y - (4) = 3 & \text{Back-substitute } z = 4 \text{ into Equation 2} \\ y = 7 & \text{Solve for } y \end{array}$$

CHECK YOUR ANSWER

$x = 3, y = 7, z = 4$:

$$\begin{aligned} (3) - 2(7) + 3(4) &= 1 \\ (3) + 2(7) - (4) &= 13 \\ 3(3) + 2(7) - 5(4) &= 3 \quad \checkmark \end{aligned}$$

Now we back-substitute $y = 7$ and $z = 4$ into the first equation and solve for x:

$$x - 2(7) + 3(4) = 1 \qquad \text{Back-substitute } y = 7 \text{ and } z = 4 \text{ into Equation 1}$$

$$x = 3 \qquad \text{Solve for } x$$

The solution of the system is $x = 3$, $y = 7$, $z = 4$, which we can write as the ordered triple $(3, 7, 4)$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17**

The Number of Solutions of a Linear System

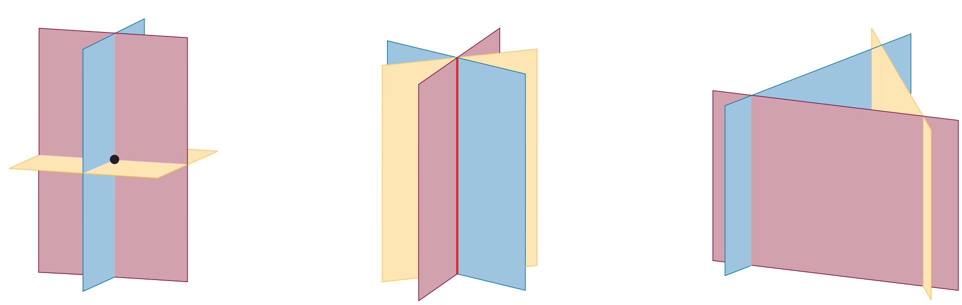
The graph of a linear equation in three variables is a plane in three-dimensional space. A system of three equations in three variables represents three planes in space. The solutions of the system are the points where all three planes intersect. Three planes may intersect in a point, a line, not at all, or all three planes may coincide. Figure 1 illustrates some of these possibilities. Checking these possibilities, we see that there are three possible outcomes when solving such a system.

NUMBER OF SOLUTIONS OF A LINEAR SYSTEM

For a system of linear equations, exactly one of the following is true.

1. The system has exactly one solution.
2. The system has no solution.
3. The system has infinitely many solutions.

A system with no solutions is said to be **inconsistent**, and a system with infinitely many solutions is said to be **dependent**. As we see in the next example, a linear system has no solution if we end up with a *false equation* after applying Gaussian elimination to the system.



(a) The three planes intersect at a single point. The system has one solution.

(b) The three planes intersect at more than one point. The system has infinitely many solutions.

(c) The three planes have no point in common. The system has no solution.

FIGURE 1

EXAMPLE 3 | A System with No Solution

Solve the following system.

$$\begin{cases} x + 2y - 2z = 1 & \text{Equation 1} \\ 2x + 2y - z = 6 & \text{Equation 2} \\ 3x + 4y - 3z = 5 & \text{Equation 3} \end{cases}$$

SOLUTION To put this in triangular form, we begin by eliminating the x-terms from the second equation and the third equation.

$$\begin{cases} x + 2y - 2z = 1 \\ \quad -2y + 3z = 4 & \text{Equation 2} + (-2) \times \text{Equation 1} = \text{new Equation 2} \\ 3x + 4y + 3z = 5 \end{cases}$$

$$\begin{cases} x + 2y - 2z = 1 \\ \quad -2y + 3z = 4 \\ \quad -2y + 3z = 2 & \text{Equation 3} + (-3) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

Now we eliminate the y-term from the third equation:

$$\begin{cases} x + 2y - 2z = 1 \\ \quad -2y + 3z = 4 \\ \qquad\qquad 0 = 2 & \text{Equation 3} + (-1) \times \text{Equation 2} = \text{new Equation 3} \end{cases}$$

The system is now in triangular form, but the third equation says $0 = 2$, which is false. No matter what values we assign to x, y, and z, the third equation will never be true. This means that the system has *no solution*.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **29**

EXAMPLE 4 | A System with Infinitely Many Solutions

Solve the following system.

$$\begin{cases} x - \;\; y + 5z = -2 & \text{Equation 1} \\ 2x + \;\; y + 4z = \;\; 2 & \text{Equation 2} \\ 2x + 4y - 2z = \;\; 8 & \text{Equation 3} \end{cases}$$

SOLUTION To put this in triangular form, we begin by eliminating the x-terms from the second equation and the third equation.

$$\begin{cases} x - \;\; y + 5z = -2 \\ \qquad 3y - 6z = \;\; 6 & \text{Equation 2} + (-2) \times \text{Equation 1} = \text{new Equation 2} \\ 2x + 4y - 2z = \;\; 8 \end{cases}$$

$$\begin{cases} x - y + \;\; 5z = -2 \\ \quad 3y - \;\; 6z = \;\; 6 \\ \quad 6y - 12z = \;\; 12 & \text{Equation 3} + (-2) \times \text{Equation 1} = \text{new Equation 3} \end{cases}$$

Now we eliminate the y-term from the third equation:

$$\begin{cases} x - y + 5z = -2 \\ \quad 3y - 6z = \;\; 6 \\ \qquad\quad 0 = \;\; 0 & \text{Equation 3} + (-2) \times \text{Equation 2} = \text{new Equation 3} \end{cases}$$

The new third equation is true, but it gives us no new information, so we can drop it from the system. Only two equations are left. We can use them to solve for x and y in terms of z, but z can take on any value, so there are infinitely many solutions.

To find the complete solution of the system, we begin by solving for y in terms of z, using the new second equation.

$$\begin{aligned} 3y - 6z &= 6 && \text{Equation 2} \\ y - 2z &= 2 && \text{Multiply by } \tfrac{1}{3} \\ y &= 2z + 2 && \text{Solve for } y \end{aligned}$$

Then we solve for x in terms of z, using the first equation:

$$x - (2z + 2) + 5z = -2 \qquad \text{Substitute } y = 2z + 2 \text{ into Equation 1}$$

$$x + 3z - 2 = -2 \qquad \text{Simplify}$$

$$x = -3z \qquad \text{Solve for } x$$

To describe the complete solution, we let t represent any real number. The solution is

$$x = -3t$$
$$y = 2t + 2$$
$$z = t$$

We can also write this as the ordered triple $(-3t, 2t + 2, t)$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **33** ■

In the solution of Example 4 the variable t is called a **parameter**. To get a specific solution, we give a specific value to the parameter t. For instance, if we set $t = 2$, we get

$$x = -3(2) = -6$$
$$y = 2(2) + 2 = 6$$
$$z = 2$$

Thus $(-6, 6, 2)$ is a solution of the system. Here are some other solutions of the system obtained by substituting other values for the parameter t.

Parameter t	**Solution $(-3t, 2t + 2, t)$**
-1	$(3, 0, -1)$
0	$(0, 2, 0)$
3	$(-9, 8, 3)$
10	$(-30, 22, 10)$

You should check that these points satisfy the original equations. There are infinitely many choices for the parameter t, so the system has infinitely many solutions.

▼ Modeling Using Linear Systems

Linear systems are used to model situations that involve several varying quantities. In the next example we consider an application of linear systems to finance.

EXAMPLE 5 | Modeling a Financial Problem Using a Linear System

Jason receives an inheritance of \$50,000. His financial advisor suggests that he invest this in three mutual funds: a money-market fund, a blue-chip stock fund, and a high-tech stock fund. The advisor estimates that the money-market fund will return 5% over the next year, the blue-chip fund 9%, and the high-tech fund 16%. Jason wants a total first-year return of \$4000. To avoid excessive risk, he decides to invest three times as much in the money-market fund as in the high-tech stock fund. How much should he invest in each fund?

SOLUTION

Let x = amount invested in the money-market fund

y = amount invested in the blue-chip stock fund

z = amount invested in the high-tech stock fund

We convert each fact given in the problem into an equation:

$$
\begin{aligned}
x + y + z &= 50{,}000 && \text{Total amount invested is \$50,000}\\
0.05x + 0.09y + 0.16z &= 4000 && \text{Total investment return is \$4000}\\
x &= 3z && \text{Money-market amount is } 3 \times \text{high-tech amount}
\end{aligned}
$$

Multiplying the second equation by 100 and rewriting the third gives the following system, which we solve using Gaussian elimination.

$$
\begin{cases}
x + y + z = 50{,}000 & \\
5x + 9y + 16z = 400{,}000 & 100 \times \text{Equation 2}\\
x - 3z = 0 & \text{Subtract } 3z
\end{cases}
$$

$$
\begin{cases}
x + y + z = 50{,}000 & \\
4y + 11z = 150{,}000 & \text{Equation 2} + (-5) \times \text{Equation 1} = \text{new Equation 2}\\
-y - 4z = -50{,}000 & \text{Equation 3} + (-1) \times \text{Equation 1} = \text{new Equation 3}
\end{cases}
$$

$$
\begin{cases}
x + y + z = 50{,}000 & \\
-5z = -50{,}000 & \text{Equation 2} + 4 \times \text{Equation 3} = \text{new Equation 3}\\
-y - 4z = -50{,}000 &
\end{cases}
$$

$$
\begin{cases}
x + y + z = 50{,}000 & \\
z = 10{,}000 & \left(-\tfrac{1}{5}\right) \times \text{Equation 2}\\
y + 4z = 50{,}000 & (-1) \times \text{Equation 3}
\end{cases}
$$

$$
\begin{cases}
x + y + z = 50{,}000 & \\
y + 4z = 50{,}000 & \text{Interchange Equations 2 and 3}\\
z = 10{,}000 &
\end{cases}
$$

Now that the system is in triangular form, we use back-substitution to find that $x = 30{,}000$, $y = 10{,}000$, and $z = 10{,}000$. This means that Jason should invest

$30,000 in the money-market fund

$10,000 in the blue-chip stock fund

$10,000 in the high-tech stock fund

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **39** ■

5.2 EXERCISES

CONCEPTS

1–2 ■ These exercises refer to the following system.

$$
\begin{cases}
x - y + z = 2\\
-x + 2y + z = -3\\
3x + y - 2z = 2
\end{cases}
$$

1. If we add 2 times the first equation to the second equation, the second equation becomes ______________ = ____.

2. To eliminate x from the third equation, we add ________ times the first equation to the third equation. The third equation becomes ______________ = ____.

SKILLS

3–6 ■ State whether the equation or system of equations is linear.

3. $6x - \sqrt{3}y + \frac{1}{2}z = 0$

4. $x^2 + y^2 + z^2 = 4$

5. $\begin{cases} xy - 3y + z = 5 \\ x - y^2 + 5z = 0 \\ 2x + yz = 3 \end{cases}$

6. $\begin{cases} x - 2y + 3z = 10 \\ 2x + 5y = 2 \\ y + 2z = 4 \end{cases}$

7–12 ■ Use back-substitution to solve the triangular system.

7. $\begin{cases} x - 3y + z = 0 \\ y - z = 3 \\ z = -2 \end{cases}$

8. $\begin{cases} 3x - 3y + z = 0 \\ y + 4z = 10 \\ z = 3 \end{cases}$

9. $\begin{cases} x + 2y + z = 7 \\ -y + 3z = 9 \\ 2z = 6 \end{cases}$

10. $\begin{cases} x - 2y + 3z = 10 \\ 2y - z = 2 \\ 3z = 12 \end{cases}$

11. $\begin{cases} 2x - y + 6z = 5 \\ y + 4z = 0 \\ -2z = 1 \end{cases}$

12. $\begin{cases} 4x + 3z = 10 \\ 2y - z = -6 \\ \frac{1}{2}z = 4 \end{cases}$

13–16 ■ Perform an operation on the given system that eliminates the indicated variable. Write the new equivalent system.

13. $\begin{cases} 3x + y + z = 4 \\ -x + y + 2z = 0 \\ x - 2y - z = -1 \end{cases}$

Eliminate the x-term from the second equation.

14. $\begin{cases} -5x + 2y - 3z = 3 \\ 10x - 3y + z = -20 \\ -x + 3y + z = 8 \end{cases}$

Eliminate the x-term from the second equation.

15. $\begin{cases} 2x + y - 3z = 5 \\ 2x + 3y + z = 13 \\ 6x - 5y - z = 7 \end{cases}$

Eliminate the x-term from the third equation.

16. $\begin{cases} x - 3y + 2z = -1 \\ y + z = -1 \\ 2y - z = 1 \end{cases}$

Eliminate the y-term from the third equation.

17–38 ■ Find the complete solution of the linear system, or show that it is inconsistent.

17. $\begin{cases} x - y - z = 4 \\ 2y + z = -1 \\ -x + y - 2z = 5 \end{cases}$

18. $\begin{cases} x - y + z = 0 \\ y + 2z = -2 \\ x + y - z = 2 \end{cases}$

19. $\begin{cases} x + 2y - z = -6 \\ y - 3z = -16 \\ x - 3y + 2z = 14 \end{cases}$

20. $\begin{cases} x - 2y + 3z = -10 \\ 3y + z = 7 \\ x + y - z = 7 \end{cases}$

21. $\begin{cases} x + y + z = 4 \\ x + 3y + 3z = 10 \\ 2x + y - z = 3 \end{cases}$

22. $\begin{cases} x + y + z = 0 \\ -x + 2y + 5z = 3 \\ 3x - y = 6 \end{cases}$

23. $\begin{cases} x - 4z = 1 \\ 2x - y - 6z = 4 \\ 2x + 3y - 2z = 8 \end{cases}$

24. $\begin{cases} x - y + 2z = 2 \\ 3x + y + 5z = 8 \\ 2x - y - 2z = -7 \end{cases}$

25. $\begin{cases} 2x + 4y - z = 2 \\ x + 2y - 3z = -4 \\ 3x - y + z = 1 \end{cases}$

26. $\begin{cases} 2x + y - z = -8 \\ -x + y + z = 3 \\ -2x + 4z = 18 \end{cases}$

27. $\begin{cases} 2y + 4z = -1 \\ -2x + y + 2z = -1 \\ 4x - 2y = 0 \end{cases}$

28. $\begin{cases} y - z = -1 \\ 6x + 2y + z = 2 \\ -x - y - 3z = -2 \end{cases}$

29. $\begin{cases} x + 2y - z = 1 \\ 2x + 3y - 4z = -3 \\ 3x + 6y - 3z = 4 \end{cases}$

30. $\begin{cases} -x + 2y + 5z = 4 \\ x - 2z = 0 \\ 4x - 2y - 11z = 2 \end{cases}$

31. $\begin{cases} 2x + 3y - z = 1 \\ x + 2y = 3 \\ x + 3y + z = 4 \end{cases}$

32. $\begin{cases} x - 2y - 3z = 5 \\ 2x + y - z = 5 \\ 4x - 3y - 7z = 5 \end{cases}$

33. $\begin{cases} x + y - z = 0 \\ x + 2y - 3z = -3 \\ 2x + 3y - 4z = -3 \end{cases}$

34. $\begin{cases} x - 2y + z = 3 \\ 2x - 5y + 6z = 7 \\ 2x - 3y - 2z = 5 \end{cases}$

35. $\begin{cases} x + 3y - 2z = 0 \\ 2x + 4z = 4 \\ 4x + 6y = 4 \end{cases}$

36. $\begin{cases} 2x + 4y - z = 3 \\ x + 2y + 4z = 6 \\ x + 2y - 2z = 0 \end{cases}$

37. $\begin{cases} x + z + 2w = 6 \\ y - 2z = -3 \\ x + 2y - z = -2 \\ 2x + y + 3z - 2w = 0 \end{cases}$

38. $\begin{cases} x + y + z + w = 0 \\ x + y + 2z + 2w = 0 \\ 2x + 2y + 3z + 4w = 1 \\ 2x + 3y + 4z + 5w = 2 \end{cases}$

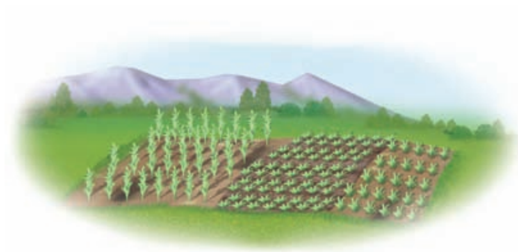
APPLICATIONS

39–40 ■ **Finance** An investor has \$100,000 to invest in three types of bonds: short-term, intermediate-term, and long-term. How much should she invest in each type to satisfy the given conditions?

39. Short-term bonds pay 4% annually, intermediate-term bonds pay 5%, and long-term bonds pay 6%. The investor wishes to realize a total annual income of 5.1%, with equal amounts invested in short- and intermediate-term bonds.

40. Short-term bonds pay 4% annually, intermediate-term bonds pay 6%, and long-term bonds pay 8%. The investor wishes to have a total annual return of \$6700 on her investment, with equal amounts invested in intermediate- and long-term bonds.

41. **Agriculture** A farmer has 1200 acres of land on which he grows corn, wheat, and soybeans. It costs \$45 per acre to grow corn, \$60 to grow wheat, and \$50 to grow soybeans. Because of market demand, the farmer will grow twice as many acres of wheat as of corn. He has allocated \$63,750 for the cost of growing his crops. How many acres of each crop should he plant?



42. Gas Station A gas station sells three types of gas: Regular for \$3.00 a gallon, Performance Plus for \$3.20 a gallon, and Premium for \$3.30 a gallon. On a particular day 6500 gallons of gas were sold for a total of \$20,050. Three times as many gallons of Regular as Premium gas were sold. How many gallons of each type of gas were sold that day?

43. Nutrition A biologist is performing an experiment on the effects of various combinations of vitamins. She wishes to feed each of her laboratory rabbits a diet that contains exactly 9 mg of niacin, 14 mg of thiamin, and 32 mg of riboflavin. She has available three different types of commercial rabbit pellets; their vitamin content (per ounce) is given in the table. How many ounces of each type of food should each rabbit be given daily to satisfy the experiment requirements?

	Type A	Type B	Type C
Niacin (mg)	2	3	1
Thiamin (mg)	3	1	3
Riboflavin (mg)	8	5	7

44. Diet Program Nicole started a new diet that requires each meal to have 460 calories, 6 grams of fiber, and 11 grams of fat. The table shows the fiber, fat, and calorie content of one serving of each of three breakfast foods. How many servings of each food should Nicole eat to follow her diet?

Food	Fiber	Fat	Calories
Toast	2	1	100
Cottage cheese	0	5	120
Fruit	2	0	60

45. Juice Blends The Juice Company offers three kinds of smoothies: Midnight Mango, Tropical Torrent, and Pineapple Power. Each smoothie contains the amounts of juices shown in the table.

Smoothie	Mango juice (oz)	Pineapple juice (oz)	Orange juice (oz)
Midnight Mango	8	3	3
Tropical Torrent	6	5	3
Pineapple Power	2	8	4

On a particular day the Juice Company used 820 oz of mango juice, 690 oz of pineapple juice, and 450 oz of orange juice. How many smoothies of each kind were sold that day?

46. Appliance Manufacturing Kitchen Korner produces refrigerators, dishwashers, and stoves at three different factories. The table gives the number of each product produced at each factory per day. Kitchen Korner receives an order for 110 refrigerators, 150 dishwashers, and 114 ovens. How many days should each plant be scheduled to fill this order?

Appliance	Factory A	Factory B	Factory C
Refrigerators	8	10	14
Dishwashers	16	12	10
Stoves	10	18	6

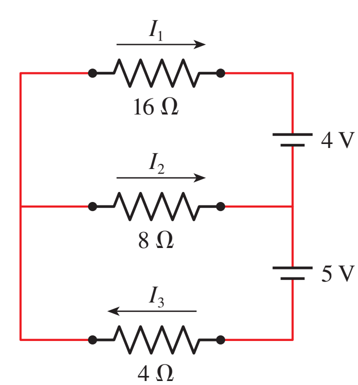
47. Stock Portfolio An investor owns three stocks: A, B, and C. The closing prices of the stocks on three successive trading days are given in the table.

	Stock A	Stock B	Stock C
Monday	\$10	\$25	\$29
Tuesday	\$12	\$20	\$32
Wednesday	\$16	\$15	\$32

Despite the volatility in the stock prices, the total value of the investor's stocks remained unchanged at \$74,000 at the end of each of these three days. How many shares of each stock does the investor own?

48. Electricity By using Kirchhoff's Laws, it can be shown that the currents I_1, I_2, and I_3 that pass through the three branches of the circuit in the figure satisfy the given linear system. Solve the system to find I_1, I_2, and I_3.

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 16I_1 - 8I_2 = 4 \\ 8I_2 + 4I_3 = 5 \end{cases}$$



DISCOVERY ▪ DISCUSSION ▪ WRITING

49. Can a Linear System Have Exactly Two Solutions?

(a) Suppose that (x_0, y_0, z_0) and (x_1, y_1, z_1) are solutions of the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

Show that $\left(\dfrac{x_0 + x_1}{2}, \dfrac{y_0 + y_1}{2}, \dfrac{z_0 + z_1}{2}\right)$ is also a solution.

(b) Use the result of part (a) to prove that if the system has two different solutions, then it has infinitely many solutions.

DISCOVERY PROJECT **Best Fit Versus Exact Fit**

In this project we use linear systems to find quadratic functions whose graphs pass through a set of given points. You can find the project at the book companion website: **www.stewartmath.com**

5.3 Partial Fractions

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find the partial fraction decomposition of a rational expression in which the denominator consists of distinct linear factors, repeated linear factors, irreducible quadratic factors, or repeated irreducible quadratic factors

To write a sum or difference of fractional expressions as a single fraction, we bring them to a common denominator. For example,

$$\frac{1}{x-1} + \frac{1}{2x+1} = \frac{(2x+1)+(x-1)}{(x-1)(2x+1)} = \frac{3x}{2x^2-x-1}$$

But for some applications of algebra to calculus we must reverse this process—that is, we must express a fraction such as $3x/(2x^2 - x - 1)$ as the sum of the simpler fractions $1/(x-1)$ and $1/(2x+1)$. These simpler fractions are called *partial fractions*.

Common denominator

$$\frac{1}{x-1} + \frac{1}{2x+1} = \frac{3x}{2x^2-x-1}$$

Partial fractions

Let r be the rational function

$$r(x) = \frac{P(x)}{Q(x)}$$

where the degree of P is less than the degree of Q. By the Linear and Quadratic Factors Theorem in Section 3.6, every polynomial with real coefficients can be factored completely into linear and irreducible quadratic factors, that is, factors of the form $ax + b$ and $ax^2 + bx + c$, where a, b, and c are real numbers. For instance,

$$x^4 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

After we have completely factored the denominator Q of r, we can express $r(x)$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax+b)^i} \quad \text{and} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

This sum is called the **partial fraction decomposition** of r. Let's examine the details of the four possible cases.

Distinct Linear Factors

We first consider the case in which the denominator factors into distinct linear factors.

CASE 1: THE DENOMINATOR IS A PRODUCT OF DISTINCT LINEAR FACTORS

Suppose that we can factor $Q(x)$ as

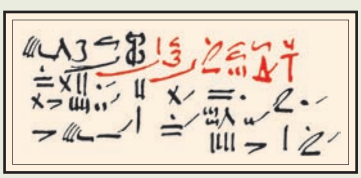
$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n)$$

with no factor repeated. In this case the partial fraction decomposition of $P(x)/Q(x)$ takes the form

$$\frac{P(x)}{Q(x)} = \frac{A}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

The Rhind Papyrus is the oldest known mathematical document. It is an Egyptian scroll written in 1650 B.C. by the scribe Ahmes, who explains that it is an exact copy of a scroll written 200 years earlier. Ahmes claims that his papyrus contains "a thorough study of all things, insight into all that exists, knowledge of all obscure secrets." Actually, the document contains rules for doing arithmetic, including multiplication and division of fractions and several exercises with solutions. The exercise shown below reads: "A heap and its seventh make nineteen; how large is the heap?" In solving problems of this sort, the Egyptians used partial fractions because their number system required all fractions to be written as sums of reciprocals of whole numbers. For example, $\frac{7}{12}$ would be written as $\frac{1}{3} + \frac{1}{4}$.

The papyrus gives a correct formula for the volume of a truncated pyramid, which the ancient Egyptians used when building the pyramids at Giza. It also gives the formula $A = (\frac{8}{9}d)^2$ for the area of a circle with diameter d. How close is this to the actual area?



The constants $A_1, A_2, \ldots, A_n$ are determined as in the following example.

EXAMPLE 1 | Distinct Linear Factors

Find the partial fraction decomposition of $\dfrac{5x + 7}{x^3 + 2x^2 - x - 2}$.

SOLUTION The denominator factors as follows.

$$\begin{aligned} x^3 + 2x^2 - x - 2 &= x^2(x + 2) - (x + 2) = (x^2 - 1)(x + 2) \\ &= (x - 1)(x + 1)(x + 2) \end{aligned}$$

This gives us the partial fraction decomposition

$$\frac{5x + 7}{x^3 + 2x^2 - x - 2} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 2}$$

Multiplying each side by the common denominator, $(x - 1)(x + 1)(x + 2)$, we get

$$\begin{aligned} 5x + 7 &= A(x + 1)(x + 2) + B(x - 1)(x + 2) + C(x - 1)(x + 1) \\ &= A(x^2 + 3x + 2) + B(x^2 + x - 2) + C(x^2 - 1) && \text{Expand} \\ &= (A + B + C)x^2 + (3A + B)x + (2A - 2B - C) && \text{Combine like terms} \end{aligned}$$

If two polynomials are equal, then their coefficients are equal. Thus, since $5x + 7$ has no x^2-term, we have $A + B + C = 0$. Similarly, by comparing the coefficients of x, we see that $3A + B = 5$, and by comparing constant terms, we get $2A - 2B - C = 7$. This leads to the following system of linear equations for A, B, and C.

$$\begin{cases} A + B + C = 0 & \text{Equation 1: Coefficients of } x^2 \\ 3A + B = 5 & \text{Equation 2: Coefficients of } x \\ 2A - 2B - C = 7 & \text{Equation 3: Constant coefficients} \end{cases}$$

We use Gaussian elimination to solve this system.

$$\begin{cases} A + B + C = 0 & \\ -2B - 3C = 5 & \text{Equation 2} + (-3) \times \text{Equation 1} \\ -4B - 3C = 7 & \text{Equation 3} + (-2) \times \text{Equation 1} \end{cases}$$

$$\begin{cases} A + B + C = 0 & \\ -2B - 3C = 5 & \\ 3C = -3 & \text{Equation 3} + (-2) \times \text{Equation 2} \end{cases}$$

From the third equation we get $C = -1$. Back-substituting, we find that $B = -1$ and $A = 2$. So the partial fraction decomposition is

$$\frac{5x + 7}{x^3 + 2x^2 - x - 2} = \frac{2}{x - 1} + \frac{-1}{x + 1} + \frac{-1}{x + 2}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **3** AND **13**

The same approach works in the remaining cases. We set up the partial fraction decomposition with the unknown constants $A, B, C, \ldots$. Then we multiply each side of the resulting equation by the common denominator, simplify the right-hand side of the equation, and equate coefficients. This gives a set of linear equations that will always have a unique solution (provided that the partial fraction decomposition has been set up correctly).

Repeated Linear Factors

We now consider the case in which the denominator factors into linear factors, some of which are repeated.

CASE 2: THE DENOMINATOR IS A PRODUCT OF LINEAR FACTORS, SOME OF WHICH ARE REPEATED

Suppose the complete factorization of $Q(x)$ contains the linear factor $ax + b$ repeated k times; that is, $(ax + b)^k$ is a factor of $Q(x)$. Then, corresponding to each such factor, the partial fraction decomposition for $P(x)/Q(x)$ contains

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}$$

EXAMPLE 2 | Repeated Linear Factors

Find the partial fraction decomposition of $\dfrac{x^2 + 1}{x(x - 1)^3}$.

SOLUTION Because the factor $x - 1$ is repeated three times in the denominator, the partial fraction decomposition has the form

$$\frac{x^2 + 1}{x(x - 1)^3} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} + \frac{D}{(x - 1)^3}$$

Multiplying each side by the common denominator, $x(x - 1)^3$, gives

$$\begin{aligned} x^2 + 1 &= A(x - 1)^3 + Bx(x - 1)^2 + Cx(x - 1) + Dx \\ &= A(x^3 - 3x^2 + 3x - 1) + B(x^3 - 2x^2 + x) + C(x^2 - x) + Dx && \text{Expand} \\ &= (A + B)x^3 + (-3A - 2B + C)x^2 + (3A + B - C + D)x - A && \text{Combine like terms} \end{aligned}$$

Equating coefficients, we get the following equations.

$$\begin{cases} A + B = 0 & \text{Coefficients of } x^3 \\ -3A - 2B + C = 1 & \text{Coefficients of } x^2 \\ 3A + B - C + D = 0 & \text{Coefficients of } x \\ -A = 1 & \text{Constant coefficients} \end{cases}$$

If we rearrange these equations by putting the last one in the first position, we can easily see (using substitution) that the solution to the system is $A = -1$, $B = 1$, $C = 0$, $D = 2$, so the partial fraction decomposition is

$$\frac{x^2 + 1}{x(x - 1)^3} = \frac{-1}{x} + \frac{1}{x - 1} + \frac{2}{(x - 1)^3}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **5** AND **29**

Irreducible Quadratic Factors

We now consider the case in which the denominator has distinct irreducible quadratic factors.

CASE 3: THE DENOMINATOR HAS IRREDUCIBLE QUADRATIC FACTORS, NONE OF WHICH IS REPEATED

Suppose the complete factorization of $Q(x)$ contains the quadratic factor $ax^2 + bx + c$ (which can't be factored further). Then, corresponding to this, the partial fraction decomposition of $P(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

EXAMPLE 3 | Distinct Quadratic Factors

Find the partial fraction decomposition of $\dfrac{2x^2 - x + 4}{x^3 + 4x}$.

SOLUTION Since $x^3 + 4x = x(x^2 + 4)$, which can't be factored further, we write

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying by $x(x^2 + 4)$, we get

$$\begin{aligned} 2x^2 - x + 4 &= A(x^2 + 4) + (Bx + C)x \\ &= (A + B)x^2 + Cx + 4A \end{aligned}$$

Equating coefficients gives us the equations

$$\begin{cases} A + B = 2 & \text{Coefficients of } x^2 \\ \phantom{A+{}} C = -1 & \text{Coefficients of } x \\ \phantom{A+{}} 4A = 4 & \text{Constant coefficients} \end{cases}$$

so $A = 1$, $B = 1$, and $C = -1$. The required partial fraction decomposition is

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **7** AND **37**

Repeated Irreducible Quadratic Factors

We now consider the case in which the denominator has irreducible quadratic factors, some of which are repeated.

CASE 4: THE DENOMINATOR HAS A REPEATED IRREDUCIBLE QUADRATIC FACTOR

Suppose the complete factorization of $Q(x)$ contains the factor $(ax^2 + bx + c)^k$, where $ax^2 + bx + c$ can't be factored further. Then the partial fraction decomposition of $P(x)/Q(x)$ will have the terms

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

EXAMPLE 4 | Repeated Quadratic Factors

Write the form of the partial fraction decomposition of

$$\frac{x^5 - 3x^2 + 12x - 1}{x^3(x^2 + x + 1)(x^2 + 2)^3}$$

SOLUTION

$$\begin{aligned} &\frac{x^5 - 3x^2 + 12x - 1}{x^3(x^2 + x + 1)(x^2 + 2)^3} \\ &\quad = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + x + 1} + \frac{Fx + G}{x^2 + 2} + \frac{Hx + I}{(x^2 + 2)^2} + \frac{Jx + K}{(x^2 + 2)^3} \end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **11** AND **41**

To find the values of $A, B, C, D, E, F, G, H, I, J$, and K in Example 4, we would have to solve a system of 11 linear equations. Although possible, this would certainly involve a great deal of work!

The techniques that we have described in this section apply only to rational functions $P(x)/Q(x)$ in which the degree of P is less than the degree of Q. If this isn't the case, we must first use long division to divide Q into P.

EXAMPLE 5 | Using Long Division to Prepare for Partial Fractions

Find the partial fraction decomposition of

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2}$$

SOLUTION Since the degree of the numerator is larger than the degree of the denominator, we use long division to obtain

$$\begin{array}{r} 2x \\ x^3 + 2x^2 - x - 2 \overline{) 2x^4 + 4x^3 - 2x^2 + x + 7} \\ \underline{2x^4 + 4x^3 - 2x^2 - 4x } \\ 5x + 7 \end{array}$$

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2} = 2x + \frac{5x + 7}{x^3 + 2x^2 - x - 2}$$

The remainder term now satisfies the requirement that the degree of the numerator is less than the degree of the denominator. At this point we proceed as in Example 1 to obtain the decomposition

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2} = 2x + \frac{2}{x - 1} + \frac{-1}{x + 1} + \frac{-1}{x + 2}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **43**

5.3 EXERCISES

CONCEPTS

1–2 ■ For each rational function r, choose from (i)–(iv) the appropriate form for its partial fraction decomposition.

1. $r(x) = \dfrac{4}{x(x-2)^2}$

(i) $\dfrac{A}{x} + \dfrac{B}{x-2}$ (ii) $\dfrac{A}{x} + \dfrac{B}{(x-2)^2}$

(iii) $\dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$ (iv) $\dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{Cx+D}{(x-2)^2}$

2. $r(x) = \dfrac{2x+8}{(x-1)(x^2+4)}$

(i) $\dfrac{A}{x-1} + \dfrac{B}{x^2+4}$

(ii) $\dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+4}$

(iii) $\dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x^2+4}$

(iv) $\dfrac{Ax+B}{x-1} + \dfrac{Cx+D}{x^2+4}$

SKILLS

3–12 ■ Write the form of the partial fraction decomposition of the function (as in Example 4). Do not determine the numerical values of the coefficients.

3. $\dfrac{1}{(x-1)(x+2)}$ **4.** $\dfrac{x}{x^2+3x-4}$

5. $\dfrac{x^2-3x+5}{(x-2)^2(x+4)}$ **6.** $\dfrac{1}{x^4-x^3}$

7. $\dfrac{x^2}{(x-3)(x^2+4)}$ **8.** $\dfrac{1}{x^4-1}$

9. $\dfrac{x^3-4x^2+2}{(x^2+1)(x^2+2)}$ **10.** $\dfrac{x^4+x^2+1}{x^2(x^2+4)^2}$

11. $\dfrac{x^3+x+1}{x(2x-5)^3(x^2+2x+5)^2}$ **12.** $\dfrac{1}{(x^3-1)(x^2-1)}$

13–44 ■ Find the partial fraction decomposition of the rational function.

13. $\dfrac{2}{(x-1)(x+1)}$ **14.** $\dfrac{2x}{(x-1)(x+1)}$

15. $\dfrac{5}{(x-1)(x+4)}$ **16.** $\dfrac{x+6}{x(x+3)}$

17. $\dfrac{12}{x^2-9}$

18. $\dfrac{x-12}{x^2-4x}$

19. $\dfrac{4}{x^2-4}$

20. $\dfrac{2x+1}{x^2+x-2}$

21. $\dfrac{x+14}{x^2-2x-8}$

22. $\dfrac{8x-3}{2x^2-x}$

23. $\dfrac{x}{8x^2-10x+3}$

24. $\dfrac{7x-3}{x^3+2x^2-3x}$

25. $\dfrac{9x^2-9x+6}{2x^3-x^2-8x+4}$

26. $\dfrac{-3x^2-3x+27}{(x+2)(2x^2+3x-9)}$

27. $\dfrac{x^2+1}{x^3+x^2}$

28. $\dfrac{3x^2+5x-13}{(3x+2)(x^2-4x+4)}$

29. $\dfrac{2x}{4x^2+12x+9}$

30. $\dfrac{x-4}{(2x-5)^2}$

31. $\dfrac{4x^2-x-2}{x^4+2x^3}$

32. $\dfrac{x^3-2x^2-4x+3}{x^4}$

33. $\dfrac{-10x^2+27x-14}{(x-1)^3(x+2)}$

34. $\dfrac{-2x^2+5x-1}{x^4-2x^3+2x-1}$

35. $\dfrac{3x^3+22x^2+53x+41}{(x+2)^2(x+3)^2}$

36. $\dfrac{3x^2+12x-20}{x^4-8x^2+16}$

37. $\dfrac{x-3}{x^3+3x}$

38. $\dfrac{3x^2-2x+8}{x^3-x^2+2x-2}$

39. $\dfrac{2x^3+7x+5}{(x^2+x+2)(x^2+1)}$

40. $\dfrac{x^2+x+1}{2x^4+3x^2+1}$

41. $\dfrac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}$

42. $\dfrac{2x^2-x+8}{(x^2+4)^2}$

43. $\dfrac{x^5-2x^4+x^3+x+5}{x^3-2x^2+x-2}$

44. $\dfrac{x^5-3x^4+3x^3-4x^2+4x+12}{(x-2)^2(x^2+2)}$

45. Determine A and B in terms of a and b.

$$\frac{ax+b}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1}$$

46. Determine A, B, C, and D in terms of a and b.

$$\frac{ax^3+bx^2}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$$

DISCOVERY ▪ DISCUSSION ▪ WRITING

47. **Recognizing Partial Fraction Decompositions** For each expression, determine whether it is already a partial fraction decomposition or whether it can be decomposed further.

(a) $\dfrac{x}{x^2+1}+\dfrac{1}{x+1}$ (b) $\dfrac{x}{(x+1)^2}$

(c) $\dfrac{1}{x+1}+\dfrac{2}{(x+1)^2}$ (d) $\dfrac{x+2}{(x^2+1)^2}$

48. **Assembling and Disassembling Partial Fractions** The following expression is a partial fraction decomposition.

$$\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{1}{x+1}$$

Use a common denominator to combine the terms into one fraction. Then use the techniques of this section to find its partial fraction decomposition. Did you get back the original expression?

5.4 Systems of Nonlinear Equations

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Solve nonlinear systems using the Substitution Method ▶ Solve nonlinear systems using the Elimination Method ▶ Solve nonlinear systems using the Graphical Method

GET READY **Prepare for this section by reviewing how to solve equations algebraically and graphically in Sections P.8, 1.4, and 1.6.**

In this section we solve systems of equations in which the equations are not all linear. The methods we learned in Section 5.1 can also be used to solve nonlinear systems.

▼ Substitution and Elimination Methods

To solve a system of nonlinear equations, we can use the substitution or elimination method, as illustrated in the next examples.

EXAMPLE 1 | Substitution Method

Find all solutions of the system.

$$\begin{cases} x^2+y^2=100 & \text{Equation 1}\\ 3x-y=10 & \text{Equation 2}\end{cases}$$

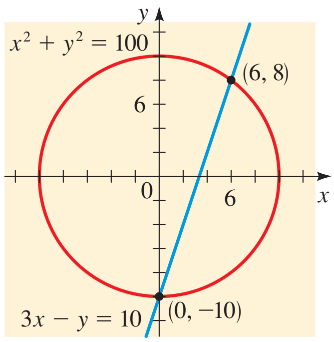


FIGURE 1

CHECK YOUR ANSWERS

$x = 0, y = -10$:

$$\begin{cases} (0)^2 + (-10)^2 = 100 \\ 3(0) - (-10) = 10 \quad \checkmark \end{cases}$$

$x = 6, y = 8$:

$$\begin{cases} (6)^2 + (8)^2 = 36 + 64 = 100 \\ 3(6) - (8) = 18 - 8 = 10 \quad \checkmark \end{cases}$$

SOLUTION Solve for one variable. We start by solving for y in the second equation:

$$y = 3x - 10 \qquad \text{Solve for } y \text{ in Equation 2}$$

Substitute. Next we substitute for y in the first equation and solve for x:

$$\begin{aligned} x^2 + (3x - 10)^2 &= 100 && \text{Substitute } y = 3x - 10 \text{ into Equation 1} \\ x^2 + (9x^2 - 60x + 100) &= 100 && \text{Expand} \\ 10x^2 - 60x &= 0 && \text{Simplify} \\ 10x(x - 6) &= 0 && \text{Factor} \\ x = 0 \quad \text{or} \quad x &= 6 && \text{Solve for } x \end{aligned}$$

Back-substitute. Now we back-substitute these values of x into the equation $y = 3x - 10$:

$$\begin{aligned} &\text{For } x = 0: && y = 3(0) - 10 = -10 && \text{Back-substitute} \\ &\text{For } x = 6: && y = 3(6) - 10 = 8 && \text{Back-substitute} \end{aligned}$$

So we have two solutions: $(0, -10)$ and $(6, 8)$.

The graph of the first equation is a circle, and the graph of the second equation is a line; Figure 1 shows that the graphs intersect at the two points $(0, -10)$ and $(6, 8)$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5**

EXAMPLE 2 | Elimination Method

Find all solutions of the system.

$$\begin{cases} 3x^2 + 2y = 26 & \text{Equation 1} \\ 5x^2 + 7y = 3 & \text{Equation 2} \end{cases}$$

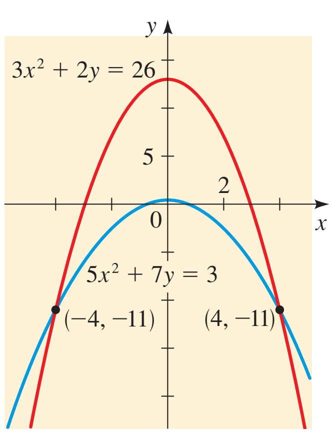


FIGURE 2

CHECK YOUR ANSWERS

$x = -4, y = -11$:

$$\begin{cases} 3(-4)^2 + 2(-11) = 26 \\ 5(-4)^2 + 7(-11) = 3 \quad \checkmark \end{cases}$$

$x = 4, y = -11$:

$$\begin{cases} 3(4)^2 + 2(-11) = 26 \\ 5(4)^2 + 7(-11) = 3 \quad \checkmark \end{cases}$$

SOLUTION We choose to eliminate the x-term, so we multiply the first equation by 5 and the second equation by -3. Then we add the two equations and solve for y:

$$\begin{aligned} &\begin{cases} 15x^2 + 10y = 130 \\ -15x^2 - 21y = -9 \end{cases} && \begin{aligned} &5 \times \text{Equation 1} \\ &(-3) \times \text{Equation 2} \end{aligned} \\ &\quad -11y = 121 && \text{Add} \\ &\quad y = -11 && \text{Solve for } y \end{aligned}$$

Now we back-substitute $y = -11$ into one of the original equations, say, $3x^2 + 2y = 26$, and solve for x:

$$\begin{aligned} 3x^2 + 2(-11) &= 26 && \text{Back-substitute } y = -11 \text{ into Equation 1} \\ 3x^2 &= 48 && \text{Add 22} \\ x^2 &= 16 && \text{Divide by 3} \\ x = -4 \quad \text{or} \quad x &= 4 && \text{Solve for } x \end{aligned}$$

So we have two solutions: $(-4, -11)$ and $(4, -11)$.

The graphs of both equations are parabolas (see Section 3.1). Figure 2 shows that the graphs intersect at the two points $(-4, -11)$ and $(4, -11)$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **11**

Graphical Method

The graphical method is particularly useful in solving systems of nonlinear equations.

EXAMPLE 3 | Graphical Method

Find all solutions of the system

$$\begin{cases} x^2 - y = 2 \\ 2x - y = -1 \end{cases}$$

SOLUTION Graph each equation. Solving for y in terms of x, we get the equivalent system

$$\begin{cases} y = x^2 - 2 \\ y = 2x + 1 \end{cases}$$

See Appendix B, *Graphing with a Graphing Calculator,* for guidelines on using a graphing calculator. See Appendix C, *Using the TI-83/84 Graphing Calculator,* for specific graphing instructions.

Find intersection points. Figure 3 shows that the graphs of these equations intersect at two points. Zooming in, we see that the solutions are

$$(-1, -1) \quad \text{and} \quad (3, 7)$$

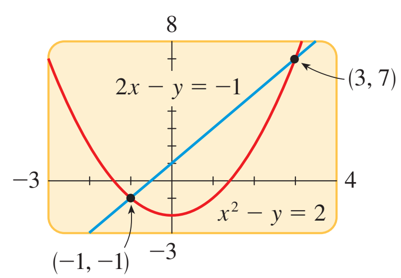


FIGURE 3

CHECK YOUR ANSWERS

$x = -1, y = -1$:

$$\begin{cases} (-1)^2 - (-1) = 2 \\ 2(-1) - (-1) = -1 \end{cases} \checkmark$$

$x = 3, y = 7$:

$$\begin{cases} 3^2 - 7 = 2 \\ 2(3) - 7 = -1 \end{cases} \checkmark$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **33**

MATHEMATICS IN THE MODERN WORLD

Global Positioning System (GPS)

Courtesy of NASA

On a cold, foggy day in 1707 a British naval fleet was sailing home at a fast clip. The fleet's navigators didn't know it, but the fleet was only a few yards from the rocky shores of England. In the ensuing disaster the fleet was totally destroyed. This tragedy could have been avoided had the navigators known their positions. In those days latitude was determined by the position of the North Star (and this could only be done at night in good weather), and longitude by the position of the sun relative to where it would be in England *at that same time.* So navigation required an accurate method of telling time on ships. (The invention of the spring-loaded clock brought about the eventual solution.)

Since then, several different methods have been developed to determine position, and all rely heavily on mathematics (see LORAN, page 547). The latest method, called the Global Positioning System (GPS), uses triangulation. In this system, 24 satellites are strategically located above the surface of the earth. A handheld GPS device measures distance from a satellite, using the travel time of radio signals emitted from the satellite. Knowing the distances to three different satellites tells us that we are at the point of intersection of three different spheres. This uniquely determines our position (see Exercise 47, page 445).

EXAMPLE 4 | Solving a System of Equations Graphically

Find all solutions of the system, rounded to one decimal place.

$$\begin{cases} x^2 + y^2 = 12 & \text{Equation 1} \\ y = 2x^2 - 5x & \text{Equation 2} \end{cases}$$

SOLUTION The graph of the first equation is a circle, and the graph of the second is a parabola. To graph the circle on a graphing calculator, we must first solve for y in terms of x (see Appendix B, *Graphing with a Graphing Calculator*).

$$\begin{aligned} x^2 + y^2 &= 12 \\ y^2 &= 12 - x^2 && \text{Isolate } y^2 \text{ on LHS} \\ y &= \pm\sqrt{12 - x^2} && \text{Take square roots} \end{aligned}$$

To graph the circle, we must graph both functions.

$$y = \sqrt{12 - x^2} \qquad \text{and} \qquad y = -\sqrt{12 - x^2}$$

In Figure 4 the graph of the circle is shown in red, and the parabola is shown in blue. The graphs intersect in Quadrants I and II. Zooming in or using the `Intersect` command, we see that the intersection points are $(-0.559, 3.419)$ and $(2.847, 1.974)$. There also appears to be an intersection point in Quadrant IV. However, when we zoom in, we see that the curves come close to each other but don't intersect (see Figure 5). Thus the system has two solutions; rounded to the nearest tenth, they are

$$(-0.6, 3.4) \quad \text{and} \quad (2.8, 2.0)$$

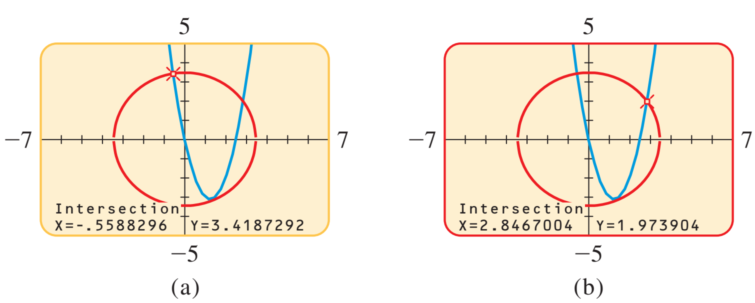


FIGURE 4 $x^2 + y^2 = 12,\ y = 2x^2 - 5x$

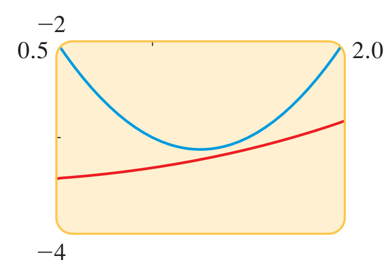


FIGURE 5 Zooming in

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **37**

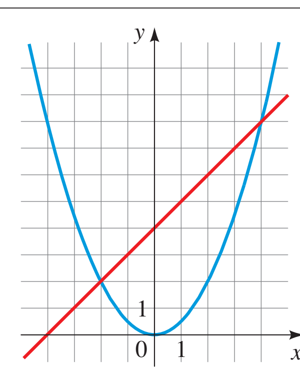
5.4 EXERCISES

CONCEPTS

1–2 ■ The system of equations

$$\begin{cases} 2y - x^2 = 0 \\ \ \ y - x \ = 4 \end{cases}$$

is graphed to the right.



1. Use the graph to find the solution(s) of the system.
2. Check that the solutions you found in Exercise 1 satisfy the system.

SKILLS

3–8 ■ Use the substitution method to find all solutions of the system of equations.

3. $\begin{cases} y = x^2 \\ y = x + 12 \end{cases}$ **4.** $\begin{cases} x^2 + y^2 = 25 \\ y = 2x \end{cases}$

5. $\begin{cases} x^2 + y^2 = 8 \\ x + y = 0 \end{cases}$ **6.** $\begin{cases} x^2 + y = 9 \\ x - y + 3 = 0 \end{cases}$

7. $\begin{cases} x + y^2 = 0 \\ 2x + 5y^2 = 75 \end{cases}$ **8.** $\begin{cases} x^2 - y = 1 \\ 2x^2 + 3y = 17 \end{cases}$

9–14 ■ Use the elimination method to find all solutions of the system of equations.

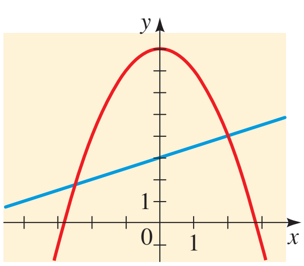
9. $\begin{cases} x^2 - 2y = 1 \\ x^2 + 5y = 29 \end{cases}$ **10.** $\begin{cases} 3x^2 + 4y = 17 \\ 2x^2 + 5y = 2 \end{cases}$

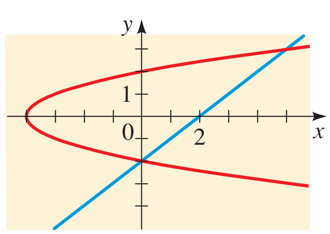
11. $\begin{cases} 3x^2 - y^2 = 11 \\ x^2 + 4y^2 = 8 \end{cases}$ **12.** $\begin{cases} 2x^2 + 4y = 13 \\ x^2 - y^2 = \frac{7}{2} \end{cases}$

13. $\begin{cases} x - y^2 + 3 = 0 \\ 2x^2 + y^2 - 4 = 0 \end{cases}$ **14.** $\begin{cases} x^2 - y^2 = 1 \\ 2x^2 - y^2 = x + 3 \end{cases}$

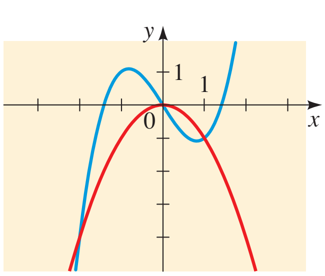
15–18 ■ Two equations and their graphs are given. Find the intersection point(s) of the graphs by solving the system.

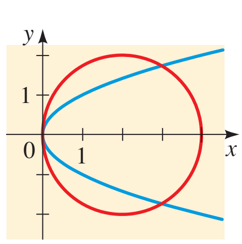
15. $\begin{cases} x^2 + y = 8 \\ x - 2y = -6 \end{cases}$ **16.** $\begin{cases} x - y^2 = -4 \\ x - y = 2 \end{cases}$





17. $\begin{cases} x^2 + y = 0 \\ x^3 - 2x - y = 0 \end{cases}$ **18.** $\begin{cases} x^2 + y^2 = 4x \\ x = y^2 \end{cases}$





19–32 ■ Find all solutions of the system of equations.

19. $\begin{cases} y + x^2 = 4x \\ y + 4x = 16 \end{cases}$ **20.** $\begin{cases} x - y^2 = 0 \\ y - x^2 = 0 \end{cases}$

21. $\begin{cases} x - 2y = 2 \\ y^2 - x^2 = 2x + 4 \end{cases}$ **22.** $\begin{cases} y = 4 - x^2 \\ y = x^2 - 4 \end{cases}$

23. $\begin{cases} x - y = 4 \\ xy = 12 \end{cases}$ **24.** $\begin{cases} xy = 24 \\ 2x^2 - y^2 + 4 = 0 \end{cases}$

25. $\begin{cases} x^2 y = 16 \\ x^2 + 4y + 16 = 0 \end{cases}$ **26.** $\begin{cases} x + \sqrt{y} = 0 \\ y^2 - 4x^2 = 12 \end{cases}$

27. $\begin{cases} x^2 + y^2 = 9 \\ x^2 - y^2 = 1 \end{cases}$ **28.** $\begin{cases} x^2 + 2y^2 = 2 \\ 2x^2 - 3y = 15 \end{cases}$

29. $\begin{cases} 2x^2 - 8y^3 = 19 \\ 4x^2 + 16y^3 = 34 \end{cases}$ **30.** $\begin{cases} x^4 + y^3 = 17 \\ 3x^4 + 5y^3 = 53 \end{cases}$

31. $\begin{cases} \dfrac{2}{x} - \dfrac{3}{y} = 1 \\ -\dfrac{4}{x} + \dfrac{7}{y} = 1 \end{cases}$ **32.** $\begin{cases} \dfrac{4}{x^2} + \dfrac{6}{y^4} = \dfrac{7}{2} \\ \dfrac{1}{x^2} - \dfrac{2}{y^4} = 0 \end{cases}$

33–40 ■ Use the graphical method to find all solutions of the system of equations, rounded to two decimal places.

33. $\begin{cases} y = x^2 + 8x \\ y = 2x + 16 \end{cases}$ **34.** $\begin{cases} y = x^2 - 4x \\ 2x - y = 2 \end{cases}$

35. $\begin{cases} x^2 + y^2 = 25 \\ x + 3y = 2 \end{cases}$ **36.** $\begin{cases} x^2 + y^2 = 17 \\ x^2 - 2x + y^2 = 13 \end{cases}$

37. $\begin{cases} \dfrac{x^2}{9} + \dfrac{y^2}{18} = 1 \\ y = -x^2 + 6x - 2 \end{cases}$

38. $\begin{cases} x^2 - y^2 = 3 \\ y = x^2 - 2x - 8 \end{cases}$

39. $\begin{cases} x^4 + 16y^4 = 32 \\ x^2 + 2x + y = 0 \end{cases}$

40. $\begin{cases} y = e^x + e^{-x} \\ y = 5 - x^2 \end{cases}$

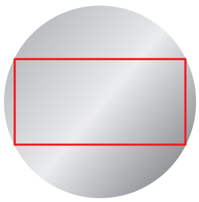
APPLICATIONS

41. **Dimensions of a Rectangle** A rectangle has an area of 180 cm^2 and a perimeter of 54 cm. What are its dimensions?

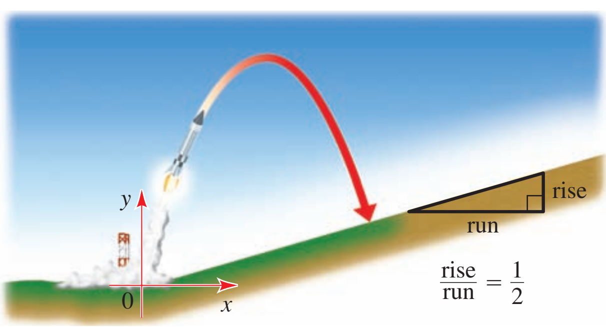
42. **Legs of a Right Triangle** A right triangle has an area of 84 ft^2 and a hypotenuse 25 ft long. What are the lengths of its other two sides?

43. Dimensions of a Rectangle The perimeter of a rectangle is 70, and its diagonal is 25. Find its length and width.

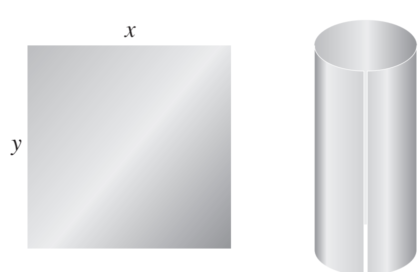
44. Dimensions of a Rectangle A circular piece of sheet metal has a diameter of 20 in. The edges are to be cut off to form a rectangle of area 160 in^2 (see the figure). What are the dimensions of the rectangle?



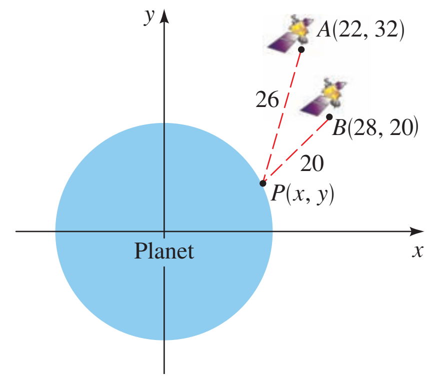
45. Flight of a Rocket A hill is inclined so that its "slope" is $\frac{1}{2}$, as shown in the figure. We introduce a coordinate system with the origin at the base of the hill and with the scales on the axes measured in meters. A rocket is fired from the base of the hill in such a way that its trajectory is the parabola $y = -x^2 + 401x$. At what point does the rocket strike the hillside? How far is this point from the base of the hill (to the nearest centimeter)?



46. Making a Stovepipe A rectangular piece of sheet metal with an area of 1200 in^2 is to be bent into a cylindrical length of stovepipe having a volume of 600 in^3. What are the dimensions of the sheet metal?



47. Global Positioning System (GPS) The Global Positioning System determines the location of an object from its distances to satellites in orbit around the earth. In the simplified, two-dimensional situation shown in the following figure, determine the coordinates of P from the fact that P is 26 units from satellite A and 20 units from satellite B.



DISCOVERY ▪ DISCUSSION ▪ WRITING

48. Intersection of a Parabola and a Line On a sheet of graph paper or using a graphing calculator, draw the parabola $y = x^2$. Then draw the graphs of the linear equation $y = x + k$ on the same coordinate plane for various values of k. Try to choose values of k so that the line and the parabola intersect at two points for some of your k's and not for others. For what value of k is there exactly one intersection point? Use the results of your experiment to make a conjecture about the values of k for which the following system has two solutions, one solution, and no solution. Prove your conjecture.

$$\begin{cases} y = x^2 \\ y = x + k \end{cases}$$

49. Some Trickier Systems Follow the hints and solve the systems.

(a) $\begin{cases} \log x + \log y = \frac{3}{2} \\ 2 \log x - \log y = 0 \end{cases}$ [*Hint:* Add the equations.]

(b) $\begin{cases} 2^x + 2^y = 10 \\ 4^x + 4^y = 68 \end{cases}$ [*Hint:* Note that $4^x = 2^{2x} = (2^x)^2$.]

(c) $\begin{cases} x - y = 3 \\ x^3 - y^3 = 387 \end{cases}$ [*Hint:* Factor the left-hand side of the second equation.]

(d) $\begin{cases} x^2 + xy = 1 \\ xy + y^2 = 3 \end{cases}$ [*Hint:* Add the equations, and factor the result.]

5.5 Systems of Inequalities

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Graph an inequality ▶ Graph a system of inequalities ▶ Graph a system of linear inequalities ▶ Find feasible regions for applications

GET READY **Prepare for this section by reviewing how to solve equations algebraically and graphically in Sections P.8, 1.4, and 1.6.**

In this section we study systems of inequalities in two variables from a graphical point of view.

▼ Graphing an Inequality

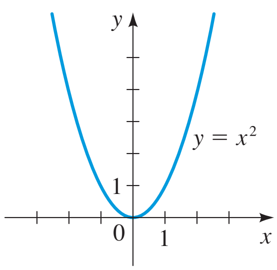


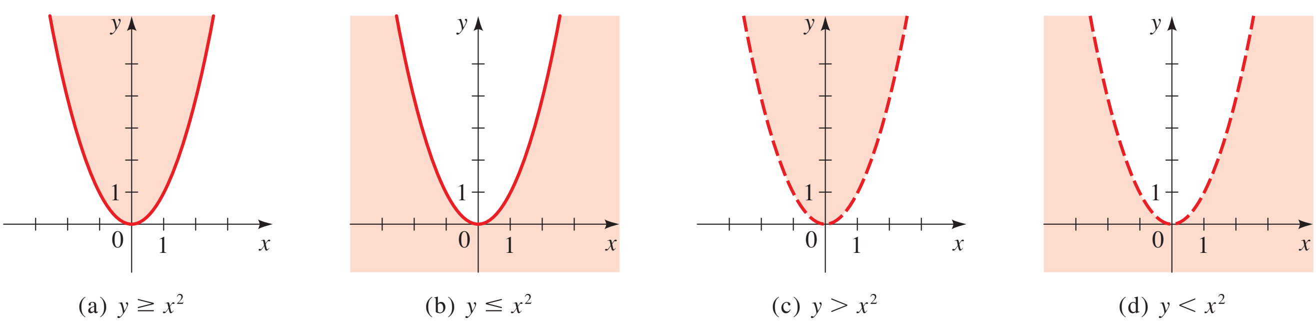
FIGURE 1

We begin by considering the graph of a single inequality in two variables. We already know that the graph of the two-variable equation $y = x^2$, for example, is the *parabola* in Figure 1. If we replace the equal sign by the symbol $\geq$, we obtain the two-variable *inequality*

$$y \geq x^2$$

Its graph consists of not just the parabola in Figure 1, but also every point whose y-coordinate is *larger* than x^2. We indicate the solution in Figure 2(a) by shading the points *above* the parabola.

Similarly, the graph of $y \leq x^2$ in Figure 2(b) consists of all points on and *below* the parabola. However, the graphs of $y > x^2$ and $y < x^2$ do not include the points on the parabola itself, as indicated by the dashed curves in Figures 2(c) and 2(d).



(a) $y \geq x^2$ (b) $y \leq x^2$ (c) $y > x^2$ (d) $y < x^2$

FIGURE 2

The graph of an inequality, in general, consists of a region in the plane whose boundary is the graph of the equation obtained by replacing the inequality sign ($\geq$, $\leq$, $>$, or $<$) with an equal sign. To determine which side of the graph gives the solution set of the inequality, we need only check **test points**.

> **GRAPHING INEQUALITIES**
>
> To graph an inequality, we carry out the following steps.
>
> 1. **Graph Equation.** Graph the equation corresponding to the inequality. Use a dashed curve for $>$ or $<$ and a solid curve for $\leq$ or $\geq$.
> 2. **Test Points.** Test one point in each region formed by the graph in Step 1. If the point satisfies the inequality, then all the points in that region satisfy the inequality. In that case, shade the region to indicate that it is part of the graph. If the test point does not satisfy the inequality, then the region isn't part of the graph.

EXAMPLE 1 | Graphs of Inequalities

Graph each inequality.

(a) $x^2 + y^2 < 25$ **(b)** $x + 2y \geq 5$

SOLUTION

(a) The graph of $x^2 + y^2 = 25$ is a circle of radius 5 centered at the origin. The points on the circle itself do not satisfy the inequality because it is of the form $<$, so we graph the circle with a dashed curve, as shown in Figure 3.

To determine whether the inside or the outside of the circle satisfies the inequality, we use the test points $(0, 0)$ on the inside and $(6, 0)$ on the outside. To do this, we substitute the coordinates of each point into the inequality and check whether the result satisfies the inequality. (Note that *any* point inside or outside the circle can serve as a test point. We have chosen these points for simplicity.)

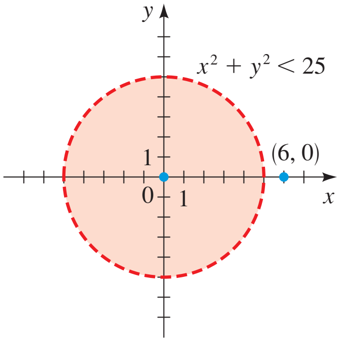


FIGURE 3

Test point	$x^2 + y^2 < 25$	Conclusion
$(0, 0)$	$0^2 + 0^2 = 0 < 25$	Part of graph
$(6, 0)$	$6^2 + 0^2 = 36 \not< 25$	Not part of graph

Thus the graph of $x^2 + y^2 < 25$ is the set of all points *inside* the circle (see Figure 3).

(b) The graph of $x + 2y = 5$ is the line shown in Figure 4. We use the test points $(0, 0)$ and $(5, 5)$ on opposite sides of the line.

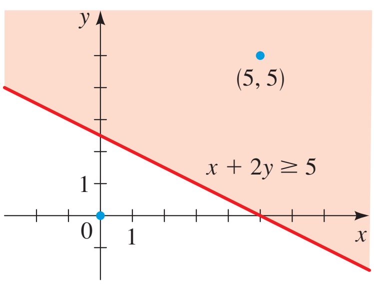


FIGURE 4

Test point	$x + 2y \geq 5$	Conclusion
$(0, 0)$	$0 + 2(0) = 0 \ngeq 5$	Not part of graph
$(5, 5)$	$5 + 2(5) = 15 \geq 5$	Part of graph

Our check shows that the points *above* the line satisfy the inequality.

Alternatively, we could put the inequality into slope-intercept form and graph it directly:

$$\begin{aligned} x + 2y &\geq 5 \\ 2y &\geq -x + 5 \\ y &\geq -\tfrac{1}{2}x + \tfrac{5}{2} \end{aligned}$$

From this form we see that the graph includes all points whose y-coordinates are *greater* than those on the line $y = -\frac{1}{2}x + \frac{5}{2}$; that is, the graph consists of the points *on or above* this line, as shown in Figure 4.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **11** AND **19**

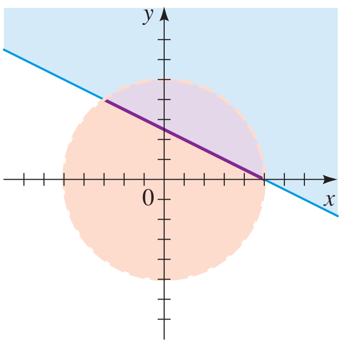
Systems of Inequalities

We now consider *systems* of inequalities. The solution of such a system is the set of all points in the coordinate plane that satisfy every inequality in the system.

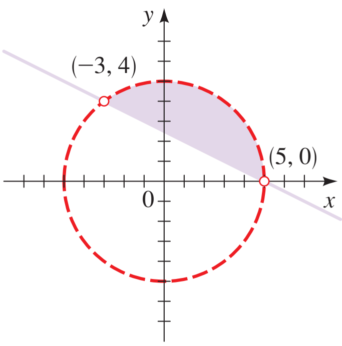
EXAMPLE 2 | A System of Two Inequalities

Graph the solution of the system of inequalities, and label its vertices.

$$\begin{cases} x^2 + y^2 < 25 \\ x + 2y \geq 5 \end{cases}$$



(a)



(b)

FIGURE 5 $\begin{cases} x^2 + y^2 < 25 \\ x + 2y \geq 5 \end{cases}$

SOLUTION These are the two inequalities of Example 1. In this example we wish to graph only those points that simultaneously satisfy both inequalities. The solution consists of the intersection of the graphs in Example 1. In Figure 5(a) we show the two regions on the same coordinate plane (in different colors), and in Figure 5(b) we show their intersection.

Vertices The points $(-3, 4)$ and $(5, 0)$ in Figure 5(b) are the **vertices** of the solution set. They are obtained by solving the system of *equations*

$$\begin{cases} x^2 + y^2 = 25 \\ x + 2y = 5 \end{cases}$$

We solve this system of equations by substitution. Solving for x in the second equation gives $x = 5 - 2y$, and substituting this into the first equation gives

$$(5 - 2y)^2 + y^2 = 25 \qquad \text{Substitute } x = 5 - 2y$$

$$(25 - 20y + 4y^2) + y^2 = 25 \qquad \text{Expand}$$

$$-20y + 5y^2 = 0 \qquad \text{Simplify}$$

$$-5y(4 - y) = 0 \qquad \text{Factor}$$

Thus $y = 0$ or $y = 4$. When $y = 0$, we have $x = 5 - 2(0) = 5$, and when $y = 4$, we have $x = 5 - 2(4) = -3$. So the points of intersection of these curves are $(5, 0)$ and $(-3, 4)$.

Note that in this case the vertices are not part of the solution set, since they don't satisfy the inequality $x^2 + y^2 < 25$ (so they are graphed as open circles in the figure). They simply show where the "corners" of the solution set lie.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **37**

Systems of Linear Inequalities

An inequality is **linear** if it can be put into one of the following forms:

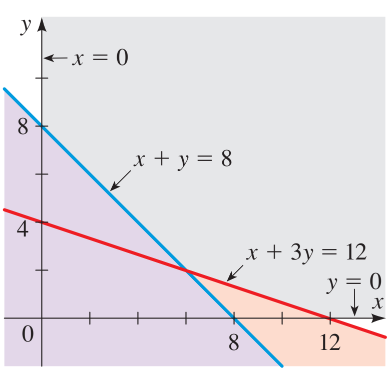
$$ax + by \geq c \qquad ax + by \leq c \qquad ax + by > c \qquad ax + by < c$$

In the next example we graph the solution set of a system of linear inequalities.

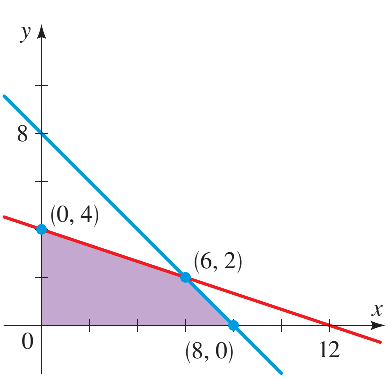
EXAMPLE 3 | A System of Four Linear Inequalities

Graph the solution set of the system, and label its vertices.

$$\begin{cases} x + 3y \leq 12 \\ x + y \leq 8 \\ x \geq 0 \\ y \geq 0 \end{cases}$$



(a)



(b)

FIGURE 6

SOLUTION In Figure 6 we first graph the lines given by the equations that correspond to each inequality. To determine the graphs of the linear inequalities, we need to check only one test point. For simplicity let's use the point $(0, 0)$.

Inequality	Test point (0, 0)	Conclusion
$x + 3y \leq 12$	$0 + 3(0) = 0 \leq 12$	Satisfies inequality
$x + y \leq 8$	$0 + 0 = 0 \leq 8$	Satisfies inequality

Since $(0, 0)$ is below the line $x + 3y = 12$, our check shows that the region on or below the line must satisfy the inequality. Likewise, since $(0, 0)$ is below the line $x + y = 8$, our check shows that the region on or below this line must satisfy the inequality. The inequalities $x \geq 0$ and $y \geq 0$ say that x and y are nonnegative. These regions are sketched in Figure 6(a), and the intersection—the solution set—is sketched in Figure 6(b).

Vertices The coordinates of each vertex are obtained by simultaneously solving the equations of the lines that intersect at that vertex. From the system

$$\begin{cases} x + 3y = 12 \\ x + y = 8 \end{cases}$$

we get the vertex (6, 2). The origin (0, 0) is also clearly a vertex. The other two vertices are at the x- and y-intercepts of the corresponding lines: (8, 0) and (0, 4). In this case all the vertices *are* part of the solution set.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45**

EXAMPLE 4 | A System of Linear Inequalities

Graph the solution set of the system.

$$\begin{cases} x + 2y \geq 8 \\ -x + 2y \leq 4 \\ 3x - 2y \leq 8 \end{cases}$$

SOLUTION We must graph the lines that correspond to these inequalities and then shade the appropriate regions, as in Example 3. We will use a graphing calculator, so we must first isolate y on the left-hand side of each inequality.

$$\begin{cases} y \geq -\frac{1}{2}x + 4 \\ y \leq \frac{1}{2}x + 2 \\ y \geq \frac{3}{2}x - 4 \end{cases}$$

See Appendix B, *Graphing with a Graphing Calculator,* for guidelines on using a graphing calculator. See Appendix C, *Using the TI-83/84 Graphing Calculator,* for specific graphing instructions.

Using the shading feature of the calculator, we obtain the graph in Figure 7. The solution set is the triangular region that is shaded in all three patterns. We then use `TRACE` or the `Intersect` command to find the vertices of the region. The solution set is graphed in Figure 8.

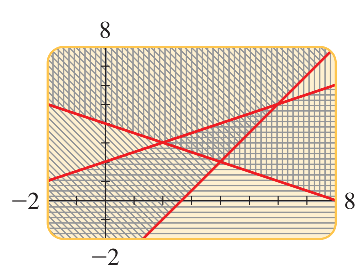


FIGURE 7

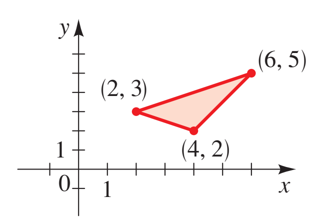


FIGURE 8

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **53**

When a region in the plane can be covered by a (sufficiently large) circle, it is said to be **bounded**. A region that is not bounded is called **unbounded**. For example, the regions graphed in Figures 3, 5(b), 6(b), and 8 are bounded, whereas those in Figures 2 and 4 are unbounded. An unbounded region cannot be "fenced in"—it extends infinitely far in at least one direction.

Application: Feasible Regions

Many applied problems involve *constraints* on the variables. For instance, a factory manager has only a certain number of workers who can be assigned to perform jobs on the factory floor. A farmer deciding what crops to cultivate has only a certain amount of land that can be seeded. Such constraints or limitations can usually be expressed as systems of inequalities. When dealing with applied inequalities, we usually refer to the solution set of a system as a *feasible region*, because the points in the solution set represent feasible (or possible) values for the quantities being studied.

EXAMPLE 5 | Restricting Pollutant Outputs

A factory produces two agricultural pesticides, A and B. For every barrel of A, the factory emits 0.25 kg of carbon monoxide (CO) and 0.60 kg of sulfur dioxide (SO_2); and for every barrel of B, it emits 0.50 kg of CO and 0.20 kg of SO_2. Pollution laws restrict the factory's output of CO to a maximum of 75 kg and its output of SO_2 to a maximum of 90 kg per day.

(a) Find a system of inequalities that describes the number of barrels of each pesticide the factory can produce and still satisfy the pollution laws. Graph the feasible region.

(b) Would it be legal for the factory to produce 100 barrels of A and 80 barrels of B per day?

(c) Would it be legal for the factory to produce 60 barrels of A and 160 barrels of B per day?

SOLUTION

(a) To set up the required inequalities, it is helpful to organize the given information into a table.

	A	B	Maximum
CO (kg)	0.25	0.50	75
SO_2 (kg)	0.60	0.20	90

We let

$$x = \text{number of barrels of A produced per day}$$

$$y = \text{number of barrels of B produced per day}$$

From the data in the table and the fact that x and y can't be negative, we obtain the following inequalities.

$$\begin{cases} 0.25x + 0.50y \le 75 & \text{CO inequality} \\ 0.60x + 0.20y \le 90 & SO_2 \text{ inequality} \\ x \ge 0, \quad y \ge 0 \end{cases}$$

Multiplying the first inequality by 4 and the second by 5 simplifies this to

$$\begin{cases} x + 2y \le 300 \\ 3x + y \le 450 \\ x \ge 0, \quad y \ge 0 \end{cases}$$

The feasible region is the solution of this system of inequalities, shown in Figure 9.

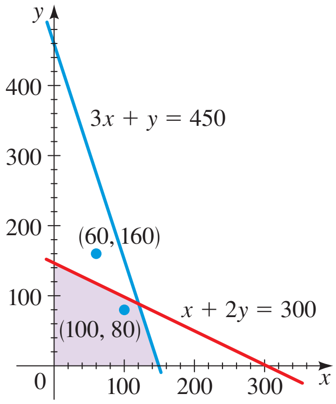


FIGURE 9

(b) Since the point (100, 80) lies inside the feasible region, this production plan is legal (see Figure 9).

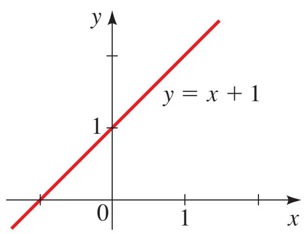
(c) Since the point (60, 160) lies outside the feasible region, this production plan is not legal. It violates the CO restriction, although it does not violate the SO_2 restriction (see Figure 9).

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **57**

5.5 EXERCISES

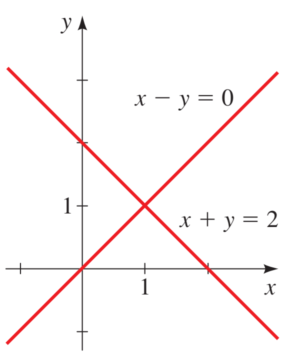
CONCEPTS

1. To graph an inequality, we first graph the corresponding ________. So to graph $y \le x + 1$, we first graph the equation ________. To decide which side of the graph of the equation is the graph of the inequality, we use ________ points. Using $(0, 0)$ as such a point, graph the inequality by shading the appropriate region.

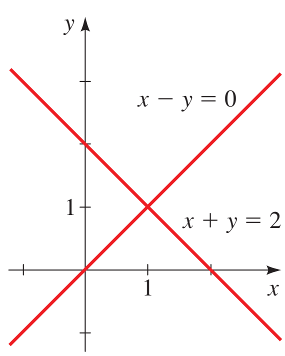


2. Shade the solution of each system of inequalities on the given graph.

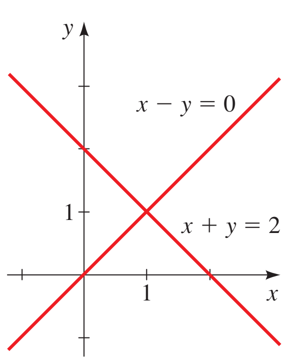
(a) $\begin{cases} x - y \ge 0 \\ x + y \ge 2 \end{cases}$



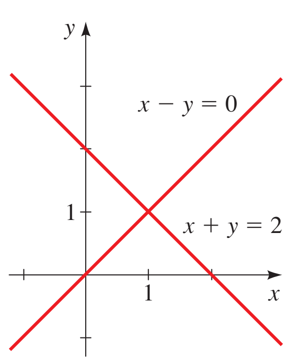
(b) $\begin{cases} x - y \le 0 \\ x + y \le 2 \end{cases}$



(c) $\begin{cases} x - y \ge 0 \\ x + y \le 2 \end{cases}$



(d) $\begin{cases} x - y \le 0 \\ x + y \ge 2 \end{cases}$



SKILLS

3–20 ■ Graph the inequality.

3. $y < 2x$ **4.** $y \ge 3x$

5. $y \ge 2$ **6.** $x \le -1$

7. $x < 2$ **8.** $y > 1$

9. $y < x - 3$ **10.** $y \le -x + 1$

11. $-2x + y \le 4$ **12.** $-3x + y > -9$

13. $-3x + 7y > 21$ **14.** $3x + 4y \le 12$

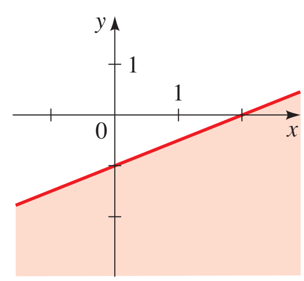
15. $2x - 3y \le 9$ **16.** $-5x - 3y < -10$

17. $x^2 + y \ge 3$ **18.** $-x^2 + y \le -1$

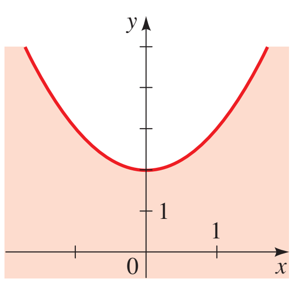
19. $x^2 + y^2 \ge 100$ **20.** $(x - 1)^2 + (y - 2)^2 < 25$

21–24 ■ An equation and its graph are given. Find an inequality whose solution is the shaded region.

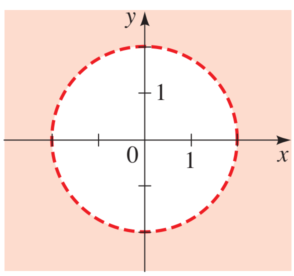
21. $y = \frac{1}{2}x - 1$



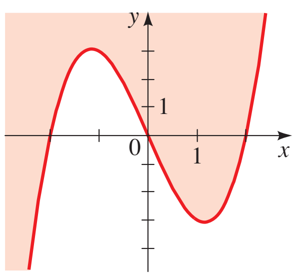
22. $y = x^2 + 2$



23. $x^2 + y^2 = 4$



24. $y = x^3 - 4x$



25–52 ■ Graph the solution of the system of inequalities. Find the coordinates of all vertices, and determine whether the solution set is bounded.

25. $\begin{cases} x + y \le 4 \\ y \ge x \end{cases}$ **26.** $\begin{cases} 2x + 3y > 12 \\ 3x - y < 21 \end{cases}$

27. $\begin{cases} y < \frac{1}{4}x + 2 \\ y \ge 2x - 5 \end{cases}$ **28.** $\begin{cases} x - y > 0 \\ 4 + y \le 2x \end{cases}$

29. $\begin{cases} y \le -2x + 8 \\ y \le -\frac{1}{2}x + 5 \\ x \ge 0, \quad y \ge 0 \end{cases}$ **30.** $\begin{cases} 4x + 3y \le 18 \\ 2x + y \le 8 \\ x \ge 0, \quad y \ge 0 \end{cases}$

31. $\begin{cases} x \ge 0 \\ y \ge 0 \\ 3x + 5y \le 15 \\ 3x + 2y \le 9 \end{cases}$ **32.** $\begin{cases} x > 2 \\ y < 12 \\ 2x - 4y > 8 \end{cases}$

33. $\begin{cases} y \le 9 - x^2 \\ x \ge 0, \quad y \ge 0 \end{cases}$

34. $\begin{cases} y \ge x^2 \\ y \le 4 \\ x \ge 0 \end{cases}$

35. $\begin{cases} y < 9 - x^2 \\ y \ge x + 3 \end{cases}$

36. $\begin{cases} y \ge x^2 \\ x + y \ge 6 \end{cases}$

37. $\begin{cases} x^2 + y^2 \le 4 \\ x - y > 0 \end{cases}$

38. $\begin{cases} x > 0 \\ y > 0 \\ x + y < 10 \\ x^2 + y^2 > 9 \end{cases}$

39. $\begin{cases} x^2 - y \le 0 \\ 2x^2 + y \le 12 \end{cases}$

40. $\begin{cases} 2x^2 + y > 4 \\ x^2 - y \le 8 \end{cases}$

41. $\begin{cases} x^2 + y^2 \le 9 \\ x^2 + 2y \le 1 \end{cases}$

42. $\begin{cases} x^2 + y^2 \le 4 \\ x^2 - 2y > 1 \end{cases}$

43. $\begin{cases} x + 2y \le 14 \\ 3x - y \ge 0 \\ x - y \ge 2 \end{cases}$

44. $\begin{cases} y < x + 6 \\ 3x + 2y \ge 12 \\ x - 2y \le 2 \end{cases}$

45. $\begin{cases} x \ge 0 \\ y \ge 0 \\ x \le 5 \\ x + y \le 7 \end{cases}$

46. $\begin{cases} x \ge 0 \\ y \ge 0 \\ y \le 4 \\ 2x + y \le 8 \end{cases}$

47. $\begin{cases} y > x + 1 \\ x + 2y \le 12 \\ x + 1 > 0 \end{cases}$

48. $\begin{cases} x + y > 12 \\ y < \frac{1}{2}x - 6 \\ 3x + y < 6 \end{cases}$

49. $\begin{cases} x^2 + y^2 \le 8 \\ x \ge 2 \\ y \ge 0 \end{cases}$

50. $\begin{cases} x^2 - y \ge 0 \\ x + y < 6 \\ x - y < 6 \end{cases}$

51. $\begin{cases} x^2 + y^2 < 9 \\ x + y > 0 \\ x \le 0 \end{cases}$

52. $\begin{cases} y \ge x^3 \\ y \le 2x + 4 \\ x + y \ge 0 \end{cases}$

53–56 ■ Use a graphing calculator to graph the solution of the system of inequalities. Find the coordinates of all vertices, rounded to one decimal place.

53. $\begin{cases} y \ge x - 3 \\ y \ge -2x + 6 \\ y \le 8 \end{cases}$

54. $\begin{cases} x + y \ge 12 \\ 2x + y \le 24 \\ x - y \ge -6 \end{cases}$

55. $\begin{cases} y \le 6x - x \\ x + y \ge 4 \end{cases}$

56. $\begin{cases} y \ge x^3 \\ 2x + y \ge 0 \\ y \le 2x + 6 \end{cases}$

APPLICATIONS

57. **Publishing Books** A publishing company publishes a total of no more than 100 books every year. At least 20 of these are nonfiction, but the company always publishes at least as much fiction as nonfiction. Find a system of inequalities that describes the possible numbers of fiction and nonfiction books that the company can produce each year consistent with these policies. Graph the solution set.

58. **Furniture Manufacturing** A man and his daughter manufacture unfinished tables and chairs. Each table requires 3 hours of sawing and 1 hour of assembly. Each chair requires 2 hours of sawing and 2 hours of assembly. Between the two of them, they can put in up to 12 hours of sawing and 8 hours of assembly work each day. Find a system of inequalities that describes all possible combinations of tables and chairs that they can make daily. Graph the solution set.

59. **Coffee Blends** A coffee merchant sells two different coffee blends. The Standard blend uses 4 oz of arabica beans and 12 oz of robusta beans per package; the Deluxe blend uses 10 oz of arabica beans and 6 oz of robusta beans per package. The merchant has 80 lb of arabica beans and 90 lb of robusta beans available. Find a system of inequalities that describes the possible number of Standard and Deluxe packages the merchant can make. Graph the solution set.

60. **Nutrition** A cat food manufacturer uses fish and beef by-products. The fish contains 12 g of protein and 3 g of fat per ounce. The beef contains 6 g of protein and 9 g of fat per ounce. Each can of cat food must contain at least 60 g of protein and 45 g of fat. Find a system of inequalities that describes the possible number of ounces of fish and beef that can be used in each can to satisfy these minimum requirements. Graph the solution set.

DISCOVERY ■ DISCUSSION ■ WRITING

61. **Shading Unwanted Regions** To graph the solution of a system of inequalities, we have shaded the solution of each inequality in a different color; the solution of the system is the region where all the shaded parts overlap. Here is a different method: For each inequality, shade the region that does *not* satisfy the inequality. Explain why the part of the plane that is left unshaded is the solution of the system. Solve the following system by both methods. Which do you prefer? Why?

$$\begin{cases} x + 2y > 4 \\ -x + y < 1 \\ x + 3y < 9 \\ x < 3 \end{cases}$$

CHAPTER 5 | REVIEW

PROPERTIES AND FORMULAS

Substitution Method (p. 416)

To solve a pair of equations in two variables by substitution:

1. Solve for one variable in terms of the other variable in one equation.
2. **Substitute** into the other equation to get an equation in one variable, and solve for this variable.
3. Substitute the value(s) of the variable you have found into either original equation, and solve for the remaining variable.

Elimination Method (p. 417)

To solve a pair of equations in two variables by elimination:

1. Multiply the equations by appropriate constants so that the term(s) involving one of the variables are of opposite sign in the equations.
2. Add the equations to **eliminate** that one variable; this gives an equation in the other variable. Solve for this variable.
3. Substitute the value(s) of the variable that you have found into either original equation, and solve for the remaining variable.

Graphical Method (p. 418)

To solve a pair of equations in two variables graphically:

1. Put each equation in function form, $y = f(x)$.
2. Use a graphing calculator to **graph** the equations on a common screen.
3. Find the points of intersection of the graphs. The solutions are the x- and y-coordinates of the points of intersection.

Gaussian Elimination (p. 428)

When we use **Gaussian elimination** to solve a system of linear equations, we use the following operations to change the system to an **equivalent** simpler system:

1. Add a nonzero multiple of one equation to another.
2. Multiply an equation by a nonzero constant.
3. Interchange the position of two equations in the system.

Number of Solutions of a System of Linear Equations (pp. 419, 429)

A system of linear equations can have:

1. A unique solution for each variable.
2. No solution, in which case the system is **inconsistent**.
3. Infinitely many solutions, in which case the system is **dependent**.

How to Determine the Number of Solutions of a Linear System (p. 429)

When we use **Gaussian elimination** to solve a system of linear equations, then we can tell that the system has:

1. **No solution** (is *inconsistent*) if we arrive at a false equation of the form $0 = c$, where c is nonzero.
2. **Infinitely many solutions** (is *dependent*) if the system is consistent but we end up with fewer equations than variables (after discarding redundant equations of the form $0 = 0$).

Partial Fractions (p. 435)

The *partial fraction decomposition* of a rational function

$$r(x) = \frac{P(x)}{Q(x)}$$

(where the degree of P is less than the degree of Q) is a sum of simpler fractional expressions that equal $r(x)$ when brought to a common denominator. The denominator of each simpler fraction is either a linear or quadratic factor of $Q(x)$ or a power of such a linear or quadratic factor. So to find the terms of the partial fraction decomposition, we first factor $Q(x)$ into linear and irreducible quadratic factors. The terms then have the following forms, depending on the factors of $Q(x)$.

1. For every **distinct linear factor** $ax + b$, there is a term of the form
$$\frac{A}{ax + b}$$
2. For every **repeated linear factor** $(ax + b)^m$, there are terms of the form
$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_m}{(ax + b)^m}$$
3. For every **distinct quadratic factor** $ax^2 + bx + c$, there is a term of the form
$$\frac{Ax + B}{ax^2 + bx + c}$$
4. For every **repeated quadratic factor** $(ax^2 + bx + c)^m$, there are terms of the form
$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

Graphing Inequalities (p. 446)

To graph an inequality:

1. Graph the equation that corresponds to the inequality. This "boundary curve" divides the coordinate plane into separate regions.
2. Use **test points** to determine which region(s) satisfy the inequality.
3. Shade the region(s) that satisfy the inequality, and use a solid line for the boundary curve if it satisfies the inequality ($\le$ or $\ge$) and a dashed line if it does not ($<$ or $>$).

Graphing Systems of Inequalities (p. 447)

To graph the solution of a system of inequalities (or **feasible region** determined by the inequalities):

1. Graph all the inequalities on the same coordinate plane.
2. The solution is the intersection of the solutions of all the inequalities, so shade the region that satisfies all the inequalities.
3. Determine the coordinates of the intersection points of all the boundary curves that touch the solution set of the system. These points are the **vertices** of the solution.

LEARNING OBJECTIVES SUMMARY

Section	After completing this chapter, you should be able to . . .	Review Exercises
5.1	■ Solve a system of linear equations in two variables using the substitution method	3–4. 6
	■ Solve a system of linear equations in two variables using the elimination method	1–2, 5, 7–8, 11
	■ Solve a system of linear equations in two variables using the graphical method	13–16
	■ Determine whether a system of two linear equations in two variables has one solution, infinitely many solutions, or no solution	1–6
	■ Model with linear systems in two variables	25–26
5.2	■ Use Gaussian elimination to solve a system of three (or more) linear equations	17–24
	■ Determine whether a system of (three or more) linear equations has one solution, infinitely many solutions, or no solution	17–24
	■ Model with linear systems in three (or more) variables	27–28
5.3	■ Find the form of the partial fraction decomposition of rational expression in the following cases:	
	Denominator contains distinct linear factors	29–30
	Denominator contains repeated linear factors	31–32
	Denominator contains distinct quadratic factors	33–34
	Denominator contains repeated quadratic factors	35–36
	■ Find the partial fractions decomposition of a rational expression in the above cases	29–36
5.4	■ Solve a system of nonlinear equations in two variables using the substitution and elimination methods	9–10, 12
	■ Solve a system of nonlinear equations in two variables using the graphical method	37–40
5.5	■ Graph the solution of an inequality	41–46
	■ Graph the solution of a system of inequalities	47–54
	■ Graph the solution of a system of linear inequalities	49–50, 53–54

EXERCISES

1–8 ■ Solve the system of equations and graph the lines.

1. $\begin{cases} 3x - y = 5 \\ 2x + y = 5 \end{cases}$

2. $\begin{cases} 4x - 12y = 8 \\ 3x - 9y = 6 \end{cases}$

3. $\begin{cases} 2x - 7y = 2 \\ y = \frac{2}{7}x - 4 \end{cases}$

4. $\begin{cases} 3x + 2y = 6 \\ y = -5x - 4 \end{cases}$

5. $\begin{cases} 6x - 8y = 16 \\ -\frac{3}{2}x + 2y = -2 \end{cases}$

6. $\begin{cases} y = 2x - 6 \\ 2y = 3 + 4x \end{cases}$

7. $\begin{cases} 2x - y = 1 \\ x + 3y = 10 \\ 3x + 4y = 15 \end{cases}$

8. $\begin{cases} 2x + 5y = 9 \\ -x + 3y = 1 \\ 7x - 2y = 14 \end{cases}$

9–12 ■ Solve the system of equations.

9. $\begin{cases} y = x^2 + 2x \\ y = 6 + x \end{cases}$

10. $\begin{cases} x^2 + y^2 = 8 \\ y = x + 2 \end{cases}$

11. $\begin{cases} 3x + \dfrac{4}{y} = 6 \\ x - \dfrac{8}{y} = 4 \end{cases}$

12. $\begin{cases} x^2 + y^2 = 10 \\ x^2 + 2y^2 - 7y = 0 \end{cases}$

13–16 ■ Use a graphing device to solve the system, rounded to the nearest hundredth.

13. $\begin{cases} 0.32x + 0.43y = 0 \\ 7x - 12y = 341 \end{cases}$

14. $\begin{cases} \sqrt{12}\,x - 3\sqrt{2}\,y = 660 \\ 7137x + 3931y = 20{,}000 \end{cases}$

15. $\begin{cases} x - y^2 = 10 \\ x = \frac{1}{22}y + 12 \end{cases}$

16. $\begin{cases} y = 5^x + x \\ y = x^5 + 5 \end{cases}$

17–24 ■ Find the complete solution of the system, or show that the system has no solution.

17. $\begin{cases} x + y + 2z = 6 \\ 2x + 5z = 12 \\ x + 2y + 3z = 9 \end{cases}$

18. $\begin{cases} x - 2y + 3z = 1 \\ x - 3y - z = 0 \\ 2x - 6z = 6 \end{cases}$

19. $\begin{cases} x - 2y + 3z = 1 \\ 2x - y + z = 3 \\ 2x - 7y + 11z = 2 \end{cases}$

20. $\begin{cases} x + y + z + w = 2 \\ 2x - 3z = 5 \\ x - 2y + 4w = 9 \\ x + y + 2z + 3w = 5 \end{cases}$

21. $\begin{cases} x - 3y + z = 4 \\ 4x - y + 15z = 5 \end{cases}$

22. $\begin{cases} 2x - 3y + 4z = 3 \\ 4x - 5y + 9z = 13 \\ 2x + 7z = 0 \end{cases}$

23. $\begin{cases} -x + 4y + z = 8 \\ 2x - 6y + z = -9 \\ x - 6y - 4z = -15 \end{cases}$

24. $\begin{cases} x - z + w = 2 \\ 2x + y - 2w = 12 \\ 3y + z + w = 4 \\ x + y - z = 10 \end{cases}$

25. Eleanor has two children, Kieran and Siobhan. Kieran is 4 years older than Siobhan, and the sum of their ages is 22. How old are the children?

26. A man invests his savings in two accounts, one paying 6% interest per year and the other paying 7%. He has twice as much invested in the 7% account as in the 6% account, and his annual interest income is $600. How much is invested in each account?

27. A piggy bank contains 50 coins, all of them nickels, dimes, or quarters. The total value of the coins is $5.60, and the value of the dimes is five times the value of the nickels. How many coins of each type are there?

28. Tornie is a commercial fisherman who trolls for salmon on the British Columbia coast. One day he catches a total of 25 fish of three salmon species: coho, sockeye, and pink. He catches three more coho than the other two species combined; moreover, he catches twice as many coho as sockeye. How many fish of each species has he caught?

29–36 ■ Find the partial fraction decomposition of the rational function.

29. $\dfrac{3x + 1}{x^2 - 2x - 15}$

30. $\dfrac{8}{x^3 - 4x}$

31. $\dfrac{2x - 4}{x(x - 1)^2}$

32. $\dfrac{6x - 4}{x^3 - 2x^2 - 4x + 8}$

33. $\dfrac{2x - 1}{x^3 + x}$

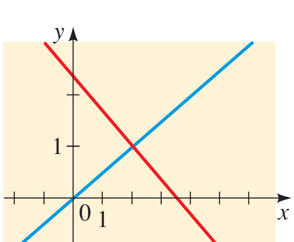
34. $\dfrac{5x^2 - 3x + 10}{x^4 + x^2 - 2}$

35. $\dfrac{3x^2 - x + 6}{(x^2 + 2)^2}$

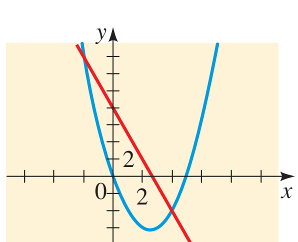
36. $\dfrac{x^2 + x + 1}{x(x^2 + 1)^2}$

37–40 ■ Two equations and their graphs are given. Find the intersection point(s) of the graphs by solving the system.

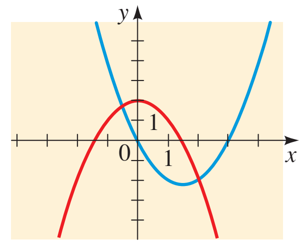
37. $\begin{cases} 2x + 3y = 7 \\ x - 2y = 0 \end{cases}$



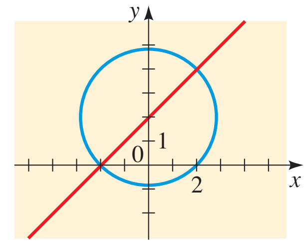
38. $\begin{cases} 3x + y = 8 \\ y = x^2 - 5x \end{cases}$



39. $\begin{cases} x^2 + y = 2 \\ x^2 - 3x - y = 0 \end{cases}$

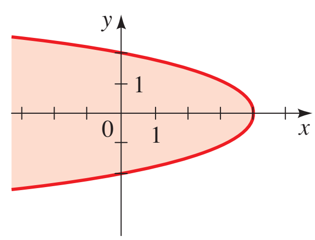


40. $\begin{cases} x - y = -2 \\ x^2 + y^2 - 4y = 4 \end{cases}$

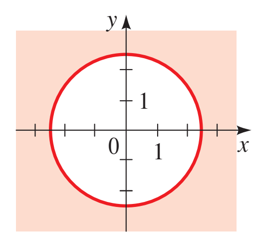


41–42 ■ An equation and its graph are given. Find an inequality whose solution is the shaded region.

41. $x + y^2 = 4$



42. $x^2 + y^2 = 8$



43–46 ■ Graph the inequality.

43. $3x + y \le 6$

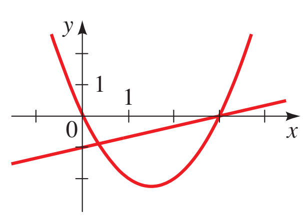
44. $y \ge x^2 - 3$

45. $x^2 + y^2 > 9$

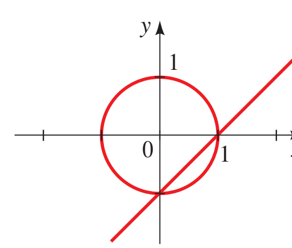
46. $y - x^2 < 4$

47–50 ■ The figure shows the graphs of the equations corresponding to the given inequalities. Shade the solution set of the system of inequalities.

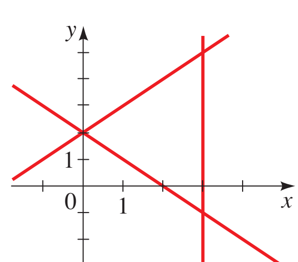
47. $\begin{cases} y \ge x^2 - 3x \\ y \le \frac{1}{3}x - 1 \end{cases}$



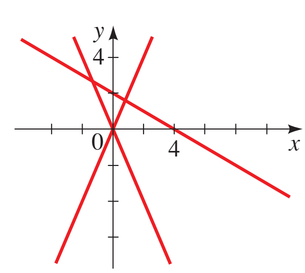
48. $\begin{cases} y \ge x - 1 \\ x^2 + y^2 \le 1 \end{cases}$



49. $\begin{cases} x + y \ge 2 \\ y - x \le 2 \\ x \le 3 \end{cases}$



50. $\begin{cases} y \ge -2x \\ y \le 2x \\ y \le -\frac{1}{2}x + 2 \end{cases}$



51–54 ■ Graph the solution set of the system of inequalities. Find the coordinates of all vertices, and determine whether the solution set is bounded or unbounded.

51. $\begin{cases} x^2 + y^2 < 9 \\ x + y < 0 \end{cases}$

52. $\begin{cases} y - x^2 \geq 4 \\ y < 20 \end{cases}$

53. $\begin{cases} x \geq 0,\ y \geq 0 \\ x + 2y \leq 12 \\ y \leq x + 4 \end{cases}$

54. $\begin{cases} x \geq 4 \\ x + y \geq 24 \\ x \leq 2y + 12 \end{cases}$

55–56 ■ Solve for x, y, and z in terms of a, b, and c.

55. $\begin{cases} -x + y + z = a \\ x - y + z = b \\ x + y - z = c \end{cases}$

56. $\begin{cases} ax + by + cz = a - b + c \\ bx + by + cz = c \\ cx + cy + cz = c \end{cases} \qquad (a \neq b,\ b \neq c,\ c \neq 0)$

57. For what values of k do the following three lines have a common point of intersection?

$$x + y = 12$$
$$kx - y = 0$$
$$y - x = 2k$$

58. For what value of k does the following system have infinitely many solutions?

$$\begin{cases} kx + y + z = 0 \\ x + 2y + kz = 0 \\ -x + 3z = 0 \end{cases}$$

CHAPTER 5 TEST

1–3 ■ A system of equations is given. **(a)** Determine whether the system is linear or nonlinear. **(b)** Find all solutions of the system.

1. $\begin{cases} 3x + 5y = 4 \\ x - 4y = 7 \end{cases}$

2. $\begin{cases} 10x - y^2 = 4 \\ 2x + y = 2 \end{cases}$

3. $\begin{cases} x^2 + y^2 = 100 \\ y = 3x \end{cases}$

4. Use a graphing device to find all solutions of the system rounded to two decimal places.

$$\begin{cases} x - 2y = 1 \\ y = x^3 - 2x^2 \end{cases}$$

5. In $2\frac{1}{2}$ hours an airplane travels 600 km against the wind. It takes 50 min to travel 300 km with the wind. Find the speed of the wind and the speed of the airplane in still air.

6–9 ■ A system of linear equations is given. **(a)** Find the complete solution of the system, or show that there is no solution. **(b)** State whether the system is inconsistent, dependent, or neither.

6. $\begin{cases} x + 2y + z = 3 \\ x + 3y + 2z = 3 \\ 2x + 3y - z = 8 \end{cases}$

7. $\begin{cases} x - y + 9z = -8 \\ -4z = 7 \\ 3x - y + z = 5 \end{cases}$

8. $\begin{cases} 2x - y + z = 0 \\ 3x + 2y - 3z = 1 \\ x - 4y + 5z = -1 \end{cases}$

9. $\begin{cases} x + y - 2z = 8 \\ 2x - y = 20 \\ 2x + 2y - 5z = 15 \end{cases}$

10. Anne, Barry, and Cathy enter a coffee shop. Anne orders two coffees, one juice, and two doughnuts and pays $6.25. Barry orders one coffee and three doughnuts and pays $3.75. Cathy orders three coffees, one juice, and four doughnuts and pays $9.25. Find the price of coffee, juice, and doughnuts at this coffee shop.

11. Graph the inequality.

(a) $3x + 4y < 6$

(b) $-x^2 + y \geq 3$

12–13 ■ Graph the solution set of the system of inequalities. Label the vertices with their coordinates.

12. $\begin{cases} 2x + y \leq 8 \\ x - y \geq -2 \\ x + 2y \geq 4 \end{cases}$

13. $\begin{cases} x^2 + y \leq 5 \\ y \geq 2x + 5 \end{cases}$

14–15 ■ Find the partial fraction decomposition of the rational function.

14. $\dfrac{4x - 1}{(x - 1)^2(x + 2)}$

15. $\dfrac{2x - 3}{x^3 + 3x}$

Linear Programming

Linear programming is a modeling technique that is used to determine the optimal allocation of resources in business, the military, and other areas of human endeavor. For example, a manufacturer who makes several different products from the same raw materials can use linear programming to determine how much of each product should be produced to maximize the profit. This modeling technique is probably the most important practical application of systems of linear inequalities. In 1975 Leonid Kantorovich and T. C. Koopmans won the Nobel Prize in economics for their work in the development of this technique.

Although linear programming can be applied to very complex problems with hundreds or even thousands of variables, we consider only a few simple examples to which the graphical methods of Section 5.5 can be applied. (For large numbers of variables a linear programming method based on matrices is used.) Let's examine a typical problem.

EXAMPLE 1 | Manufacturing for Maximum Profit

A small shoe manufacturer makes two styles of shoes: oxfords and loafers. Two machines are used in the process: a cutting machine and a sewing machine. Each type of shoe requires 15 min per pair on the cutting machine. Oxfords require 10 min of sewing per pair, and loafers require 20 min of sewing per pair. Because the manufacturer can hire only one operator for each machine, each process is available for just 8 hours per day. If the profit is $15 on each pair of oxfords and $20 on each pair of loafers, how many pairs of each type should be produced per day for maximum profit?

Because loafers produce more profit, it would seem best to manufacture only loafers. Surprisingly, this does not turn out to be the most profitable solution.

SOLUTION First we organize the given information into a table. To be consistent, let's convert all times to hours.

	Oxfords	Loafers	Time available
Time on cutting machine (h)	$\frac{1}{4}$	$\frac{1}{4}$	8
Time on sewing machine (h)	$\frac{1}{6}$	$\frac{1}{3}$	8
Profit	$15	$20	

We describe the model and solve the problem in four steps.

▶ **Choose the Variables.** To make a mathematical model, we first give names to the variable quantities. For this problem we let

$$x = \text{number of pairs of oxfords made daily}$$

$$y = \text{number of pairs of loafers made daily}$$

▶ **Find the Objective Function.** Our goal is to determine which values for x and y give maximum profit. Since each pair of oxfords provides $15 profit and each pair of loafers $20, the total profit is given by

$$P = 15x + 20y$$

This function is called the *objective function*.

▶ **Graph the Feasible Region.** The larger x and y are, the greater is the profit. But we cannot choose arbitrarily large values for these variables because of the restrictions, or *constraints*, in the problem. Each restriction is an inequality in the variables.

In this problem the total number of cutting hours needed is $\frac{1}{4}x + \frac{1}{4}y$. Since only 8 hours are available on the cutting machine, we have

$$\tfrac{1}{4}x + \tfrac{1}{4}y \le 8$$

Similarly, by considering the amount of time needed and available on the sewing machine, we get

$$\tfrac{1}{6}x + \tfrac{1}{3}y \le 8$$

We cannot produce a negative number of shoes, so we also have

$$x \ge 0 \qquad \text{and} \qquad y \ge 0$$

Thus x and y must satisfy the constraints

$$\begin{cases} \frac{1}{4}x + \frac{1}{4}y \le 8 \\ \frac{1}{6}x + \frac{1}{3}y \le 8 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

If we multiply the first inequality by 4 and the second by 6, we obtain the simplified system

$$\begin{cases} x + y \le 32 \\ x + 2y \le 48 \\ x \le 0 \\ y \le 0 \end{cases}$$

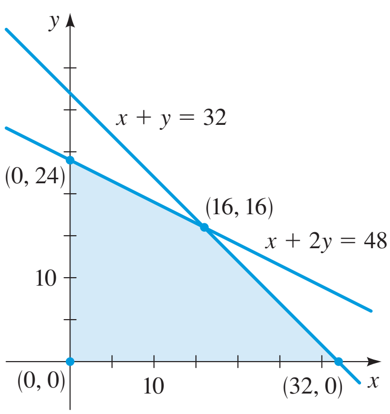


FIGURE 1

The solution of this system (with vertices labeled) is sketched in Figure 1. The only values that satisfy the restrictions of the problem are the ones that correspond to points of the shaded region in Figure 1. This is called the *feasible region* for the problem.

▶ **Find the Maximum Profit.** As x or y increases, profit increases as well. Thus it seems reasonable that the maximum profit will occur at a point on one of the outside edges of the feasible region, where it is impossible to increase x or y without going outside the region. In fact, it can be shown that the maximum value occurs at a vertex. This means that we need to check the profit only at the vertices. The largest value of P occurs at the point (16, 16), where $P = \$560$. Thus the manufacturer should make 16 pairs of oxfords and 16 pairs of loafers, for a maximum daily profit of \$560.

Vertex	$P = 15x + 20y$
(0, 0)	0
(0, 24)	$15(0) + 20(24) = \$480$
(16, 16)	$15(16) + 20(16) = \$560$
(32, 0)	$15(32) + 20(0) = \$480$

Maximum profit

Linear Programming helps the telephone industry to determine the most efficient way to route telephone calls. The computerized routing decisions must be made very rapidly so that callers are not kept waiting for connections. Since the database of customers and routes is huge, an extremely fast method for solving linear programming problems is essential. In 1984 the 28-year-old mathematician **Narendra Karmarkar**, working at Bell Labs in Murray Hill, New Jersey, discovered just such a method. His idea is so ingenious and his method so fast that the discovery caused a sensation in the mathematical world. Although mathematical discoveries rarely make the news, this one was reported in *Time,* on December 3, 1984. Today airlines routinely use Karmarkar's technique to minimize costs in scheduling passengers, flight personnel, fuel, baggage, and maintenance workers.

The linear programming problems that we consider all follow the pattern of Example 1. Each problem involves two variables. The problem describes restrictions, called **constraints**, that lead to a system of linear inequalities whose solution is called the **feasible region**. The function that we wish to maximize or minimize is called the **objective function**. This function always attains its largest and smallest values at the **vertices** of the feasible region. This modeling technique involves four steps, summarized in the following box.

GUIDELINES FOR LINEAR PROGRAMMING

1. **Choose the Variables.** Decide what variable quantities in the problem should be named x and y.
2. **Find the Objective Function.** Write an expression for the function we want to maximize or minimize.
3. **Graph the Feasible Region.** Express the constraints as a system of inequalities, and graph the solution of this system (the feasible region).
4. **Find the Maximum or Minimum.** Evaluate the objective function at the vertices of the feasible region to determine its maximum or minimum value.

EXAMPLE 2 | A Shipping Problem

A car dealer has warehouses in Millville and Trenton and dealerships in Camden and Atlantic City. Every car that is sold at the dealerships must be delivered from one of the warehouses. On a certain day the Camden dealers sell 10 cars, and the Atlantic City dealers sell 12. The Millville warehouse has 15 cars available, and the Trenton warehouse has 10. The cost of shipping one car is \$50 from Millville to Camden, \$40 from Millville to Atlantic City, \$60 from Trenton to Camden, and \$55 from Trenton to Atlantic City. How many cars should be moved from each warehouse to each dealership to fill the orders at minimum cost?

SOLUTION Our first step is to organize the given information. Rather than constructing a table, we draw a diagram to show the flow of cars from the warehouses to the dealerships (see Figure 2 below). The diagram shows the number of cars available at each warehouse or required at each dealership and the cost of shipping between these locations.

▶ **Choose the Variables.** The arrows in Figure 2 indicate four possible routes, so the problem seems to involve four variables. But we let

x = number of cars to be shipped from Millville to Camden

y = number of cars to be shipped from Millville to Atlantic City

To fill the orders, we must have

$10 - x$ = number of cars shipped from Trenton to Camden

$12 - y$ = number of cars shipped from Trenton to Atlantic City

So the only variables in the problem are x and y.

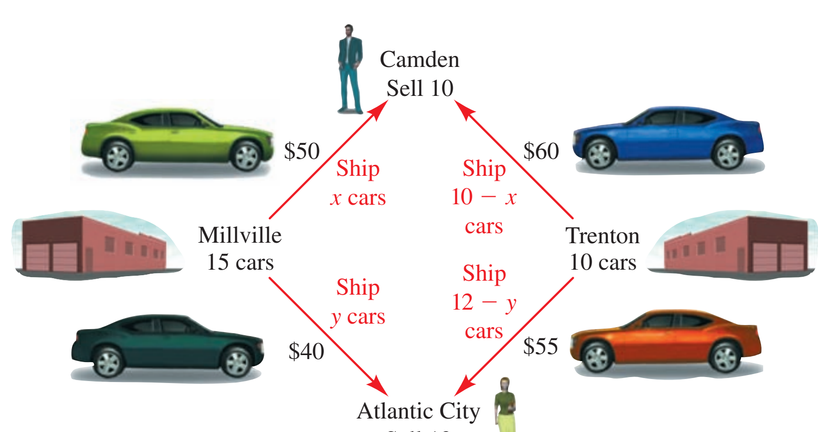


FIGURE 2

▶ **Find the Objective Function.** The objective of this problem is to minimize cost. From Figure 2 we see that the total cost C of shipping the cars is

$$\begin{aligned} C &= 50x + 40y + 60(10 - x) + 55(12 - y) \\ &= 50x + 40y + 600 - 60x + 660 - 55y \\ &= 1260 - 10x - 15y \end{aligned}$$

This is the objective function.

▶ **Graph the Feasible Region.** Now we derive the constraint inequalities that define the feasible region. First, the number of cars shipped on each route can't be negative, so we have

$$\begin{aligned} x &\geq 0 & y &\geq 0 \\ 10 - x &\geq 0 & 12 - y &\geq 0 \end{aligned}$$

Second, the total number of cars shipped from each warehouse can't exceed the number of cars available there, so

$$\begin{aligned} x + y &\leq 15 \\ (10 - x) + (12 - y) &\leq 10 \end{aligned}$$

Simplifying the latter inequality, we get

$$\begin{aligned} 22 - x - y &\leq 10 \\ -x - y &\leq -12 \\ x + y &\geq 12 \end{aligned}$$

The inequalities $10 - x \geq 0$ and $12 - y \geq 0$ can be rewritten as $x \leq 10$ and $y \leq 12$. Thus the feasible region is described by the constraints

$$\begin{cases} x + y \leq 15 \\ x + y \geq 12 \\ 0 \leq x \leq 10 \\ 0 \leq y \leq 12 \end{cases}$$

The feasible region is graphed in Figure 3.

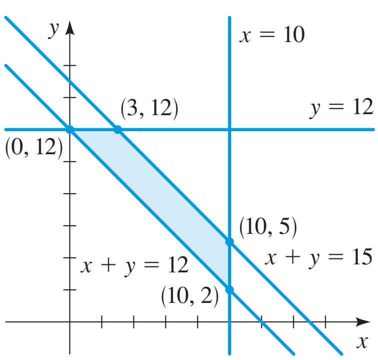


FIGURE 3

▶ **Find the Minimum Cost.** We check the value of the objective function at each vertex of the feasible region.

Vertex	$C = 1260 - 10x - 15y$
(0, 12)	$1260 - 10(0) - 15(12) = \$1080$
(3, 12)	$1260 - 10(3) - 15(12) = \$1050$ ← Minimum cost
(10, 5)	$1260 - 10(10) - 15(5) = \$1085$
(10, 2)	$1260 - 10(10) - 15(2) = \$1130$

The lowest cost is incurred at the point (3, 12). Thus the dealer should ship

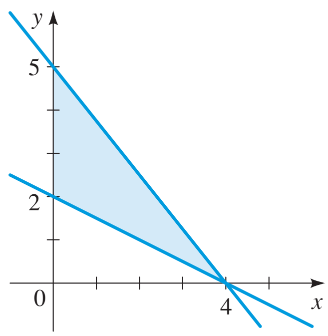
3 cars from Millville to Camden
12 cars from Millville to Atlantic City
7 cars from Trenton to Camden
0 cars from Trenton to Atlantic City ■

In the 1940s mathematicians developed matrix methods for solving linear programming problems that involve more than two variables. These methods were first used by the Allies in World War II to solve supply problems similar to (but, of course, much more complicated than) Example 2. Improving such matrix methods is an active and exciting area of current mathematical research.

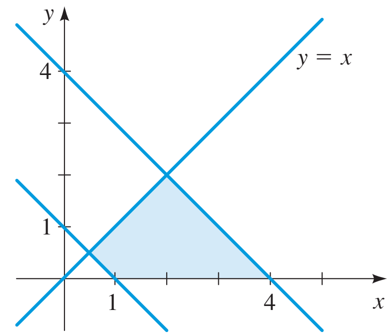
PROBLEMS

1–4 ■ Find the maximum and minimum values of the given objective function on the indicated feasible region.

1. $M = 200 - x - y$



2. $N = \frac{1}{2}x + \frac{1}{4}y + 40$



3. $P = 140 - x + 3y$

$$\begin{cases} x \geq 0,\ y \geq 0 \\ 2x + \ \ y \leq 10 \\ 2x + 4y \leq 28 \end{cases}$$

4. $Q = 70x + 82y$

$$\begin{cases} x \geq 0,\ y \geq 0 \\ x \leq 10,\ y \leq 20 \\ x + \ \ y \geq 5 \\ x + 2y \leq 18 \end{cases}$$

5. Making Furniture A furniture manufacturer makes wooden tables and chairs. The production process involves two basic types of labor: carpentry and finishing. A table requires 2 hours of carpentry and 1 hour of finishing, and a chair requires 3 hours of carpentry and $\frac{1}{2}$ hour of finishing. The profit is \$35 per table and \$20 per chair. The manufacturer's employees can supply a maximum of 108 hours of carpentry work and 20 hours of finishing work per day. How many tables and chairs should be made each day to maximize profit?

6. A Housing Development A housing contractor has subdivided a farm into 100 building lots. She has designed two types of homes for these lots: colonial and ranch style. A colonial requires \$30,000 of capital and produces a profit of \$4000 when sold. A ranch-style house requires \$40,000 of capital and provides an \$8000 profit. If the contractor has \$3.6 million of capital on hand, how many houses of each type should she build for maximum profit? Will any of the lots be left vacant?

7. Hauling Fruit A trucker hauls citrus fruit from Florida to Montreal. Each crate of oranges is 4 ft^3 in volume and weighs 80 lb. Each crate of grapefruit has a volume of 6 ft^3 and weighs 100 lb. His truck has a maximum capacity of 300 ft^3 and can carry no more than 5600 lb. Moreover, he is not permitted to carry more crates of grapefruit than crates of oranges. If his profit is \$2.50 on each crate of oranges and \$4 on each crate of grapefruit, how many crates of each fruit should he carry for maximum profit?

8. Manufacturing Calculators A manufacturer of calculators produces two models: standard and scientific. Long-term demand for the two models mandates that the company manufacture at least 100 standard and 80 scientific calculators each day. However, because of limitations on production capacity, no more than 200 standard and 170 scientific calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators must be shipped every day.

(a) If the production cost is \$5 for a standard calculator and \$7 for a scientific one, how many of each model should be produced daily to minimize this cost?

(b) If each standard calculator results in a \$2 loss but each scientific one produces a \$5 profit, how many of each model should be made daily to maximize profit?

9. Shipping Stereos An electronics discount chain has a sale on a certain brand of stereo. The chain has stores in Santa Monica and El Toro and warehouses in Long Beach and Pasadena. To satisfy rush orders, 15 sets must be shipped from the warehouses to the Santa Monica store, and 19 must be shipped to the El Toro store. The cost of shipping a set is \$5 from Long Beach to Santa Monica, \$6 from Long Beach to El Toro, \$4 from Pasadena to Santa Monica, and \$5.50 from Pasadena to El Toro. If the Long Beach warehouse has 24 sets and the Pasadena warehouse has 18 sets in stock, how many sets should be shipped from each warehouse to each store to fill the orders at a minimum shipping cost?

10. Delivering Plywood A man owns two building supply stores, one on the east side and one on the west side of a city. Two customers order some $\frac{1}{2}$-inch plywood. Customer A needs 50 sheets, and customer B needs 70 sheets. The east-side store has 80 sheets, and the west-side store has 45 sheets of this plywood in stock. The east-side store's delivery costs per sheet are \$0.50 to customer A and \$0.60 to customer B. The west-side store's delivery costs per sheet are \$0.40 to customer A and \$0.55 to customer B. How many sheets should be shipped from each store to each customer to minimize delivery costs?

11. Packaging Nuts A confectioner sells two types of nut mixtures. The standard-mixture package contains 100 g of cashews and 200 g of peanuts and sells for \$1.95. The deluxe-mixture package contains 150 g of cashews and 50 g of peanuts and sells for \$2.25. The confectioner has 15 kg of cashews and 20 kg of peanuts available. On the basis of past sales, the confectioner needs to have at least as many standard as deluxe packages available. How many bags of each mixture should he package to maximize his revenue?

12. Feeding Lab Rabbits A biologist wishes to feed laboratory rabbits a mixture of two types of foods. Type I contains 8 g of fat, 12 g of carbohydrate, and 2 g of protein per ounce. Type II contains 12 g of fat, 12 g of carbohydrate, and 1 g of protein per ounce. Type I costs \$0.20 per ounce and type II costs \$0.30 per ounce. The rabbits each receive a daily minimum of 24 g of fat, 36 g of carbohydrate, and 4 g of protein, but get no more than 5 oz of food per day. How many ounces of each food type should be fed to each rabbit daily to satisfy the dietary requirements at minimum cost?

13. Investing in Bonds A woman wishes to invest \$12,000 in three types of bonds: municipal bonds paying 7% interest per year, bank investment certificates paying 8%, and high-risk bonds paying 12%. For tax reasons she wants the amount invested in municipal bonds to be at least three times the amount invested in bank certificates. To keep her level of risk manageable, she will invest no more than \$2000 in high-risk bonds. How much should she invest in each type of bond to maximize her annual interest yield? [*Hint:* Let x = amount in municipal bonds and y = amount in bank certificates. Then the amount in high-risk bonds will be $12{,}000 - x - y$.]

14. Annual Interest Yield Refer to Problem 13. Suppose the investor decides to increase the maximum invested in high-risk bonds to \$3000 but leaves the other conditions unchanged. By how much will her maximum possible interest yield increase?

15. Business Strategy A small software company publishes computer games and educational and utility software. Their business strategy is to market a total of 36 new programs each year, at least four of these being games. The number of utility programs published is never more than twice the number of educational programs. On average, the company makes an annual profit of \$5000 on each computer game, \$8000 on each educational program, and \$6000 on each utility program. How many of each type of software should the company publish annually for maximum profit?

16. Feasible Region All parts of this problem refer to the following feasible region and objective function:

$$\begin{cases} x \geq 0 \\ x \geq y \\ x + 2y \leq 12 \\ x + y \leq 10 \end{cases}$$

$$P = x + 4y$$

(a) Graph the feasible region.

(b) On your graph from part (a), sketch the graphs of the linear equations obtained by setting P equal to 40, 36, 32, and 28.

(c) If you continue to decrease the value of P, at which vertex of the feasible region will these lines first touch the feasible region?

(d) Verify that the maximum value of P on the feasible region occurs at the vertex you chose in part (c).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CHAPTER 6

Matrices and Determinants

Information in Categories Much of the information we see in newspapers, magazines, books, and other sources is presented in the form of a table. The rows and columns of a table represent different categories of information. For example, the results of a survey might be presented as a table where the columns are the political affiliations of the respondents (Democrat, Republican, Independent) and the rows represent sex (male, female). The table presents the results of the survey in a very efficient manner. From the table we can immediately determine the number of respondents in each category (the number of male Democrats, female Independents, etc.).

In mathematics a rectangular array (or table) of numbers is called a *matrix*. We'll see that a linear system of equations can be represented by a matrix. But in mathematics, presenting information is not enough; we'll learn to perform operations on matrices (such as addition, subtraction, multiplication, and inversion). These operations are powerful tools for getting additional information from a matrix. For example, we'll see how to solve a system of equations by applying special operations to the matrix that represents the system.

In *Focus on Modeling* at the end of the chapter we represent a figure in the plane as a matrix and then perform operations on the matrix that transform the figure in different ways.

6.1 Matrices and Systems of Linear Equations

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find the augmented matrix of a linear system ▶ Solve a linear system using elementary row operations ▶ Solve a linear system using the row-echelon form of its matrix ▶ Solve a linear system using the reduced row-echelon form of its matrix ▶ Determine the number of solutions of a linear system from the row-echelon form of its matrix ▶ Model using linear systems

A *matrix* is simply a rectangular array of numbers. Matrices* are used to organize information into categories that correspond to the rows and columns of the matrix. For example, a scientist might organize information on a population of endangered whales as follows:

$$\begin{array}{r c} & \begin{array}{ccc} \textbf{Immature} & \textbf{Juvenile} & \textbf{Adult} \end{array} \\ \begin{array}{r} \textbf{Male} \\ \textbf{Female} \end{array} & \begin{bmatrix} 12 & 52 & 18 \\ 15 & 42 & 11 \end{bmatrix} \end{array}$$

This is a compact way of saying that there are 12 immature males, 15 immature females, 18 adult males, and so on.

In this section we represent a linear system by a matrix, called the *augmented matrix* of the system:

Linear system **Augmented matrix**

$$\begin{cases} 2x - y = 5 \\ x + 4y = 7 \end{cases} \qquad \begin{array}{c} \text{Equation 1} \\ \text{Equation 2} \end{array} \qquad \begin{array}{c} \begin{bmatrix} 2 & -1 & 5 \\ 1 & 4 & 7 \end{bmatrix} \\ \begin{array}{ccc} x & y & \end{array} \end{array}$$

The augmented matrix contains the same information as the system, but in a simpler form. The operations we learned for solving systems of equations can now be performed on the augmented matrix.

▼ Matrices

We begin by defining the various elements that make up a matrix.

DEFINITION OF MATRIX

An $m \times n$ **matrix** is a rectangular array of numbers with m **rows** and n **columns**.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \left.\begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \\ \leftarrow \end{array}\right\} m \text{ rows}$$

We say that the matrix has **dimension** $m \times n$. The numbers a_{ij} are the **entries** of the matrix. The subscript on the entry a_{ij} indicates that it is in the ith row and the jth column.

*The plural of *matrix* is *matrices*.

Here are some examples of matrices:

Matrix	Dimension	
$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix}$	2×3	2 rows by 3 columns
$\begin{bmatrix} 6 & -5 & 0 & 1 \end{bmatrix}$	1×4	1 row by 4 columns

The Augmented Matrix of a Linear System

We can write a system of linear equations as a matrix, called the **augmented matrix** of the system, by writing only the coefficients and constants that appear in the equations. Here is an example.

Linear system	Augmented matrix
$\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + 4z = 11 \end{cases}$	$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$

Notice that a missing variable in an equation corresponds to a 0 entry in the augmented matrix.

EXAMPLE 1 | Finding the Augmented Matrix of a Linear System

Write the augmented matrix of the system of equations.

$$\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 1 \\ 7y + z = 5 \end{cases}$$

SOLUTION First we write the linear system with the variables lined up in columns:

$$\begin{cases} 6x - 2y - z = 4 \\ x \phantom{{}-2y} + 3z = 1 \\ \phantom{6x-{}} 7y + z = 5 \end{cases}$$

The augmented matrix is the matrix whose entries are the coefficients and the constants in this system.

$$\begin{bmatrix} 6 & -2 & -1 & 4 \\ 1 & 0 & 3 & 1 \\ 0 & 7 & 1 & 5 \end{bmatrix}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **11**

Elementary Row Operations

The operations that we used in Section 5.2 to solve linear systems correspond to operations on the rows of the augmented matrix of the system. For example, adding a multiple of one equation to another corresponds to adding a multiple of one row to another.

ELEMENTARY ROW OPERATIONS

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

Note that performing any of these operations on the augmented matrix of a system does not change its solution. We use the following notation to describe the elementary row operations:

Symbol	**Description**
$R_i + kR_j \rightarrow R_i$	Change the ith row by adding k times row j to it, and then put the result back in row i.
kR_i	Multiply the ith row by k.
$R_i \leftrightarrow R_j$	Interchange the ith and jth rows.

In the next example we compare the two ways of writing systems of linear equations.

EXAMPLE 2 | Using Elementary Row Operations to Solve a Linear System

Solve the system of linear equations.

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

SOLUTION Our goal is to eliminate the x-term from the second equation and the x- and y-terms from the third equation. For comparison we write both the system of equations and its augmented matrix.

System | **Augmented matrix**

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases} \qquad \begin{bmatrix} 1 & -1 & 3 & 4 \\ 1 & 2 & -2 & 10 \\ 3 & -1 & 5 & 14 \end{bmatrix}$$

Add $(-1) \times$ Equation 1 to Equation 2.
Add $(-3) \times$ Equation 1 to Equation 3.

$$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ 2y - 4z = 2 \end{cases} \qquad \xrightarrow[R_3 - 3R_1 \rightarrow R_3]{R_2 - R_1 \rightarrow R_2} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$$

Multiply Equation 3 by $\frac{1}{2}$.

$$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ y - 2z = 1 \end{cases} \qquad \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

Add $(-3) \times$ Equation 3 to Equation 2 (to eliminate y from Equation 2).

$$\begin{cases} x - y + 3z = 4 \\ z = 3 \\ y - 2z = 1 \end{cases} \qquad \xrightarrow{R_2 - 3R_3 \rightarrow R_2} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

Interchange Equations 2 and 3.

$$\begin{cases} x - y + 3z = 4 \\ y - 2z = 1 \\ z = 3 \end{cases} \qquad \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now we use back-substitution to find that $x = 2$, $y = 7$, and $z = 3$. The solution is $(2, 7, 3)$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **29**

Gaussian Elimination

In general, to solve a system of linear equations using its augmented matrix, we use elementary row operations to arrive at a matrix in a certain form. This form is described in the following box.

ROW-ECHELON FORM AND REDUCED ROW-ECHELON FORM OF A MATRIX

A matrix is in **row-echelon form** if it satisfies the following conditions.

1. The first nonzero number in each row (reading from left to right) is 1. This is called the **leading entry**.
2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
3. All rows consisting entirely of zeros are at the bottom of the matrix.

A matrix is in **reduced row-echelon** form if it is in row-echelon form and also satisfies the following condition.

4. Every number above and below each leading entry is a 0.

In the following matrices the first one is not in row-echelon form. The second one *is* in row-echelon form, and the third one is in reduced row-echelon form. The entries in red are the leading entries.

Not in row-echelon form

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2} & 0 & 6 \\ 1 & 0 & 3 & 4 & -5 \\ 0 & 0 & 0 & 1 & 0.4 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Leading 1's do *not* shift to the right in successive rows

Row-echelon form

$$\begin{bmatrix} 1 & 3 & -6 & 10 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's shift to the right in successive rows

Reduced row-echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's have 0's above and below them

Here is a systematic way to put a matrix in row-echelon form using elementary row operations:

- Start by obtaining 1 in the top left corner. Then obtain zeros below that 1 by adding appropriate multiples of the first row to the rows below it.
- Next, obtain a leading 1 in the next row, and then obtain zeros below that 1.
- At each stage make sure that every leading entry is to the right of the leading entry in the row above it—rearrange the rows if necessary.
- Continue this process until you arrive at a matrix in row-echelon form.

This is how the process might work for a 3×4 matrix:

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \qquad \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare \end{bmatrix} \qquad \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Once an augmented matrix is in row-echelon form, we can solve the corresponding linear system using back-substitution. This technique is called **Gaussian elimination**, in honor of its inventor, the German mathematician C. F. Gauss (see page 306).

SOLVING A SYSTEM USING GAUSSIAN ELIMINATION

1. **Augmented Matrix.** Write the augmented matrix of the system.
2. **Row-Echelon Form.** Use elementary row operations to change the augmented matrix to row-echelon form.
3. **Back-Substitution.** Write the new system of equations that corresponds to the row-echelon form of the augmented matrix and solve by back-substitution.

EXAMPLE 3 | Solving a System Using Row-Echelon Form

Solve the system of linear equations using Gaussian elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

SOLUTION We first write the augmented matrix of the system, and then we use elementary row operations to put it in row-echelon form.

Augmented matrix:

Need a 1 here

$$\left[\begin{array}{ccc|c} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

Need 0's here

$$\xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

Need a 1 here

$$\xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 2 & 8 & -14 \\ 0 & 5 & 10 & -15 \end{array}\right]$$

Need a 0 here

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 5 & 10 & -15 \end{array}\right]$$

Need a 1 here

$$\xrightarrow{R_3 - 5R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -10 & 20 \end{array}\right]$$

Row-echelon form:

$$\xrightarrow{-\frac{1}{10}R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

We now have an equivalent matrix in row-echelon form, and the corresponding system of equations is

$$\begin{cases} x + 2y - z = 1 \\ y + 4z = -7 \\ z = -2 \end{cases}$$

Back-substitute: We use back-substitution to solve the system:

$$\begin{aligned} y + 4(-2) &= -7 && \text{Back-substitute } z = -2 \text{ into Equation 2} \\ y &= 1 && \text{Solve for } y \\ x + 2(1) - (-2) &= 1 && \text{Back-substitute } y = 1 \text{ and } z = -2 \text{ into Equation 1} \\ x &= -3 && \text{Solve for } x \end{aligned}$$

So the solution of the system is $(-3, 1, -2)$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **31** ■

```
ref([A])
   [[1  2  -1  1 ]
    [0  1  2   -3]
    [0  0  1   -2]]
```

FIGURE 1

Graphing calculators have a "row-echelon form" command that puts a matrix in row-echelon form. (On the TI-83 this command is `ref`.) For the augmented matrix in Example 3 the `ref` command gives the output shown in Figure 1. Notice that the

See Appendix C, *Using the TI-83/84 Graphing Calculator*, for specific instructions on working with matrices.

row-echelon form that is obtained by the calculator differs from the one we got in Example 3. This is because the calculator used different row operations than we did. You should check that your calculator's row-echelon form leads to the same solution as ours.

Gauss-Jordan Elimination

If we put the augmented matrix of a linear system in *reduced* row-echelon form, then we don't need to back-substitute to solve the system. To put a matrix in reduced row-echelon form, we use the following steps.

- Use the elementary row operations to put the matrix in row-echelon form.
- Obtain zeros above each leading entry by adding multiples of the row containing that entry to the rows above it. Begin with the last leading entry and work up.

Here is how the process works for a 3×4 matrix:

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \blacksquare & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Using the reduced row-echelon form to solve a system is called **Gauss-Jordan elimination**. The process is illustrated in the next example.

EXAMPLE 4 | Solving a System Using Reduced Row-Echelon Form

Solve the system of linear equations, using Gauss-Jordan elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

SOLUTION In Example 3 we used Gaussian elimination on the augmented matrix of this system to arrive at an equivalent matrix in row-echelon form. We continue using elementary row operations on the last matrix in Example 3 to arrive at an equivalent matrix in reduced row-echelon form.

Need 0's here

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Need a 0 here

$$\xrightarrow[R_1 + R_3 \rightarrow R_1]{R_2 - 4R_3 \rightarrow R_2} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2 \rightarrow R_1} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

We now have an equivalent matrix in reduced row-echelon form, and the corresponding system of equations is

$$\begin{cases} x = -3 \\ y = 1 \\ z = -2 \end{cases}$$

Since the system is in reduced row-echelon form, back-subsitution is not required to get the solution.

Hence we immediately arrive at the solution $(-3, 1, -2)$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **33**

```
rref([A])
    [[1 0 0 -3]
     [0 1 0 1 ]
     [0 0 1 -2]]
```

FIGURE 2

See Appendix C, *Using the TI-83/84 Graphing Calculator*, for specific instructions on working with matrices.

Graphing calculators also have a command that puts a matrix in reduced row-echelon form. (On the TI-83 this command is `rref`.) For the augmented matrix in Example 4, the `rref` command gives the output shown in Figure 2. The calculator gives the same reduced row-echelon form as the one we got in Example 4. This is because every matrix has a *unique* reduced row-echelon form.

Inconsistent and Dependent Systems

The systems of linear equations that we considered in Examples 1–4 had exactly one solution. But as we know from Section 5.2, a linear system may have one solution, no solution, or infinitely many solutions. Fortunately, the row-echelon form of a system allows us to determine which of these cases applies, as described in the following box.

First we need some terminology. A **leading variable** in a linear system is one that corresponds to a leading entry in the row-echelon form of the augmented matrix of the system.

THE SOLUTIONS OF A LINEAR SYSTEM IN ROW-ECHELON FORM

Suppose the augmented matrix of a system of linear equations has been transformed by Gaussian elimination into row-echelon form. Then exactly one of the following is true.

1. **No solution.** If the row-echelon form contains a row that represents the equation $0 = c$, where c is not zero, then the system has no solution. A system with no solution is called **inconsistent**.
2. **One solution.** If each variable in the row-echelon form is a leading variable, then the system has exactly one solution, which we find using back-substitution or Gauss-Jordan elimination.
3. **Infinitely many solutions.** If the variables in the row-echelon form are not all leading variables and if the system is not inconsistent, then it has infinitely many solutions. In this case the system is called **dependent**. We solve the system by putting the matrix in reduced row-echelon form and then expressing the leading variables in terms of the nonleading variables. The nonleading variables may take on any real numbers as their values.

The matrices below, all in row-echelon form, illustrate the three cases described above.

No solution

$$\begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Last equation says $0 = 1$

One solution

$$\begin{bmatrix} 1 & 6 & -1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

Each variable is a leading variable

Infinitely many solutions

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

z is not a leading variable

EXAMPLE 5 | A System with No Solution

Solve the system.

$$\begin{cases} x - 3y + 2z = 12 \\ 2x - 5y + 5z = 14 \\ x - 2y + 3z = 20 \end{cases}$$

SOLUTION We transform the system into row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 12 \\ 2 & -5 & 5 & 14 \\ 1 & -2 & 3 & 20 \end{array}\right] \xrightarrow[\mathsf{R_3 - R_1 \to R_3}]{\mathsf{R_2 - 2R_1 \to R_2}} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 1 & 1 & 8 \end{array}\right]$$

$$\xrightarrow{\mathsf{R_3 - R_2 \to R_3}} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 18 \end{array}\right] \xrightarrow{\frac{1}{18}\mathsf{R_3}} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This last matrix is in row-echelon form, so we can stop the Gaussian elimination process. Now if we translate the last row back into equation form, we get $0x + 0y + 0z = 1$, or $0 = 1$, which is false. No matter what values we pick for x, y, and z, the last equation will never be a true statement. This means that the system *has no solution.*

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **39** ■

```
ref([A])
[[1  -2.5  2.5  7   ]
 [0  1     1    -10]
 [0  0     0    1   ]]
```

FIGURE 3

Figure 3 shows the row-echelon form produced by a TI-83 calculator for the augmented matrix in Example 5. You should check that this gives the same solution.

EXAMPLE 6 | A System with Infinitely Many Solutions

Find the complete solution of the system.

$$\begin{cases} -3x - 5y + 36z = 10 \\ -x + 7z = 5 \\ x + y - 10z = -4 \end{cases}$$

SOLUTION We transform the system into reduced row-echelon form. (The `rref` command on a TI-83 calculator gives the same result, as shown in Figure 4.)

```
rref([A])
   [[1 0 -7 -5]
    [0 1 -3 1 ]
    [0 0 0  0 ]]
```

FIGURE 4 Reduced row-echelon form on the TI-83 calculator

$$\left[\begin{array}{ccc|c} -3 & -5 & 36 & 10 \\ -1 & 0 & 7 & 5 \\ 1 & 1 & -10 & -4 \end{array}\right] \xrightarrow{\mathsf{R_1 \leftrightarrow R_3}} \left[\begin{array}{ccc|c} 1 & 1 & -10 & -4 \\ -1 & 0 & 7 & 5 \\ -3 & -5 & 36 & 10 \end{array}\right]$$

$$\xrightarrow[\mathsf{R_3 + 3R_1 \to R_3}]{\mathsf{R_2 + R_1 \to R_2}} \left[\begin{array}{ccc|c} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & -2 & 6 & -2 \end{array}\right] \xrightarrow{\mathsf{R_3 + 2R_2 \to R_3}} \left[\begin{array}{ccc|c} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\mathsf{R_1 - R_2 \to R_1}} \left[\begin{array}{ccc|c} 1 & 0 & -7 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The third row corresponds to the equation $0 = 0$. This equation is always true, no matter what values are used for x, y, and z. Since the equation adds no new information about the variables, we can drop it from the system. So the last matrix corresponds to the system

$$\begin{cases} x - 7z = -5 & \text{Equation 1} \\ y - 3z = 1 & \text{Equation 2} \end{cases}$$

Leading variables

Now we solve for the leading variables x and y in terms of the nonleading variable z:

$$x = 7z - 5 \qquad \text{Solve for } x \text{ in Equation 1}$$

$$y = 3z + 1 \qquad \text{Solve for } y \text{ in Equation 2}$$

To obtain the complete solution, we let t represent any real number, and we express x, y, and z in terms of t:

$$\begin{aligned} x &= 7t - 5 \\ y &= 3t + 1 \\ z &= t \end{aligned}$$

We can also write the solution as the ordered triple $(7t - 5, 3t + 1, t)$, where t is any real number.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41**

In Example 6, to get specific solutions, we give a specific value to t. For example, if $t = 1$, then

$$\begin{aligned} x &= 7(1) - 5 = 2 \\ y &= 3(1) + 1 = 4 \\ z &= 1 \end{aligned}$$

Here are some other solutions of the system obtained by substituting other values for the parameter t.

Parameter t	**Solution $(7t - 5, 3t + 1, t)$**
-1	$(-12, -2, -1)$
0	$(-5, 1, 0)$
2	$(9, 7, 2)$
5	$(30, 16, 5)$

EXAMPLE 7 | A System with Infinitely Many Solutions

Find the complete solution of the system.

$$\begin{cases} x + 2y - 3z - 4w = 10 \\ x + 3y - 3z - 4w = 15 \\ 2x + 2y - 6z - 8w = 10 \end{cases}$$

SOLUTION We transform the system into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & -3 & -4 & 10 \\ 1 & 3 & -3 & -4 & 15 \\ 2 & 2 & -6 & -8 & 10 \end{bmatrix} \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & 2 & -3 & -4 & 10 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & -2 & 0 & 0 & -10 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_2 \to R_3} \begin{bmatrix} 1 & 2 & -3 & -4 & 10 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2 \to R_1} \begin{bmatrix} 1 & 0 & -3 & -4 & 0 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in reduced row-echelon form. Since the last row represents the equation $0 = 0$, we may discard it. So the last matrix corresponds to the system

$$\begin{cases} x \qquad - 3z - 4w = 0 \\ \quad y \qquad\qquad\quad = 5 \end{cases}$$

Leading variables

To obtain the complete solution, we solve for the leading variables x and y in terms of the nonleading variables z and w, and we let z and w be any real numbers. Thus the complete solution is

$$\begin{aligned} x &= 3s + 4t \\ y &= 5 \\ z &= s \\ w &= t \end{aligned}$$

where s and t are any real numbers.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **61**

Note that s and t do *not* have to be the *same* real number in the solution for Example 7. We can choose arbitrary values for each if we wish to construct a specific solution to the system. For example, if we let $s = 1$ and $t = 2$, then we get the solution $(11, 5, 1, 2)$. You should check that this does indeed satisfy all three of the original equations in Example 7.

Examples 6 and 7 illustrate this general fact: If a system in row-echelon form has n nonzero equations in m variables $(m > n)$, then the complete solution will have $m - n$ nonleading variables. For instance, in Example 6 we arrived at *two* nonzero equations in the *three* variables x, y, and z, which gave us $3 - 2 = 1$ nonleading variable.

Modeling with Linear Systems

Linear equations, often containing hundreds or even thousands of variables, occur frequently in the applications of algebra to the sciences and to other fields. For now, let's consider an example that involves only three variables.

EXAMPLE 8 | Nutritional Analysis Using a System of Linear Equations

A nutritionist is performing an experiment on student volunteers. He wishes to feed one of his subjects a daily diet that consists of a combination of three commercial diet foods: MiniCal, LiquiFast, and SlimQuick. For the experiment it is important that the subject consume exactly 500 mg of potassium, 75 g of protein, and 1150 units of vitamin D every day. The amounts of these nutrients in one ounce of each food are given in the table. How many ounces of each food should the subject eat every day to satisfy the nutrient requirements exactly?

	MiniCal	LiquiFast	SlimQuick
Potassium (mg)	50	75	10
Protein (g)	5	10	3
Vitamin D (units)	90	100	50

SOLUTION Let x, y, and z represent the number of ounces of MiniCal, LiquiFast, and SlimQuick, respectively, that the subject should eat every day. This means that he will get $50x$ mg of potassium from MiniCal, $75y$ mg from LiquiFast, and $10z$ mg from SlimQuick, for a total of $50x + 75y + 10z$ mg potassium in all. Since the potassium requirement is 500 mg, we get the first equation below. Similar reasoning for the protein and vitamin D requirements leads to the system

$$\begin{cases} 50x + 75y + 10z = 500 & \text{Potassium} \\ 5x + 10y + 3z = 75 & \text{Protein} \\ 90x + 100y + 50z = 1150 & \text{Vitamin D} \end{cases}$$

Dividing the first equation by 5 and the third one by 10 gives the system

$$\begin{cases} 10x + 15y + 2z = 100 \\ 5x + 10y + 3z = 75 \\ 9x + 10y + 5z = 115 \end{cases}$$

We can solve this system using Gaussian elimination, or we can use a graphing calculator to find the reduced row-echelon form of the augmented matrix of the system. Using the `rref` command on the TI-83, we get the output in Figure 5. From the reduced row-echelon form we see that $x = 5$, $y = 2$, $z = 10$. The subject should be fed 5 oz of Mini-Cal, 2 oz of LiquiFast, and 10 oz of SlimQuick every day.

```
rref([A])
  [[1 0 0 5 ]
   [0 1 0 2 ]
   [0 0 1 10]]
```

FIGURE 5

CHECK YOUR ANSWER

$x = 5$, $y = 2$, $z = 10$:

$$\begin{cases} 10(5) + 15(2) + 2(10) = 100 \\ 5(5) + 10(2) + 3(10) = 75 \\ 9(5) + 10(2) + 5(10) = 115 \end{cases} \checkmark$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **69**

A more practical application might involve dozens of foods and nutrients rather than just three. Such problems lead to systems with large numbers of variables and equations. Computers or graphing calculators are essential for solving such large systems.

6.1 EXERCISES

CONCEPTS

1. If a system of linear equations has infinitely many solutions, then the system is called ________. If a system of linear equations has no solution, then the system is called ________.

2. Write the augmented matrix of the following system of equations.

System

$$\begin{cases} x + y - z = 1 \\ x + 2z = -3 \\ 2y - z = 3 \end{cases}$$

Augmented matrix

$$\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$$

3. The following matrix is the augmented matrix of a system of linear equations in the variables x, y, and z. (It is given in reduced row-echelon form.)

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) The leading variables are ________.

(b) Is the system inconsistent or dependent?________

(c) The solution of the system is:

$x =$ _____, $y =$ _____, $z =$ _____

4. The augmented matrix of a system of linear equations is given in reduced row-echelon form. Find the solution of the system.

(a) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$x =$ _____ $x =$ _____ $x =$ _____

$y =$ _____ $y =$ _____ $y =$ _____

$z =$ _____ $z =$ _____ $z =$ _____

SKILLS

5–10 ■ State the dimension of the matrix.

5. $\begin{bmatrix} 2 & 7 \\ 0 & -1 \\ 5 & -3 \end{bmatrix}$

6. $\begin{bmatrix} -1 & 5 & 4 & 0 \\ 0 & 2 & 11 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 12 \\ 35 \end{bmatrix}$

8. $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 4 & 7 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

11–12 ■ Write the augmented matrix for the system of linear equations.

11. $\begin{cases} 3x + y - z = 2 \\ 2x - y = 1 \\ x - z = 3 \end{cases}$

12. $\begin{cases} -x + z = -1 \\ 3y - 2z = 7 \\ x - y + 3z = 3 \end{cases}$

13–20 ■ A matrix is given. **(a)** Determine whether the matrix is in row-echelon form. **(b)** Determine whether the matrix is in reduced row-echelon form. **(c)** Write the system of equations for which the given matrix is the augmented matrix.

13. $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 5 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & 8 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

21–24 ■ Perform the indicated elementary row operation.

21. $\begin{bmatrix} -1 & 1 & 2 & 0 \\ 3 & 1 & 1 & 4 \\ 1 & -2 & -1 & -1 \end{bmatrix}$

Add 3 times Row 1 to Row 2.

22. $\begin{bmatrix} -5 & 2 & -3 & 3 \\ 10 & -3 & 1 & -20 \\ -1 & 3 & 1 & 8 \end{bmatrix}$

Add 2 times Row 1 to Row 2.

23. $\begin{bmatrix} 2 & 1 & -3 & 5 \\ 2 & 3 & 1 & 13 \\ 6 & -5 & -1 & 7 \end{bmatrix}$

Add -3 times Row 1 to Row 3.

24. $\begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & -1 & 1 \end{bmatrix}$

Add -2 times Row 2 to Row 3.

25–28 ■ A matrix is given in row-echelon form. **(a)** Write the system of equations for which the given matrix is the augmented matrix. **(b)** Use back-substitution to solve the system.

25. $\begin{bmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 1 & -3 & 8 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 2 & 3 & -1 & 7 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 0 & -2 & 2 & 5 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

29–38 ■ The system of linear equations has a unique solution. Find the solution using Gaussian elimination or Gauss-Jordan elimination.

29. $\begin{cases} x - 2y + z = 1 \\ y + 2z = 5 \\ x + y + 3z = 8 \end{cases}$

30. $\begin{cases} x + y + 6z = 3 \\ x + y + 3z = 3 \\ x + 2y + 4z = 7 \end{cases}$

31. $\begin{cases} x + y + z = 2 \\ 2x - 3y + 2z = 4 \\ 4x + y - 3z = 1 \end{cases}$

32. $\begin{cases} x + y + z = 4 \\ -x + 2y + 3z = 17 \\ 2x - y = -7 \end{cases}$

33. $\begin{cases} x + 2y - z = -2 \\ x + z = 0 \\ 2x - y - z = -3 \end{cases}$

34. $\begin{cases} 2y + z = 4 \\ x + y = 4 \\ 3x + 3y - z = 10 \end{cases}$

35. $\begin{cases} x_1 + 2x_2 - x_3 = 9 \\ 2x_1 - x_3 = -2 \\ 3x_1 + 5x_2 + 2x_3 = 22 \end{cases}$

36. $\begin{cases} 2x_1 + x_2 = 7 \\ 2x_1 - x_2 + x_3 = 6 \\ 3x_1 - 2x_2 + 4x_3 = 11 \end{cases}$

37. $\begin{cases} 2x - 3y - z = 13 \\ -x + 2y - 5z = 6 \\ 5x - y - z = 49 \end{cases}$

38. $\begin{cases} 10x + 10y - 20z = 60 \\ 15x + 20y + 30z = -25 \\ -5x + 30y - 10z = 45 \end{cases}$

39–48 ■ Determine whether the system of linear equations is inconsistent or dependent. If it is dependent, find the complete solution.

39. $\begin{cases} x + y + z = 2 \\ y - 3z = 1 \\ 2x + y + 5z = 0 \end{cases}$

40. $\begin{cases} x + 3z = 3 \\ 2x + y - 2z = 5 \\ -y + 8z = 8 \end{cases}$

41. $\begin{cases} 2x - 3y - 9z = -5 \\ x + 3z = 2 \\ -3x + y - 4z = -3 \end{cases}$

42. $\begin{cases} x - 2y + 5z = 3 \\ -2x + 6y - 11z = 1 \\ 3x - 16y - 20z = -26 \end{cases}$

43. $\begin{cases} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{cases}$

44. $\begin{cases} -2x + 6y - 2z = -12 \\ x - 3y + 2z = 10 \\ -x + 3y + 2z = 6 \end{cases}$

45. $\begin{cases} x + 4y - 2z = -3 \\ 2x - y + 5z = 12 \\ 8x + 5y + 11z = 30 \end{cases}$

46. $\begin{cases} 3r + 2s - 3t = 10 \\ r - s - t = -5 \\ r + 4s - t = 20 \end{cases}$

47. $\begin{cases} 2x + y - 2z = 12 \\ -x - \frac{1}{2}y + z = -6 \\ 3x + \frac{3}{2}y - 3z = 18 \end{cases}$

48. $\begin{cases} y - 5z = 7 \\ 3x + 2y = 12 \\ 3x + 10z = 80 \end{cases}$

49–64 ■ Solve the system of linear equations.

49. $\begin{cases} 4x - 3y + z = -8 \\ -2x + y - 3z = -4 \\ x - y + 2z = 3 \end{cases}$

50. $\begin{cases} 2x - 3y + 5z = 14 \\ 4x - y - 2z = -17 \\ -x - y + z = 3 \end{cases}$

51. $\begin{cases} 2x + y + 3z = 9 \\ -x - 7z = 10 \\ 3x + 2y - z = 4 \end{cases}$

52. $\begin{cases} -4x - y + 36z = 24 \\ x - 2y + 9z = 3 \\ -2x + y + 6z = 6 \end{cases}$

53. $\begin{cases} x + 2y - 3z = -5 \\ -2x - 4y - 6z = 10 \\ 3x + 7y - 2z = -13 \end{cases}$

54. $\begin{cases} 3x + y = 2 \\ -4x + 3y + z = 4 \\ 2x + 5y + z = 0 \end{cases}$

55. $\begin{cases} x - y + 6z = 8 \\ x + z = 5 \\ x + 3y - 14z = -4 \end{cases}$

56. $\begin{cases} 3x - y + 2z = -1 \\ 4x - 2y + z = -7 \\ -x + 3y - 2z = -1 \end{cases}$

57. $\begin{cases} -x + 2y + z - 3w = 3 \\ 3x - 4y + z + w = 9 \\ -x - y + z + w = 0 \\ 2x + y + 4z - 2w = 3 \end{cases}$

58. $\begin{cases} x + y - z - w = 6 \\ 2x + z - 3w = 8 \\ x - y + 4w = -10 \\ 3x + 5y - z - w = 20 \end{cases}$

59. $\begin{cases} x + y + 2z - w = -2 \\ 3y + z + 2w = 2 \\ x + y + 3w = 2 \\ -3x + z + 2w = 5 \end{cases}$

60. $\begin{cases} x - 3y + 2z + w = -2 \\ x - 2y - 2w = -10 \\ z + 5w = 15 \\ 3x + 2z + w = -3 \end{cases}$

61. $\begin{cases} x - y + w = 0 \\ 3x - z + 2w = 0 \\ x - 4y + z + 2w = 0 \end{cases}$

62. $\begin{cases} 2x - y + 2z + w = 5 \\ -x + y + 4z - w = 3 \\ 3x - 2y - z = 0 \end{cases}$

63. $\begin{cases} x + z + w = 4 \\ y - z = -4 \\ x - 2y + 3z + w = 12 \\ 2x - 2z + 5w = -1 \end{cases}$

64. $\begin{cases} y - z + 2w = 0 \\ 3x + 2y + w = 0 \\ 2x + 4w = 12 \\ -2x - 2z + 5w = 6 \end{cases}$

65–68 ■ Solve the system of linear equations by using the `rref` command on a graphing calculator. State your answer rounded to two decimal places.

65. $\begin{cases} 0.75x - 3.75y + 2.95z = 4.0875 \\ 0.95x - 8.75y = 3.375 \\ 1.25x - 0.15y + 2.75z = 3.6625 \end{cases}$

66. $\begin{cases} 1.31x + 2.72y - 3.71z = -13.9534 \\ -0.21x + 3.73z = 13.4322 \\ 2.34y - 4.56z = -21.3984 \end{cases}$

67. $\begin{cases} 42x - 31y - 42w = -0.4 \\ -6x - 9w = 4.5 \\ 35x - 67z + 32w = 348.8 \\ 31y + 48z - 52w = -76.6 \end{cases}$

68. $\begin{cases} 49x - 27y + 52z = -145 \\ 27y + 43w = -118.7 \\ -31y + 42z = -72.1 \\ 73x - 54y = -132.7 \end{cases}$

APPLICATIONS

69. Nutrition A doctor recommends that a patient take 50 mg each of niacin, riboflavin, and thiamin daily to alleviate a vitamin deficiency. In his medicine chest at home the patient finds three brands of vitamin pills. The amounts of the relevant vitamins per pill are given in the table. How many pills of each type should he take every day to get 50 mg of each vitamin?

	VitaMax	Vitron	VitaPlus
Niacin (mg)	5	10	15
Riboflavin (mg)	15	20	0
Thiamin (mg)	10	10	10

70. Mixtures A chemist has three acid solutions at various concentrations. The first is 10% acid, the second is 20%, and the third is 40%. How many milliliters of each should she use to make 100 mL of 18% solution, if she has to use four times as much of the 10% solution as the 40% solution?

71. Distance, Speed, and Time Amanda, Bryce, and Corey enter a race in which they have to run, swim, and cycle over a marked course. Their average speeds are given in the table. Corey finishes first with a total time of 1 h 45 min. Amanda comes in second with a time of 2 h 30 min. Bryce finishes last with a time of 3 h. Find the distance (in miles) for each part of the race.

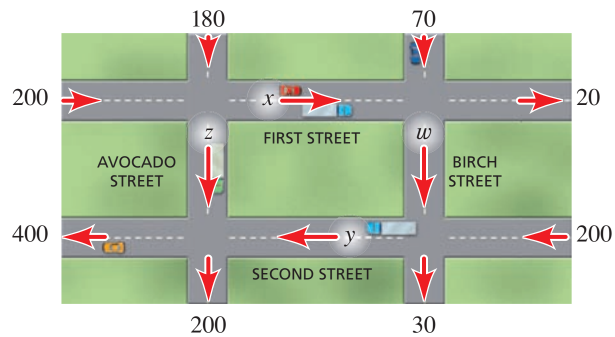
	Average speed (mi/h)		
	Running	**Swimming**	**Cycling**
Amanda	10	4	20
Bryce	$7\frac{1}{2}$	6	15
Corey	15	3	40

72. Classroom Use A small school has 100 students who occupy three classrooms: A, B, and C. After the first period of the school day, half the students in room A move to room B, one-fifth of the students in room B move to room C, and one-third of the students in room C move to room A. Nevertheless, the total number of students in each room is the same for both periods. How many students occupy each room?

73. Manufacturing Furniture A furniture factory makes wooden tables, chairs, and armoires. Each piece of furniture requires three operations: cutting the wood, assembling, and finishing. Each operation requires the number of hours (h) given in the table. The workers in the factory can provide 300 hours of cutting, 400 hours of assembling, and 590 hours of finishing each work week. How many tables, chairs, and armoires should be produced so that all available labor-hours are used? Or is this impossible?

	Table	**Chair**	**Armoire**
Cutting (h)	$\frac{1}{2}$	1	1
Assembling (h)	$\frac{1}{2}$	$1\frac{1}{2}$	1
Finishing (h)	1	$1\frac{1}{2}$	2

74. Traffic Flow A section of a city's street network is shown in the figure. The arrows indicate one-way streets, and the numbers show how many cars enter or leave this section of the city via the indicated street in a certain one-hour period. The variables x, y, z, and w represent the number of cars that travel along the portions of First, Second, Avocado, and Birch Streets during this period. Find x, y, z, and w, assuming that none of the cars stop or park on any of the streets shown.



DISCOVERY ■ DISCUSSION ■ WRITING

75. Polynomials Determined by a Set of Points We all know that two points uniquely determine a line $y = ax + b$ in the coordinate plane. Similarly, three points uniquely determine a quadratic (second-degree) polynomial

$$y = ax^2 + bx + c$$

four points uniquely determine a cubic (third-degree) polynomial

$$y = ax^3 + bx^2 + cx + d$$

and so on. (Some exceptions to this rule are if the three points actually lie on a line, or the four points lie on a quadratic or line, and so on.) For the following set of five points, find the line that contains the first two points, the quadratic that contains the first three points, the cubic that contains the first four points, and the fourth-degree polynomial that contains all five points.

$$(0, 0), \quad (1, 12), \quad (2, 40), \quad (3, 6), \quad (-1, -14)$$

Graph the points and functions in the same viewing rectangle using a graphing device.

6.2 THE ALGEBRA OF MATRICES

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Determine whether two matrices are equal ▶ Perform addition, subtraction, and scalar multiplication of matrices ▶ Perform matrix multiplication ▶ Express a linear system in matrix form

Thus far, we have used matrices simply for notational convenience when solving linear systems. Matrices have many other uses in mathematics and the sciences, and for most of these applications a knowledge of matrix algebra is essential. Like numbers, matrices can be added, subtracted, multiplied, and divided. In this section we learn how to perform these algebraic operations on matrices.

▼ Equality of Matrices

Two matrices are equal if they have the same entries in the same positions.

Equal matrices

$$\begin{bmatrix} \sqrt{4} & 2^2 & e^0 \\ 0.5 & 1 & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ \frac{1}{2} & \frac{2}{2} & 0 \end{bmatrix}$$

Unequal matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

EQUALITY OF MATRICES

The matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if and only if they have the same dimension $m \times n$, and corresponding entries are equal, that is,

$$a_{ij} = b_{ij}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

EXAMPLE 1 | Equal Matrices

Find a, b, c, and d, if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$$

SOLUTION Since the two matrices are equal, corresponding entries must be the same. So we must have $a = 1$, $b = 3$, $c = 5$, and $d = 2$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **5** AND **7**

Addition, Subtraction, and Scalar Multiplication of Matrices

Two matrices can be added or subtracted if they have the same dimension. (Otherwise, their sum or difference is undefined.) We add or subtract the matrices by adding or subtracting corresponding entries. To multiply a matrix by a number, we multiply every element of the matrix by that number. This is called the *scalar product*.

SUM, DIFFERENCE, AND SCALAR PRODUCT OF MATRICES

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the same dimension $m \times n$, and let c be any real number.

1. The **sum** $A + B$ is the $m \times n$ matrix obtained by adding corresponding entries of A and B.

$$A + B = [a_{ij} + b_{ij}]$$

2. The **difference** $A - B$ is the $m \times n$ matrix obtained by subtracting corresponding entries of A and B.

$$A - B = [a_{ij} - b_{ij}]$$

3. The **scalar product** cA is the $m \times n$ matrix obtained by multiplying each entry of A by c.

$$cA = [ca_{ij}]$$

EXAMPLE 2 | Performing Algebraic Operations on Matrices

Let

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix}$$


Courtesy UC Berkeley Office of Media Relations

JULIA ROBINSON (1919–1985) was born in St. Louis, Missouri, and grew up at Point Loma, California. Because of an illness, Robinson missed two years of school, but later, with the aid of a tutor, she completed fifth, sixth, seventh, and eighth grades, all in one year. Later, at San Diego State University, reading biographies of mathematicians in E. T. Bell's *Men of Mathematics* awakened in her what became a lifelong passion for mathematics. She said, "I cannot overemphasize the importance of such books . . . in the intellectual life of a student." Robinson is famous for her work on Hilbert's tenth problem (page 502), which asks for a general procedure for determining whether an equation has integer solutions. Her ideas led to a complete answer to the problem. Interestingly, the answer involved certain properties of the Fibonacci numbers (page 574) discovered by the then 22-year-old Russian mathematician Yuri Matijasevič. As a result of her brilliant work on Hilbert's tenth problem, Robinson was offered a professorship at the University of California, Berkeley, and became the first woman mathematician elected to the National Academy of Sciences. She also served as president of the American Mathematical Society.

Carry out each indicated operation, or explain why it cannot be performed.

(a) $A + B$ **(b)** $C - D$ **(c)** $C + A$ **(d)** $5A$

SOLUTION

(a) $A + B = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 6 \\ 9 & \frac{3}{2} \end{bmatrix}$

(b) $C - D = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -3 & 6 \\ -8 & 0 & -4 \end{bmatrix}$$

(c) $C + A$ is undefined because we can't add matrices of different dimensions.

(d) $5A = 5\begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 0 & 25 \\ 35 & -\frac{5}{2} \end{bmatrix}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **23** AND **25**

The properties in the box follow from the definitions of matrix addition and scalar multiplication and the corresponding properties of real numbers.

PROPERTIES OF ADDITION AND SCALAR MULTIPLICATION OF MATRICES

Let A, B, and C be $m \times n$ matrices, and let c and d be scalars.

$A + B = B + A$	Commutative Property of Matrix Addition
$(A + B) + C = A + (B + C)$	Associative Property of Matrix Addition
$c(dA) = cdA$	Associative Property of Scalar Multiplication
$(c + d)A = cA + dA$ $c(A + B) = cA + cB$	Distributive Properties of Scalar Multiplication

EXAMPLE 3 | Solving a Matrix Equation

Solve the matrix equation

$$2X - A = B$$

for the unknown matrix X, where

$$A = \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$$

SOLUTION We use the properties of matrices to solve for X:

$2X - A = B$	Given equation
$2X = B + A$	Add the matrix A to each side
$X = \frac{1}{2}(B + A)$	Multiply each side by the scalar $\frac{1}{2}$

So $$X = \frac{1}{2}\left(\begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix}\right)$$ Substitute the matrices A and B

$$= \frac{1}{2}\begin{bmatrix} 6 & 2 \\ -4 & 4 \end{bmatrix}$$ Add matrices

$$= \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$$ Multiply by the scalar $\frac{1}{2}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17**

Multiplication of Matrices

Multiplying two matrices is more difficult to describe than other matrix operations. In later examples we will see why multiplying matrices involves a rather complex procedure, which we now describe.

First, the product AB (or $A \cdot B$) of two matrices A and B is defined only when the number of columns in A is equal to the number of rows in B. This means that if we write their dimensions side by side, the two inner numbers must match:

Matrices	A	B
Dimensions	$m \times n$	$n \times k$
	Columns in A	Rows in B

If the dimensions of A and B match in this fashion, then the product AB is a matrix of dimension $m \times k$. Before describing the procedure for obtaining the elements of AB, we define the *inner product* of a row of A and a column of B.

If $[a_1 \quad a_2 \quad \cdots \quad a_n]$ is a row of A, and if $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is a column of B, then their **inner product**

is the number $a_1b_1 + a_2b_2 + \cdots + a_nb_n$. For example, taking the inner product of

$[2 \quad -1 \quad 0 \quad 4]$ and $\begin{bmatrix} 5 \\ 4 \\ -3 \\ \frac{1}{2} \end{bmatrix}$ gives

$$2 \cdot 5 + (-1) \cdot 4 + 0 \cdot (-3) + 4 \cdot \tfrac{1}{2} = 8$$

We now define the **product** AB of two matrices.

MATRIX MULTIPLICATION

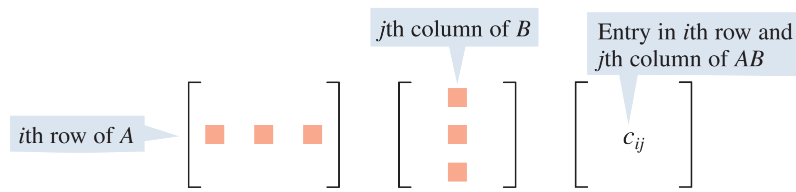
If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ an $n \times k$ matrix, then their product is the $m \times k$ matrix

$$C = [c_{ij}]$$

where c_{ij} is the inner product of the ith row of A and the jth column of B. We write the product as

$$C = AB$$

This definition of matrix product says that each entry in the matrix AB is obtained from a *row* of A and a *column* of B as follows: The entry c_{ij} in the ith row and jth column of the matrix AB is obtained by multiplying the entries in the ith row of A with the corresponding entries in the jth column of B and adding the results.



EXAMPLE 4 | Multiplying Matrices

Let

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$$

Calculate, if possible, the products AB and BA.

SOLUTION Since A has dimension 2×2 and B has dimension 2×3, the product AB is defined and has dimension 2×3. We can therefore write

Inner numbers match, so product is defined

$2 \times 2 \quad 2 \times 3$

Outer numbers give dimension of product: 2×3

$$AB = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

where the question marks must be filled in using the rule defining the product of two matrices. If we define $C = AB = [c_{ij}]$, then the entry c_{11} is the inner product of the first row of A and the first column of B:

$$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} \qquad 1 \cdot (-1) + 3 \cdot 0 = -1$$

Similarly, we calculate the remaining entries of the product as follows:

Entry	**Inner product of:**	**Value**	**Product matrix**
c_{12}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1 \cdot 5 + 3 \cdot 4 = 17$	$\begin{bmatrix} -1 & 17 & \\ & & \end{bmatrix}$
c_{13}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$1 \cdot 2 + 3 \cdot 7 = 23$	$\begin{bmatrix} -1 & 17 & 23 \\ & & \end{bmatrix}$
c_{21}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1) \cdot (-1) + 0 \cdot 0 = 1$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & & \end{bmatrix}$
c_{22}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1) \cdot 5 + 0 \cdot 4 = -5$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & \end{bmatrix}$
c_{23}	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	$(-1) \cdot 2 + 0 \cdot 7 = -2$	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$

Thus we have

$$AB = \begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$$

Not equal, so product not defined

$2 \times 3 \quad 2 \times 2$

The product BA is not defined, however, because the dimensions of B and A are

$$2 \times 3 \quad \text{and} \quad 2 \times 2$$

The inner two numbers are not the same, so the rows and columns won't match up when we try to calculate the product.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **27**

```
[A]*[B]
    [[-1 17 23]
     [1  -5 -2]]
```

FIGURE 1

See Appendix C, *Using the TI-83/84 Graphing Calculator*, for specific instructions on working with matrices.

Graphing calculators and computers are capable of performing matrix algebra. For instance, if we enter the matrices in Example 4 into the matrix variables [A] and [B] on a TI-83 calculator, then the calculator finds their product as shown in Figure 1.

Properties of Matrix Multiplication

Although matrix multiplication is not commutative, it does obey the Associative and Distributive Properties.

PROPERTIES OF MATRIX MULTIPLICATION

Let A, B, and C be matrices for which the following products are defined. Then

$A(BC) = (AB)C$	Associative Property
$A(B + C) = AB + AC$ $(B + C)A = BA + CA$	Distributive Property

The next example shows that even when both AB and BA are defined, they aren't necessarily equal. This result proves that matrix multiplication is *not* commutative.

EXAMPLE 5 | Matrix Multiplication Is Not Commutative

Let

$$A = \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}$$

Calculate the products AB and BA.

SOLUTION Since both matrices A and B have dimension 2×2, both products AB and BA are defined, and each product is also a 2×2 matrix.

$$\begin{aligned} AB &= \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 + 7 \cdot 9 & 5 \cdot 2 + 7 \cdot (-1) \\ (-3) \cdot 1 + 0 \cdot 9 & (-3) \cdot 2 + 0 \cdot (-1) \end{bmatrix} \\ &= \begin{bmatrix} 68 & 3 \\ -3 & -6 \end{bmatrix} \\ BA &= \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot (-3) & 1 \cdot 7 + 2 \cdot 0 \\ 9 \cdot 5 + (-1) \cdot (-3) & 9 \cdot 7 + (-1) \cdot 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 7 \\ 48 & 63 \end{bmatrix} \end{aligned}$$

This shows that, in general, $AB \neq BA$. In fact, in this example AB and BA don't even have an entry in common.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **29**

Applications of Matrix Multiplication

We now consider some applied examples that give some indication of why mathematicians chose to define the matrix product in such an apparently bizarre fashion. Example 6 shows how our definition of matrix product allows us to express a system of linear equations as a single matrix equation.

EXAMPLE 6 | Writing a Linear System as a Matrix Equation

Show that the following matrix equation is equivalent to the system of equations in Example 2 of Section 6.1.

Matrix equations like this one are studied in more detail on page 495.

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & -2 \\ 3 & -1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix}$$

SOLUTION If we perform matrix multiplication on the left side of the equation, we get

$$\begin{bmatrix} x - y + 3z \\ x + 2y - 2z \\ 3x - y + 5z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix}$$

Because two matrices are equal only if their corresponding entries are equal, we equate entries to get

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

This is exactly the system of equations in Example 2 of Section 6.1.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **47**

EXAMPLE 7 | Representing Demographic Data by Matrices

In a certain city the proportions of voters in each age group who are registered as Democrats, Republicans, or Independents are given by the following matrix.

$$\begin{array}{r|ccc|}
 & \multicolumn{3}{c}{\textbf{Age}} \\
 & \textbf{18–30} & \textbf{31–50} & \textbf{Over 50}
\end{array}$$

$$\begin{array}{r} \textbf{Democrat} \\ \textbf{Republican} \\ \textbf{Independent} \end{array}\begin{bmatrix} 0.30 & 0.60 & 0.50 \\ 0.50 & 0.35 & 0.25 \\ 0.20 & 0.05 & 0.25 \end{bmatrix} = A$$

The next matrix gives the distribution, by age and sex, of the voting population of this city.

$$\begin{array}{cc} & \begin{array}{cc} \textbf{Male} & \textbf{Female} \end{array} \\ \textbf{Age} \begin{array}{r} \textbf{18–30} \\ \textbf{31–50} \\ \textbf{Over 50} \end{array} & \begin{bmatrix} 5{,}000 & 6{,}000 \\ 10{,}000 & 12{,}000 \\ 12{,}000 & 15{,}000 \end{bmatrix} = B \end{array}$$

For this problem, let's make the (highly unrealistic) assumption that within each age group, political preference is not related to gender. That is, the percentage of Democrat males in the 18–30 group, for example, is the same as the percentage of Democrat females in this group.

(a) Calculate the product AB.

(b) How many males are registered as Democrats in this city?

(c) How many females are registered as Republicans?

SOLUTION

(a) $$AB = \begin{bmatrix} 0.30 & 0.60 & 0.50 \\ 0.50 & 0.35 & 0.25 \\ 0.20 & 0.05 & 0.25 \end{bmatrix}\begin{bmatrix} 5{,}000 & 6{,}000 \\ 10{,}000 & 12{,}000 \\ 12{,}000 & 15{,}000 \end{bmatrix} = \begin{bmatrix} 13{,}500 & 16{,}500 \\ 9{,}000 & 10{,}950 \\ 4{,}500 & 5{,}550 \end{bmatrix}$$

Courtesy of the Archives, California Institute of Technology

OLGA TAUSSKY-TODD (1906–1995) was instrumental in developing applications of matrix theory. Described as "in love with anything matrices can do," she successfully applied matrices to aerodynamics, a field used in the design of airplanes and rockets. Taussky-Todd was also famous for her work in number theory, which deals with prime numbers and divisibility. Although number theory has often been called the least applicable branch of mathematics, it is now used in significant ways throughout the computer industry.

Taussky-Todd studied mathematics at a time when young women rarely aspired to be mathematicians. She said, "When I entered university I had no idea what it meant to study mathematics." One of the most respected mathematicians of her day, she was for many years a professor of mathematics at Caltech in Pasadena.

(b) When we take the inner product of a row in A with a column in B, we are adding the number of people in each age group who belong to the category in question. For example, the entry c_{21} of AB (the 9000) is obtained by taking the inner product of the Republican row in A with the Male column in B. This number is therefore the total number of male Republicans in this city. We can label the rows and columns of AB as follows:

$$\begin{array}{l} \\ \textbf{Democrat} \\ \textbf{Republican} \\ \textbf{Independent} \end{array} \begin{array}{c} \begin{array}{cc} \textbf{Male} & \textbf{Female} \end{array} \\ \begin{bmatrix} 13{,}500 & 16{,}500 \\ 9{,}000 & 10{,}950 \\ 4{,}500 & 5{,}550 \end{bmatrix} \end{array} = AB$$

Thus 13,500 males are registered as Democrats in this city.

(c) There are 10,950 females registered as Republicans.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **53**

In Example 7 the entries in each column of A add up to 1. (Can you see why this has to be true, given what the matrix describes?) A matrix with this property is called **stochastic**. Stochastic matrices are used extensively in statistics, where they arise frequently in situations like the one described here.

Computer Graphics

One important use of matrices is in the digital representation of images. A digital camera or a scanner converts an image into a matrix by dividing the image into a rectangular array of elements called pixels. Each pixel is assigned a value that represents the color, brightness, or some other feature of that location. For example, in a 256-level gray-scale image each pixel is assigned a value between 0 and 255, where 0 represents white, 255 represents black, and the numbers in between represent increasing gradations of gray. The gradations of a much simpler 8-level gray scale are shown in Figure 2. We use this eight-level gray scale to illustrate the process.

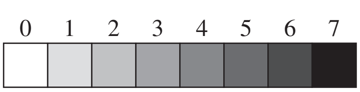


FIGURE 2

To digitize the black and white image in Figure 3(a), we place a grid over the picture as shown in Figure 3(b). Each cell in the grid is compared to the gray scale and then assigned a value between 0 and 7 depending on which gray square in the scale most closely matches the "darkness" of the cell. (If the cell is not uniformly gray, an average value is assigned.) The values are stored in the matrix shown in Figure 3(c). The digital image corresponding to this matrix is shown in Figure 3(d). Obviously, the grid that we have used is far too coarse to provide good image resolution. In practice, currently available high-resolution digital cameras use matrices with dimension 4096×4096 or larger.

Once the image is stored as a matrix, it can be manipulated by using matrix operations. For example, to darken the image, we add a constant to each entry in the matrix; to lighten the image, we subtract a constant. To increase the contrast, we darken the darker areas and

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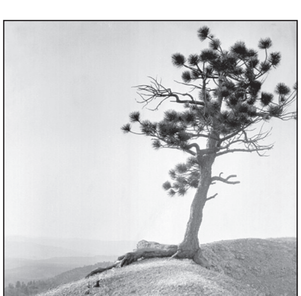
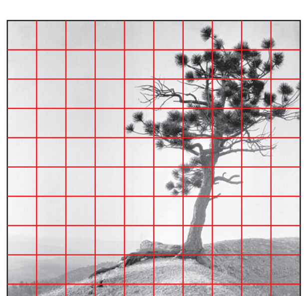


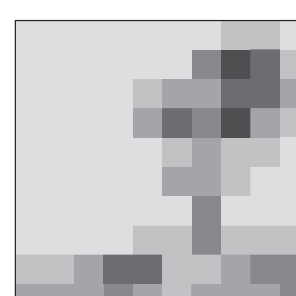
(a) Original image



(b) 10×10 grid

$$\begin{bmatrix} 1&1&1&1&1&1&1&2&2&1 \\ 1&1&1&1&1&1&4&6&5&2 \\ 1&1&1&1&2&3&3&5&5&3 \\ 1&1&1&1&3&5&4&6&3&2 \\ 1&1&1&1&1&2&3&2&2&1 \\ 1&1&1&1&1&3&3&2&1&1 \\ 1&1&1&1&1&1&4&1&1&1 \\ 1&1&1&1&2&2&4&2&2&2 \\ 2&2&3&5&5&2&2&3&4&4 \\ 3&3&3&4&3&2&3&3&3&4 \end{bmatrix}$$

(c) Matrix representation



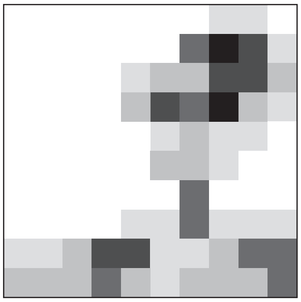
(d) Digital image

FIGURE 3

lighten the lighter areas, so we could add 1 to each entry that is 4, 5, or 6, and subtract 1 from each entry that is 1, 2, or 3. (Note that we cannot darken an entry of 7 or lighten a 0.) Applying this process to the matrix in Figure 3(c) produces the new matrix in Figure 4(a). This generates the high-contrast image shown in Figure 4(b).

$$\begin{bmatrix} 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&5&7&6&1\\ 0&0&0&0&1&2&2&6&6&2\\ 0&0&0&0&2&6&5&7&2&1\\ 0&0&0&0&0&1&2&1&1&0\\ 0&0&0&0&0&2&2&1&0&0\\ 0&0&0&0&0&0&5&0&0&0\\ 0&0&0&0&1&1&5&1&1&1\\ 1&1&2&6&6&1&1&2&5&5\\ 2&2&2&5&2&1&2&2&2&5 \end{bmatrix}$$

(a) Matrix modified to increase contrast



(b) High contrast image

FIGURE 4

Other ways of representing and manipulating images using matrices are discussed in *Focus on Modeling* on pages 518–521.

6.2 EXERCISES

CONCEPTS

1. We can add (or subtract) two matrices only if they have the same ________.

2. (a) We can multiply two matrices only if the number of ________ in the first matrix is the same as the number of ________ in the second matrix.

(b) If A is a 3×3 matrix and B is a 4×3 matrix, which of the following matrix multiplications are possible?

(i) AB (ii) BA (iii) AA (iv) BB

3. Which of the following operations can we perform for a matrix A of any dimension?

(i) $A + A$ (ii) $2A$ (iii) $A \cdot A$

4. Fill in the missing entries in the product matrix.

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 2 & 0 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 3 & -2 \\ 3 & -2 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & \blacksquare & -7 \\ 7 & -7 & \blacksquare \\ \blacksquare & -5 & -5 \end{bmatrix}$$

SKILLS

5–6 ■ Determine whether the matrices A and B are equal.

5. $A = \begin{bmatrix} 1 & -2 & 0 \\ \frac{1}{2} & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 \\ \frac{1}{2} & 6 \end{bmatrix}$

6. $A = \begin{bmatrix} \frac{1}{4} & \ln 1 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0.25 & 0 \\ \sqrt{4} & \frac{6}{2} \end{bmatrix}$

7–8 ■ Find the values of a and b that make the matrices A and B equal.

7. $A = \begin{bmatrix} 3 & 4 \\ -1 & a \end{bmatrix} \qquad B = \begin{bmatrix} b & 4 \\ -1 & -5 \end{bmatrix}$

8. $A = \begin{bmatrix} 3 & 5 & 7 \\ -4 & a & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 5 & b \\ -4 & -5 & 2 \end{bmatrix}$

9–16 ■ Perform the matrix operation, or if it is impossible, explain why.

9. $\begin{bmatrix} 2 & 6 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 6 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

11. $3\begin{bmatrix} 1 & 2 \\ 4 & -1 \\ 1 & 0 \end{bmatrix}$

12. $2\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 2 & 1 & 2 \\ 6 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 6 \\ -2 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 2 & -1 \end{bmatrix}$

16. $\begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

17–22 ■ Solve the matrix equation for the unknown matrix X, or explain why no solution exists.

$$A = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 10 & 20 \\ 30 & 20 \\ 10 & 0 \end{bmatrix}$$

17. $2X + A = B$

18. $3X - B = C$

19. $2(B - X) = D$ **20.** $5(X - C) = D$

21. $\frac{1}{5}(X + D) = C$ **22.** $2A = B - 3X$

23–36 ■ The matrices A, B, C, D, E, F, G and H are defined as follows.

$$A = \begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 3 & \frac{1}{2} & 5 \\ 1 & -1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -\frac{5}{2} & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 5 & -3 & 10 \\ 6 & 1 & 0 \\ -5 & 2 & 2 \end{bmatrix} \quad H = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$$

Carry out the indicated algebraic operation, or explain why it cannot be performed.

23. **(a)** $B + C$ **(b)** $B + F$

24. **(a)** $C - B$ **(b)** $2C - 6B$

25. **(a)** $5A$ **(b)** $C - 5A$

26. **(a)** $3B + 2C$ **(b)** $2H + D$

27. **(a)** AD **(b)** DA

28. **(a)** DH **(b)** HD

29. **(a)** AH **(b)** HA

30. **(a)** BC **(b)** BF

31. **(a)** GF **(b)** GE

32. **(a)** B^2 **(b)** F^2

33. **(a)** A^2 **(b)** A^3

34. **(a)** $(DA)B$ **(b)** $D(AB)$

35. **(a)** ABE **(b)** AHE

36. **(a)** $DB + DC$ **(b)** $BF + FE$

37–42 ■ The matrices A, B, and C are defined as follows.

$$A = \begin{bmatrix} 0.3 & 1.1 & 2.4 \\ 0.9 & -0.1 & 0.4 \\ -0.7 & 0.3 & -0.5 \end{bmatrix} \quad B = \begin{bmatrix} 1.2 & -0.1 \\ 0 & -0.5 \\ 0.5 & -2.1 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.2 & 0.2 & 0.1 \\ 1.1 & 2.1 & -2.1 \end{bmatrix}$$

Use a graphing calculator to carry out the indicated algebraic operation, or explain why it cannot be performed.

37. AB **38.** BA **39.** BC **40.** CB **41.** $B + C$ **42.** A^2

43–46 ■ Solve for x and y.

43. $\begin{bmatrix} x & 2y \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2x & -6y \end{bmatrix}$ **44.** $3\begin{bmatrix} x & y \\ y & x \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ -9 & 6 \end{bmatrix}$

45. $2\begin{bmatrix} x & y \\ x + y & x - y \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -2 & 6 \end{bmatrix}$

46. $\begin{bmatrix} x & y \\ -y & x \end{bmatrix} - \begin{bmatrix} y & x \\ x & -y \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -6 & 6 \end{bmatrix}$

47–50 ■ Write the system of equations as a matrix equation (see Example 6).

47. $\begin{cases} 2x - 5y = 7 \\ 3x + 2y = 4 \end{cases}$ **48.** $\begin{cases} 6x - y + z = 12 \\ 2x + z = 7 \\ y - 2z = 4 \end{cases}$

49. $\begin{cases} 3x_1 + 2x_2 - x_3 + x_4 = 0 \\ x_1 - x_3 = 5 \\ 3x_2 + x_3 - x_4 = 4 \end{cases}$

50. $\begin{cases} x - y + z = 2 \\ 4x - 2y - z = 2 \\ x + y + 5z = 2 \\ -x - y - z = 2 \end{cases}$

51. The matrices A, B, and C are defined as follows

$$A = \begin{bmatrix} 1 & 0 & 6 & -1 \\ 2 & \frac{1}{2} & 4 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 7 & -9 & 2 \end{bmatrix}$$

Determine which of the following products are defined, and calculate the ones that are.

ABC ACB BAC
BCA CAB CBA

52. **(a)** Prove that if A and B are 2×2 matrices, then

$$(A + B)^2 = A^2 + AB + BA + B^2$$

(b) If A and B are 2×2 matrices, is it necessarily true that

$$(A + B)^2 \stackrel{?}{=} A^2 + 2AB + B^2$$

APPLICATIONS

53. **Fast-Food Sales** A small fast-food chain with restaurants in Santa Monica, Long Beach, and Anaheim sells only hamburgers, hot dogs, and milk shakes. On a certain day, sales were distributed according to the following matrix.

$$\begin{array}{r} \\ \\ \textbf{Hamburgers} \\ \textbf{Hot dogs} \\ \textbf{Milk shakes} \end{array} \begin{array}{c} \textbf{Number of items sold} \\ \begin{array}{ccc} \textbf{Santa Monica} & \textbf{Long Beach} & \textbf{Anaheim} \end{array} \\ \begin{bmatrix} 4000 & 1000 & 3500 \\ 400 & 300 & 200 \\ 700 & 500 & 9000 \end{bmatrix} \end{array} = A$$

The price of each item is given by the following matrix.

$$\begin{array}{c} \begin{array}{ccc} \textbf{Hamburger} & \textbf{Hot dog} & \textbf{Milk shake} \end{array} \\ \begin{bmatrix} \$0.90 & \$0.80 & \$1.10 \end{bmatrix} \end{array} = B$$

(a) Calculate the product BA.
(b) Interpret the entries in the product matrix BA.

54. **Car-Manufacturing Profits** A specialty-car manufacturer has plants in Auburn, Biloxi, and Chattanooga. Three models are produced, with daily production given in the following matrix.

$$\begin{array}{r} \\ \\ \textbf{Auburn} \\ \textbf{Biloxi} \\ \textbf{Chattanooga} \end{array} \begin{array}{c} \textbf{Cars produced each day} \\ \begin{array}{ccc} \textbf{Model K} & \textbf{Model R} & \textbf{Model W} \end{array} \\ \begin{bmatrix} 12 & 10 & 0 \\ 4 & 4 & 20 \\ 8 & 9 & 12 \end{bmatrix} \end{array} = A$$

Because of a wage increase, February profits are lower than January profits. The profit per car is tabulated by model in the following matrix.

$$\begin{array}{r} \\ \textbf{Model K} \\ \textbf{Model R} \\ \textbf{Model W} \end{array} \begin{array}{c} \begin{array}{cc} \textbf{January} & \textbf{February} \end{array} \\ \begin{bmatrix} \$1000 & \$500 \\ \$2000 & \$1200 \\ \$1500 & \$1000 \end{bmatrix} \end{array} = B$$

(a) Calculate AB.
(b) Assuming that all cars produced were sold, what was the daily profit in January from the Biloxi plant?
(c) What was the total daily profit (from all three plants) in February?

55. Canning Tomato Products Jaeger Foods produces tomato sauce and tomato paste, canned in small, medium, large, and giant sized cans. The matrix A gives the size (in ounces) of each container.

$$\begin{array}{r} \\ \textbf{Ounces} \end{array} \begin{array}{c} \begin{array}{cccc} \textbf{Small} & \textbf{Medium} & \textbf{Large} & \textbf{Giant} \end{array} \\ \begin{bmatrix} 6 & 10 & 14 & 28 \end{bmatrix} \end{array} = A$$

The matrix B tabulates one day's production of tomato sauce and tomato paste.

$$\begin{array}{r} \\ \textbf{Small} \\ \textbf{Medium} \\ \textbf{Large} \\ \textbf{Giant} \end{array} \begin{array}{c} \begin{array}{cc} \textbf{Cans of sauce} & \textbf{Cans of paste} \end{array} \\ \begin{bmatrix} 2000 & 2500 \\ 3000 & 1500 \\ 2500 & 1000 \\ 1000 & 500 \end{bmatrix} \end{array} = B$$

(a) Calculate the product of AB.
(b) Interpret the entries in the product matrix AB.

56. Produce Sales A farmer's three children, Amy, Beth, and Chad, run three roadside produce stands during the summer months. One weekend they all sell watermelons, yellow squash, and tomatoes. The matrices A and B tabulate the number of pounds of each product sold by each sibling on Saturday and Sunday.

$$\begin{array}{r} \\ \\ \textbf{Amy} \\ \textbf{Beth} \\ \textbf{Chad} \end{array} \begin{array}{c} \textbf{Saturday} \\ \begin{array}{ccc} \textbf{Melons} & \textbf{Squash} & \textbf{Tomatoes} \end{array} \\ \begin{bmatrix} 120 & 50 & 60 \\ 40 & 25 & 30 \\ 60 & 30 & 20 \end{bmatrix} \end{array} = A$$

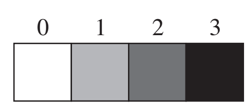
$$\begin{array}{r} \\ \\ \textbf{Amy} \\ \textbf{Beth} \\ \textbf{Chad} \end{array} \begin{array}{c} \textbf{Sunday} \\ \begin{array}{ccc} \textbf{Melons} & \textbf{Squash} & \textbf{Tomatoes} \end{array} \\ \begin{bmatrix} 100 & 60 & 30 \\ 35 & 20 & 20 \\ 60 & 25 & 30 \end{bmatrix} \end{array} = B$$

The matrix C gives the price per pound (in dollars) for each type of produce that they sell.

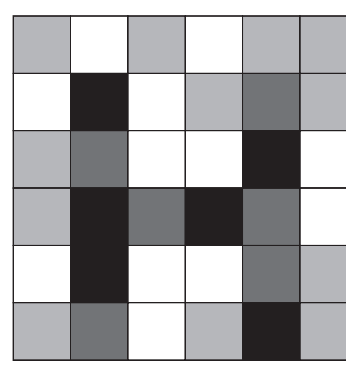
$$\begin{array}{r} \\ \textbf{Melons} \\ \textbf{Squash} \\ \textbf{Tomatoes} \end{array} \begin{array}{c} \textbf{Price per pound} \\ \begin{bmatrix} 0.10 \\ 0.50 \\ 1.00 \end{bmatrix} \end{array} = C$$

Perform each of the following matrix operations, and interpret the entries in each result.
(a) AC **(b)** BC **(c)** $A + B$ **(d)** $(A + B)C$

57. Digital Images A four-level gray scale is shown below.



(a) Use the gray scale to find a 6×6 matrix that digitally represents the image in the figure.



(b) Find a matrix that represents a darker version of the image in the figure.
(c) The **negative** of an image is obtained by reversing light and dark, as in the negative of a photograph. Find the matrix that represents the negative of the image in the figure. How do you change the elements of the matrix to create the negative?
(d) Increase the contrast of the image by changing each 1 to a 0 and each 2 to a 3 in the matrix you found in part (b). Draw the image represented by the resulting matrix. Does this clarify the image?

(e) Draw the image represented by the matrix I. Can you recognize what this is? If you don't, try increasing the contrast.

$$I = \begin{bmatrix} 1 & 2 & 3 & 3 & 2 & 0 \\ 0 & 3 & 0 & 1 & 0 & 1 \\ 1 & 3 & 2 & 3 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 1 \\ 1 & 3 & 3 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

DISCOVERY ■ DISCUSSION ■ WRITING

58. When Are Both Products Defined? What must be true about the dimensions of the matrices A and B if both products AB and BA are defined?

59. Powers of a Matrix Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Calculate $A^2, A^3, A^4, \ldots$ until you detect a pattern. Write a general formula for A^n.

60. Powers of a Matrix Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Calculate $A^2, A^3, A^4, \ldots$ until you detect a pattern. Write a general formula for A^n.

61. Square Roots of Matrices A **square root** of a matrix B is a matrix A with the property that $A^2 = B$. (This is the same definition as for a square root of a number.) Find as many square roots as you can of each matrix:

$$\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 5 \\ 0 & 9 \end{bmatrix}$$

[*Hint:* If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, write the equations that a, b, c, and d would have to satisfy if A is the square root of the given matrix.]

DISCOVERY PROJECT **Will the Species Survive?**

In this project we investigate matrix models for species populations and how multiplication by a transition matrix can predict future population trends. You can find the project at the book companion website: **www.stewartmath.com**

6.3 Inverses of Matrices and Matrix Equations

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Determine whether two matrices are inverses of each other ▶ Find the inverse of a 2×2 matrix ▶ Find the inverse of an $n \times n$ matrix ▶ Solve a matrix equation ▶ Solve a linear system by expressing it as a matrix equation ▶ Model using matrix equations

In the preceding section we saw that when the dimensions are appropriate, matrices can be added, subtracted, and multiplied. In this section we investigate division of matrices. With this operation we can solve equations that involve matrices.

▼ The Inverse of a Matrix

First, we define *identity matrices*, which play the same role for matrix multiplication as the number 1 does for ordinary multiplication of numbers; that is, $1 \cdot a = a \cdot 1 = a$ for all numbers a. A **square matrix** is one that has the same number of rows as columns. The **main diagonal** of a square matrix consists of the entries whose row and column numbers are the same. These entries stretch diagonally down the matrix, from top left to bottom right.

IDENTITY MATRIX

The **identity matrix** I_n is the $n \times n$ matrix for which each main diagonal entry is a 1 and for which all other entries are 0.

Thus the 2×2, 3×3, and 4×4 identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity matrices behave like the number 1 in the sense that

$$A \cdot I_n = A \qquad \text{and} \qquad I_n \cdot B = B$$

whenever these products are defined.

EXAMPLE 1 | Identity Matrices

The following matrix products show how multiplying a matrix by an identity matrix of the appropriate dimension leaves the matrix unchanged.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **1(a), (b)**

If A and B are $n \times n$ matrices, and if $AB = BA = I_n$, then we say that B is the *inverse* of A, and we write $B = A^{-1}$. The concept of the inverse of a matrix is analogous to that of the reciprocal of a real number.

INVERSE OF A MATRIX

Let A be a square $n \times n$ matrix. If there exists an $n \times n$ matrix A^{-1} with the property that

$$AA^{-1} = A^{-1}A = I_n$$

then we say that A^{-1} is the **inverse** of A.

EXAMPLE 2 | Verifying That a Matrix Is an Inverse

Verify that B is the inverse of A, where

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

SOLUTION We perform the matrix multiplications to show that $AB = I$ and $BA = I$:

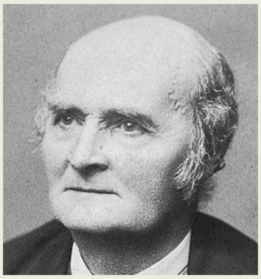
$$AB = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 1(-5) & 2(-1) + 1 \cdot 2 \\ 5 \cdot 3 + 3(-5) & 5(-1) + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + (-1)5 & 3 \cdot 1 + (-1)3 \\ (-5)2 + 2 \cdot 5 & (-5)1 + 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **3**

Finding the Inverse of a 2 × 2 Matrix

The following rule provides a simple way for finding the inverse of a 2×2 matrix, when it exists. For larger matrices there is a more general procedure for finding inverses, which we consider later in this section.



The Granger Collection, New York

ARTHUR CAYLEY (1821–1895) was an English mathematician who was instrumental in developing the theory of matrices. He was the first to use a single symbol such as A to represent a matrix, thereby introducing the idea that a matrix is a single entity rather than just a collection of numbers. Cayley practiced law until the age of 42, but his primary interest from adolescence was mathematics, and he published almost 200 articles on the subject in his spare time. In 1863 he accepted a professorship in mathematics at Cambridge, where he taught until his death. Cayley's work on matrices was of purely theoretical interest in his day, but in the 20th century many of his results found application in physics, the social sciences, business, and other fields. One of the most common uses of matrices today is in computers, where matrices are employed for data storage, error correction, image manipulation, and many other purposes. These applications have made matrix algebra more useful than ever.

INVERSE OF A 2 × 2 MATRIX

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A has no inverse.

EXAMPLE 3 | Finding the Inverse of a 2 × 2 Matrix

Let

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

Find A^{-1}, and verify that $AA^{-1} = A^{-1}A = I_2$.

SOLUTION Using the rule for the inverse of a 2×2 matrix, we get

$$A^{-1} = \frac{1}{4 \cdot 3 - 5 \cdot 2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}$$

To verify that this is indeed the inverse of A, we calculate AA^{-1} and $A^{-1}A$:

$$AA^{-1} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot \frac{3}{2} + 5(-1) & 4(-\frac{5}{2}) + 5 \cdot 2 \\ 2 \cdot \frac{3}{2} + 3(-1) & 2(-\frac{5}{2}) + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \cdot 4 + (-\frac{5}{2})2 & \frac{3}{2} \cdot 5 + (-\frac{5}{2})3 \\ (-1)4 + 2 \cdot 2 & (-1)5 + 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **7**

The quantity $ad - bc$ that appears in the rule for calculating the inverse of a 2×2 matrix is called the **determinant** of the matrix. If the determinant is 0, then the matrix does not have an inverse (since we cannot divide by 0).

Finding the Inverse of an $n \times n$ Matrix

For 3×3 and larger square matrices the following technique provides the most efficient way to calculate their inverses. If A is an $n \times n$ matrix, we first construct the $n \times 2n$ matrix that has the entries of A on the left and of the identity matrix I_n on the right.

$$\left[\begin{array}{cccc:cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

We then use the elementary row operations on this new large matrix to change the left side into the identity matrix. (This means that we are changing the large matrix to reduced row-echelon form.) The right side is transformed automatically into A^{-1}. (We omit the proof of this fact.)

EXAMPLE 4 | Finding the Inverse of a 3 × 3 Matrix

Let A be the matrix

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$$

(a) Find A^{-1}.

(b) Verify that $AA^{-1} = A^{-1}A = I_3$.

SOLUTION

(a) We begin with the 3×6 matrix whose left half is A and whose right half is the identity matrix.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right]$$

We then transform the left half of this new matrix into the identity matrix by performing the following sequence of elementary row operations on the *entire* new matrix.

$$\xrightarrow[\mathrm{R}_3 + 3\mathrm{R}_1 \to \mathrm{R}_3]{\mathrm{R}_2 - 2\mathrm{R}_1 \to \mathrm{R}_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}\mathrm{R}_3} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{\mathrm{R}_1 + 2\mathrm{R}_2 \to \mathrm{R}_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{\mathrm{R}_2 - 2\mathrm{R}_3 \to \mathrm{R}_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -4 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

We have now transformed the left half of this matrix into an identity matrix. (This means that we have put the entire matrix in reduced row-echelon form.) Note that to do this in as systematic a fashion as possible, we first changed the elements below the main diagonal to zeros, just as we would if we were using Gaussian elimination. We then changed each main diagonal element to a 1 by multiplying by the appropriate constant(s). Finally, we completed the process by changing the remaining entries on the left side to zeros.

The right half is now A^{-1}:

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

(b) We calculate AA^{-1} and $A^{-1}A$ and verify that both products give the identity matrix I_3:

$$AA^{-1} = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **9** AND **19**

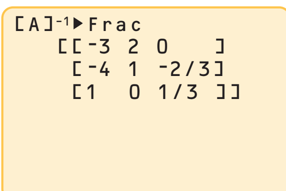


FIGURE 1

See Appendix C, *Using the TI-83/84 Graphing Calculator*, for specific instructions on working with matrices.

Graphing calculators are also able to calculate matrix inverses. On the TI-83 and TI-84 calculators, matrices are stored in memory using names such as `[A]`, `[B]`, `[C]`, ... To find the inverse of `[A]`, we key in

`[A]` `x⁻¹` `ENTER`

For the matrix of Example 4 this results in the output shown in Figure 1 (where we have also used the `▶Frac` command to display the output in fraction form rather than in decimal form).

The next example shows that not every square matrix has an inverse.

EXAMPLE 5 | A Matrix That Does Not Have an Inverse

Find the inverse of the matrix.

$$\begin{bmatrix} 2 & -3 & -7 \\ 1 & 2 & 7 \\ 1 & 1 & 4 \end{bmatrix}$$

SOLUTION We proceed as follows.

$$\left[\begin{array}{ccc|ccc} 2 & -3 & -7 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -3 & -7 & 1 & 0 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 0 & -7 & -21 & 1 & -2 & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{7}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 - 2R_2 \to R_1]{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & -\frac{5}{7} & 1 \end{array}\right]$$

At this point we would like to change the 0 in the (3, 3) position of this matrix to a 1 without changing the zeros in the (3, 1) and (3, 2) positions. But there is no way to accomplish this, because no matter what multiple of rows 1 and/or 2 we add to row 3, we can't change the third zero in row 3 without changing the first or second zero as well. Thus we cannot change the left half to the identity matrix, so the original matrix doesn't have an inverse.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **21**

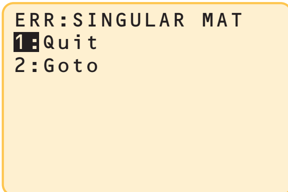


FIGURE 2

If we encounter a row of zeros on the left when trying to find an inverse, as in Example 5, then the original matrix does not have an inverse. If we try to calculate the inverse of the matrix from Example 5 on a TI-83 calculator, we get the error message shown in Figure 2. (A matrix that has no inverse is called *singular*.)

Matrix Equations

We saw in Example 6 in Section 6.2 that a system of linear equations can be written as a single matrix equation. For example, the system

$$\begin{cases} x - 2y - 4z = 7 \\ 2x - 3y - 6z = 5 \\ -3x + 6y + 15z = 0 \end{cases}$$

is equivalent to the matrix equation

$$\underbrace{\begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}}_{B}$$

If we let

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$$

then this matrix equation can be written as

$$AX = B$$

The matrix A is called the **coefficient matrix**.

We solve this matrix equation by multiplying each side by the inverse of A (provided that this inverse exists).

Solving the matrix equation $AX = B$ is very similar to solving the simple real-number equation

$$3x = 12$$

which we do by multiplying each side by the reciprocal (or inverse) of 3.

$$\tfrac{1}{3}(3x) = \tfrac{1}{3}(12)$$

$$x = 4$$

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B && \text{Multiply on left by } A^{-1} \\ (A^{-1}A)X &= A^{-1}B && \text{Associative Property} \\ I_3X &= A^{-1}B && \text{Property of inverses} \\ X &= A^{-1}B && \text{Property of identity matrix} \end{aligned}$$

In Example 4 we showed that

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

So from $X = A^{-1}B$ we have

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}}_{B} = \begin{bmatrix} -11 \\ -23 \\ 7 \end{bmatrix}$$

Thus $x = -11$, $y = -23$, $z = 7$ is the solution of the original system.

We have proved that the matrix equation $AX = B$ can be solved by the following method.

SOLVING A MATRIX EQUATION

If A is a square $n \times n$ matrix that has an inverse A^{-1} and if X is a variable matrix and B a known matrix, both with n rows, then the solution of the matrix equation

$$AX = B$$

is given by

$$X = A^{-1}B$$

EXAMPLE 6 | Solving a System Using a Matrix Inverse

A system of equations is given.

(a) Write the system of equations as a matrix equation.

(b) Solve the system by solving the matrix equation.

$$\begin{cases} 2x - 5y = 15 \\ 3x - 6y = 36 \end{cases}$$

SOLUTION

(a) We write the system as a matrix equation of the form $AX = B$:

$$\underbrace{\begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 15 \\ 36 \end{bmatrix}}_{B}$$

(b) Using the rule for finding the inverse of a 2×2 matrix, we get

$$A^{-1} = \begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix}^{-1} = \frac{1}{2(-6) - (-5)3}\begin{bmatrix} -6 & -(-5) \\ -3 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & 5 \\ -3 & 2 \end{bmatrix}$$

Multiplying each side of the matrix equation by this inverse matrix, we get

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\frac{1}{3}\begin{bmatrix} -6 & 5 \\ -3 & 2 \end{bmatrix}}_{A^{-1}}\underbrace{\begin{bmatrix} 15 \\ 36 \end{bmatrix}}_{B} = \begin{bmatrix} 30 \\ 9 \end{bmatrix}$$

So $x = 30$ and $y = 9$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **39** ■

▼ Modeling with Matrix Equations

Suppose we need to solve several systems of equations with the same coefficient matrix. Then converting the systems to matrix equations provides an efficient way to obtain the solutions, because we need to find the inverse of the coefficient matrix only once. This procedure is particularly convenient if we use a graphing calculator to perform the matrix operations, as in the next example.

EXAMPLE 7 | Modeling Nutritional Requirements Using Matrix Equations

A pet-store owner feeds his hamsters and gerbils different mixtures of three types of rodent food: KayDee Food, Pet Pellets, and Rodent Chow. He wishes to feed his animals the correct amount of each brand to satisfy their daily requirements for protein, fat, and carbohydrates exactly. Suppose that hamsters require 340 mg of protein, 280 mg of fat, and 440 mg of carbohydrates, and gerbils need 480 mg of protein, 360 mg of fat, and 680 mg of carbohydrates each day. The amount of each nutrient (in mg) in one gram of each brand is given in the following table. How many grams of each food should the storekeeper feed his hamsters and gerbils daily to satisfy their nutrient requirements?

	KayDee Food	Pet Pellets	Rodent Chow
Protein (mg)	10	0	20
Fat (mg)	10	20	10
Carbohydrates (mg)	5	10	30

SOLUTION We let x_1, x_2, and x_3 be the respective amounts (in grams) of KayDee Food, Pet Pellets, and Rodent Chow that the hamsters should eat and y_1, y_2, and y_3 be the corresponding amounts for the gerbils. Then we want to solve the matrix equations

$$\begin{bmatrix} 10 & 0 & 20 \\ 10 & 20 & 10 \\ 5 & 10 & 30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 340 \\ 280 \\ 440 \end{bmatrix} \qquad \text{Hamster equation}$$

$$\begin{bmatrix} 10 & 0 & 20 \\ 10 & 20 & 10 \\ 5 & 10 & 30 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 480 \\ 360 \\ 680 \end{bmatrix} \qquad \text{Gerbil equation}$$

MATHEMATICS IN THE MODERN WORLD

Art Wolfe/Stone/Getty Images

Mathematical Ecology

In the 1970s humpback whales became a center of controversy. Environmentalists believed that whaling threatened the whales with imminent extinction; whalers saw their livelihood threatened by any attempt to stop whaling. Are whales really threatened to extinction by whaling? What level of whaling is safe to guarantee survival of the whales? These questions motivated mathematicians to study population patterns of whales and other species more closely.

As early as the 1920s Lotka and Volterra had founded the field of mathematical biology by creating predator-prey models. Their models, which draw on a branch of mathematics called differential equations, take into account the rates at which predator eats prey and the rates of growth of each population. Note that as predator eats prey, the prey population decreases; this means less food supply for the predators, so their population begins to decrease; with fewer predators the prey population begins to increase, and so on. Normally, a state of equilibrium develops, and the two populations alternate between a minimum and a maximum. Notice that if the predators eat the prey too fast, they will be left without food and will thus ensure their own extinction.

Since Lotka and Volterra's time, more detailed mathematical models of animal populations have been developed. For many species the population is divided into several stages: immature, juvenile, adult, and so on. The proportion of each stage that survives or reproduces in a given time period is entered into a matrix (called a transition matrix); matrix multiplication is then used to predict the population in succeeding time periods. (See the Discovery Project *Will the Species Survive?* at the book companion website: www.stewartmath.com.)

As you can see, the power of mathematics to model and predict is an invaluable tool in the ongoing debate over the environment.

Let

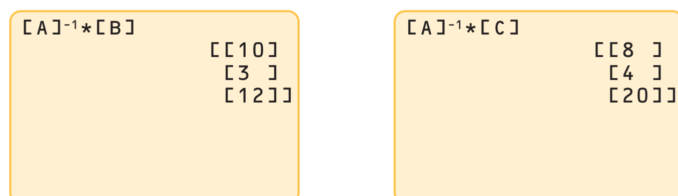
$$A = \begin{bmatrix} 10 & 0 & 20 \\ 10 & 20 & 10 \\ 5 & 10 & 30 \end{bmatrix} \qquad B = \begin{bmatrix} 340 \\ 280 \\ 440 \end{bmatrix} \qquad C = \begin{bmatrix} 480 \\ 360 \\ 680 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Then we can write these matrix equations as

$$AX = B \qquad \text{Hamster equation}$$

$$AY = C \qquad \text{Gerbil equation}$$

We want to solve for X and Y, so we multiply both sides of each equation by A^{-1}, the inverse of the coefficient matrix. We could find A^{-1} by hand, but it is more convenient to use a graphing calculator as shown in Figure 3.



(a) (b)

FIGURE 3

So

$$X = A^{-1}B = \begin{bmatrix} 10 \\ 3 \\ 12 \end{bmatrix} \qquad Y = A^{-1}C = \begin{bmatrix} 8 \\ 4 \\ 20 \end{bmatrix}$$

Thus each hamster should be fed 10 g of KayDee Food, 3 g of Pet Pellets, and 12 g of Rodent Chow; and each gerbil should be fed 8 g of KayDee Food, 4 g of Pet Pellets, and 20 g of Rodent Chow daily.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **61**

6.3 EXERCISES

CONCEPTS

1. **(a)** The matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called an ________ matrix.

(b) If A is a 2×2 matrix, then $A \times I =$ ________ and $I \times A =$ ________.

(c) If A and B are 2×2 matrices with $AB = I$, then B is the ________ of A.

2. **(a)** Write the following system as a matrix equation $AX = B$.

System	Matrix equation
	$A \cdot X = B$
$5x + 3y = 4$ $3x + 2y = 3$	$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}\begin{bmatrix} \square \\ \square \end{bmatrix} = \begin{bmatrix} \square \\ \square \end{bmatrix}$

(b) The inverse of A is $A^{-1} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$.

(c) The solution of the matrix equation is $X = A^{-1}B$.

$$\begin{array}{ccccc} X & = & A^{-1} & B & \\ \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} & \begin{bmatrix} \square \\ \square \end{bmatrix} & = \begin{bmatrix} \square \\ \square \end{bmatrix} \end{array}$$

(d) The solution of the system is $x =$ ________, $y =$ ________.

SKILLS

3–6 ■ Calculate the products AB and BA to verify that B is the inverse of A.

3. $A = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix} \quad B = \begin{bmatrix} \frac{7}{2} & -\frac{3}{2} \\ 2 & -1 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \\ -1 & -3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 8 & -3 & 4 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 1 & -6 \\ 2 & 1 & 12 \end{bmatrix} \quad B = \begin{bmatrix} 9 & -10 & -8 \\ -12 & 14 & 11 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

7–8 ■ Find the inverse of the matrix and verify that $A^{-1}A = AA^{-1} = I_2$ and $B^{-1}B = BB^{-1} = I_3$.

7. $A = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$

8. $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \\ -2 & -1 & 0 \end{bmatrix}$

9–10 ■ Use a graphing calculator to find the inverse of the matrix and to verify that $A^{-1}A = AA^{-1} = I_2$ and $B^{-1}B = BB^{-1} = I_3$. (On a TI-83, use the ▶`Frac` command to obtain the answer in fractions.)

9. $A = \begin{bmatrix} 1.2 & 0.3 \\ -1.2 & 0.2 \end{bmatrix}$

10. $B = \begin{bmatrix} 5 & -1 & 3 \\ 6 & -1 & 3 \\ 7 & 1 & -2 \end{bmatrix}$

11–26 ■ Find the inverse of the matrix if it exists.

11. $\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 5 \\ -5 & -13 \end{bmatrix}$

14. $\begin{bmatrix} -7 & 4 \\ 8 & -5 \end{bmatrix}$

15. $\begin{bmatrix} 6 & -3 \\ -8 & 4 \end{bmatrix}$

16. $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 5 & 4 \end{bmatrix}$

17. $\begin{bmatrix} 0.4 & -1.2 \\ 0.3 & 0.6 \end{bmatrix}$

18. $\begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 1 & 4 & 0 \end{bmatrix}$

20. $\begin{bmatrix} 5 & 7 & 4 \\ 3 & -1 & 3 \\ 6 & 7 & 5 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 1 & -1 & -10 \end{bmatrix}$

22. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

23. $\begin{bmatrix} 0 & -2 & 2 \\ 3 & 1 & 3 \\ 1 & -2 & 3 \end{bmatrix}$

24. $\begin{bmatrix} 3 & -2 & 0 \\ 5 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

27–34 ■ Use a graphing calculator to find the inverse of the matrix, if it exists. (On a TI-83, use the ▶`Frac` command to obtain the answer in fractions.)

27. $\begin{bmatrix} -3 & 2 & 3 \\ 0 & -1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$

28. $\begin{bmatrix} -5 & 2 & 1 \\ 5 & 1 & 0 \\ 0 & -1 & -2 \end{bmatrix}$

29. $\begin{bmatrix} -1 & -4 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 4 & 1 & -2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

30. $\begin{bmatrix} -3 & 0 & -1 & 1 \\ 3 & -1 & 1 & -1 \\ 1 & 3 & 0 & 1 \\ -2 & -3 & 1 & 0 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 7 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 4 & 2 & 3 & 0 \\ 5 & 1 & 2 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

34. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

35–38 ■ The matrices A and B are defined as follows.

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -2 & -1 \\ 4 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & -2 \\ 0 & 3 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

Use a graphing calculator to carry out the indicated algebraic operations, or explain why they cannot be performed. State the answer using fractions. (On a TI-83, use the ▶`Frac` command to obtain the answer in fractions.)

35. $A^{-1}B$ **36.** AB^{-1} **37.** BAB^{-1} **38.** $B^{-1}AB$

39–46 ■ Solve the system of equations by converting to a matrix equation and using the inverse of the coefficient matrix, as in Example 6. Use the inverses from Exercises 11–14, 19, 20, 23, and 25.

39. $\begin{cases} -3x - 5y = 4 \\ 2x + 3y = 0 \end{cases}$

40. $\begin{cases} 3x + 4y = 10 \\ 7x + 9y = 20 \end{cases}$

41. $\begin{cases} 2x + 5y = 2 \\ -5x - 13y = 20 \end{cases}$

42. $\begin{cases} -7x + 4y = 0 \\ 8x - 5y = 100 \end{cases}$

43. $\begin{cases} 2x + 4y + z = 7 \\ -x + y - z = 0 \\ x + 4y = -2 \end{cases}$

44. $\begin{cases} 5x + 7y + 4z = 1 \\ 3x - y + 3z = 1 \\ 6x + 7y + 5z = 1 \end{cases}$

45. $\begin{cases} -2y + 2z = 12 \\ 3x + y + 3z = -2 \\ x - 2y + 3z = 8 \end{cases}$

46. $\begin{cases} x + 2y + 3w = 0 \\ y + z + w = 1 \\ y + w = 2 \\ x + 2y + 2w = 3 \end{cases}$

47–52 ■ Solve the system of equations by converting to a matrix equation. Use a graphing calculator to perform the necessary matrix operations, as in Example 7.

47. $\begin{cases} x + y - 2z = 3 \\ 2x + 5z = 11 \\ 2x + 3y = 12 \end{cases}$

48. $\begin{cases} 3x + 4y - z = 2 \\ 2x - 3y + z = -5 \\ 5x - 2y + 2z = -3 \end{cases}$

49. $\begin{cases} 12x + \frac{1}{2}y - 7z = 21 \\ 11x - 2y + 3z = 43 \\ 13x + y - 4z = 29 \end{cases}$

50. $\begin{cases} x + \frac{1}{2}y - \frac{1}{3}z = 4 \\ x - \frac{1}{4}y + \frac{1}{6}z = 7 \\ x + y - z = -6 \end{cases}$

51. $\begin{cases} x + y - 3w = 0 \\ x - 2z = 8 \\ 2y - z + w = 5 \\ 2x + 3y - 2w = 13 \end{cases}$

52. $\begin{cases} x + y + z + w = 15 \\ x - y + z - w = 5 \\ x + 2y + 3z + 4w = 26 \\ x - 2y + 3z - 4w = 2 \end{cases}$

53–54 ■ Solve the matrix equation by multiplying each side by the appropriate inverse matrix.

53. $\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

54. $\begin{bmatrix} 0 & -2 & 2 \\ 3 & 1 & 3 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 12 \\ 0 & 0 \end{bmatrix}$

55–56 ■ Find the inverse of the matrix.

55. $\begin{bmatrix} a & -a \\ a & a \end{bmatrix}$ $(a \neq 0)$

56. $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$ $(abcd \neq 0)$

57–60 ■ Find the inverse of the matrix. For what value(s) of x, if any, does the matrix have no inverse?

57. $\begin{bmatrix} 2 & x \\ x & x^2 \end{bmatrix}$

58. $\begin{bmatrix} e^x & -e^{2x} \\ e^{2x} & e^{3x} \end{bmatrix}$

59. $\begin{bmatrix} 1 & e^x & 0 \\ e^x & -e^{2x} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

60. $\begin{bmatrix} x & 1 \\ -x & \dfrac{1}{x-1} \end{bmatrix}$

APPLICATIONS

61. Nutrition A nutritionist is studying the effects of the nutrients folic acid, choline, and inositol. He has three types of food available, and each type contains the following amounts of these nutrients per ounce.

	Type A	Type B	Type C
Folic acid (mg)	3	1	3
Choline (mg)	4	2	4
Inositol (mg)	3	2	4

(a) Find the inverse of the matrix

$$\begin{bmatrix} 3 & 1 & 3 \\ 4 & 2 & 4 \\ 3 & 2 & 4 \end{bmatrix}$$

and use it to solve the remaining parts of this problem.

(b) How many ounces of each food should the nutritionist feed his laboratory rats if he wants their daily diet to contain 10 mg of folic acid, 14 mg of choline, and 13 mg of inositol?

(c) How much of each food is needed to supply 9 mg of folic acid, 12 mg of choline, and 10 mg of inositol?

(d) Will any combination of these foods supply 2 mg of folic acid, 4 mg of choline, and 11 mg of inositol?

62. Nutrition Refer to Exercise 61. Suppose food type C has been improperly labeled, and it actually contains 4 mg of folic acid, 6 mg of choline, and 5 mg of inositol per ounce. Would it still be possible to use matrix inversion to solve parts (b), (c), and (d) of Exercise 61? Why or why not?

63. Sales Commissions A saleswoman works at a kiosk that offers three different models of cell phones: standard with 16 GB capacity, deluxe with 32 GB capacity, and super-deluxe with 64 GB capacity. For each phone that she sells, she earns a commission based on the cell phone model. One week she sells 9 standard, 11 deluxe, and 8 super-deluxe and makes \$740 in commission. The next week she sells 13 standard, 15 deluxe, and 16 super-deluxe for a \$1204 commission. The third week she sells 8 standard, 7 deluxe, and 14 super-deluxe, earning \$828 in commission.

(a) Let x, y, and z represent the commission she earns on standard, deluxe, and super-deluxe, respectively. Translate the given information into a system of equations in x, y, and z.

(b) Express the system of equations you found in part (a) as a matrix equation of the form $AX = B$.

(c) Find the inverse of the coefficient matrix A and use it to solve the matrix equation in part (b). How much commission does the saleswoman earn on each model of cell phone?

DISCOVERY ■ DISCUSSION ■ WRITING

64. No Zero-Product Property for Matrices We have used the Zero-Product Property to solve algebraic equations. Matrices do *not* have this property. Let O represent the **2 × 2 zero matrix**

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Find 2×2 matrices $A \neq O$ and $B \neq O$ such that $AB = O$. Can you find a matrix $A \neq O$ such that $A^2 = O$?

6.4 Determinants and Cramer's Rule

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find the determinant of a 2 × 2 matrix ▶ Find the determinant of an $n \times n$ matrix ▶ Use the Invertibility Criterion ▶ Use row and column transformations in finding the determinant of a matrix ▶ Use Cramer's Rule to solve a linear system ▶ Use determinants to find the area of a triangle in the coordinate plane

If a matrix is **square** (that is, if it has the same number of rows as columns), then we can assign to it a number called its *determinant*. Determinants can be used to solve systems of linear equations, as we will see later in this section. They are also useful in determining whether a matrix has an inverse.

▼ Determinant of a 2 × 2 Matrix

We denote the determinant of a square matrix A by the symbol $\det(A)$ or $|A|$. We first define $\det(A)$ for the simplest cases. If $A = [a]$ is a 1×1 matrix, then $\det(A) = a$. The following box gives the definition of a 2×2 determinant.

We will use both notations, $\det(A)$ and $|A|$, for the determinant of A. Although the symbol $|A|$ looks like the absolute value symbol, it will be clear from the context which meaning is intended.

DETERMINANT OF A 2 × 2 MATRIX

The **determinant** of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

EXAMPLE 1 | Determinant of a 2 × 2 Matrix

Evaluate $|A|$ for $A = \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$.

SOLUTION

To evaluate a 2×2 determinant, we take the product of the diagonal from top left to bottom right and subtract the product from top right to bottom left, as indicated by the arrows.

$$\begin{vmatrix} 6 & -3 \\ 2 & 3 \end{vmatrix} = 6 \cdot 3 - (-3)2 = 18 - (-6) = 24$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5** ■

▼ Determinant of an $n \times n$ Matrix

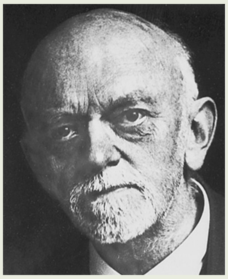
To define the concept of determinant for an arbitrary $n \times n$ matrix, we need the following terminology.

MINORS AND COFACTORS

Let A be an $n \times n$ matrix.

1. The **minor** M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A.
2. The **cofactor** A_{ij} of the element a_{ij} is

$$A_{ij} = (-1)^{i+j} M_{ij}$$



DAVID HILBERT (1862–1943) was born in Königsberg, Germany, and became a professor at Göttingen University. He is considered by many to be the greatest mathematician of the 20th century. At the International Congress of Mathematicians held in Paris in 1900, Hilbert set the direction of mathematics for the about-to-dawn 20th century by posing 23 problems that he believed to be of crucial importance. He said that "these are problems whose solutions we expect from the future." Most of Hilbert's problems have now been solved (see Julia Robinson, page 481, and Alan Turing, page 105), and their solutions have led to important new areas of mathematical research. Yet as we proceed into the new millennium, some of his problems remain unsolved. In his work, Hilbert emphasized structure, logic, and the foundations of mathematics. Part of his genius lay in his ability to see the most general possible statement of a problem. For instance, Euler proved that every whole number is the sum of four squares; Hilbert proved a similar statement for all powers of positive integers.

For example, if A is the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

then the minor M_{12} is the determinant of the matrix obtained by deleting the first row and second column from A. Thus

$$M_{12} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} = 0(6) - 4(-2) = 8$$

So the cofactor $A_{12} = (-1)^{1+2}M_{12} = -8$. Similarly,

$$M_{33} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = 2 \cdot 2 - 3 \cdot 0 = 4$$

So $A_{33} = (-1)^{3+3}M_{33} = 4$.

Note that the cofactor of a_{ij} is simply the minor of a_{ij} multiplied by either 1 or -1, depending on whether $i + j$ is even or odd. Thus in a 3×3 matrix we obtain the cofactor of any element by prefixing its minor with the sign obtained from the following checkerboard pattern.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

We are now ready to define the determinant of any square matrix.

THE DETERMINANT OF A SQUARE MATRIX

If A is an $n \times n$ matrix, then the **determinant** of A is obtained by multiplying each element of the first row by its cofactor and then adding the results. In symbols,

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

EXAMPLE 2 | Determinant of a 3 × 3 Matrix

Evaluate the determinant of the matrix.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

SOLUTION

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix}$$

$$= 2(2\cdot 6 - 4\cdot 5) - 3[0\cdot 6 - 4(-2)] - [0\cdot 5 - 2(-2)]$$

$$= -16 - 24 - 4$$

$$= -44$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **21** AND **29**

In our definition of the determinant we used the cofactors of elements in the first row only. This is called **expanding the determinant by the first row**. In fact, *we can expand the determinant by any row or column in the same way and obtain the same result in each case* (although we won't prove this). The next example illustrates this principle.

EXAMPLE 3 | Expanding a Determinant About a Row and a Column

Let A be the matrix of Example 2. Evaluate the determinant of A by expanding

(a) by the second row

(b) by the third column

Verify that each expansion gives the same value.

SOLUTION

(a) Expanding by the second row, we get

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = -0\begin{vmatrix} 3 & -1 \\ 5 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -2 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix}$$

$$= 0 + 2[2\cdot 6 - (-1)(-2)] - 4[2\cdot 5 - 3(-2)]$$

$$= 0 + 20 - 64 = -44$$

(b) Expanding by the third column gives

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix}$$

$$= -1\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix}$$

$$= -[0\cdot 5 - 2(-2)] - 4[2\cdot 5 - 3(-2)] + 6(2\cdot 2 - 3\cdot 0)$$

$$= -4 - 64 + 24 = -44$$

In both cases we obtain the same value for the determinant as when we expanded by the first row in Example 2.

We can also use a graphing calculator to compute determinants as shown in Figure 1.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **39**

See Appendix C, *Using the TI-83/84 Graphing Calculator*, for specific instructions on calculating determinants.

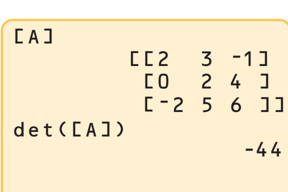


FIGURE 1 Finding the determinant on the TI-83 calculator

The following criterion allows us to determine whether a square matrix has an inverse without actually calculating the inverse. This is one of the most important uses of the determinant in matrix algebra, and it is the reason for the name *determinant*.

INVERTIBILITY CRITERION

If A is a square matrix, then A has an inverse if and only if $\det(A) \neq 0$.

We will not prove this fact, but from the formula for the inverse of a 2×2 matrix (page 492) you can see why it is true in the 2×2 case.

EXAMPLE 4 | Using the Determinant to Show That a Matrix Is Not Invertible

Show that the matrix A has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 5 & 6 & 2 & 6 \\ 2 & 4 & 0 & 9 \end{bmatrix}$$

SOLUTION We begin by calculating the determinant of A. Since all but one of the elements of the second row is zero, we expand the determinant by the second row. If we do this, we see from the following equation that only the cofactor A_{24} will have to be calculated.

$$\det(A) = \begin{vmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 5 & 6 & 2 & 6 \\ 2 & 4 & 0 & 9 \end{vmatrix}$$

$$= -0 \cdot A_{21} + 0 \cdot A_{22} - 0 \cdot A_{23} + 3 \cdot A_{24} = 3A_{24}$$

$$= 3\begin{vmatrix} 1 & 2 & 0 \\ 5 & 6 & 2 \\ 2 & 4 & 0 \end{vmatrix}$$ Expand this by column 3

$$= 3(-2)\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

$$= 3(-2)(1 \cdot 4 - 2 \cdot 2) = 0$$

Since the determinant of A is zero, A cannot have an inverse, by the Invertibility Criterion.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **25** ■

▼ Row and Column Transformations

The preceding example shows that if we expand a determinant about a row or column that contains many zeros, our work is reduced considerably because we don't have to evaluate the cofactors of the elements that are zero. The following principle often simplifies the process of finding a determinant by introducing zeros into the matrix without changing the value of the determinant.

EMMY NOETHER (1882–1935) was one of the foremost mathematicians of the early 20th century. Her groundbreaking work in abstract algebra provided much of the foundation for this field, and her work in invariant theory was essential in the development of Einstein's theory of general relativity. Although women weren't allowed to study at German universities at that time, she audited courses unofficially and went on to receive a doctorate at Erlangen *summa cum laude,* despite the opposition of the academic senate, which declared that women students would "overthrow all academic order." She subsequently taught mathematics at Göttingen, Moscow, and Frankfurt. In 1933 she left Germany to escape Nazi persecution, accepting a position at Bryn Mawr College in suburban Philadelphia. She lectured there and at the Institute for Advanced Study in Princeton, New Jersey, until her untimely death in 1935.

ROW AND COLUMN TRANSFORMATIONS OF A DETERMINANT

If A is a square matrix and if the matrix B is obtained from A by adding a multiple of one row to another or a multiple of one column to another, then $\det(A) = \det(B)$.

EXAMPLE 5 | Using Row and Column Transformations to Calculate a Determinant

Find the determinant of the matrix A. Does it have an inverse?

$$A = \begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 24 & 6 & 1 & -12 \\ 2 & 2 & 7 & -1 \end{bmatrix}$$

SOLUTION If we add -3 times row 1 to row 3, we change all but one element of row 3 to zeros:

$$\begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 0 & 0 & 4 & 0 \\ 2 & 2 & 7 & -1 \end{bmatrix}$$

This new matrix has the same determinant as A, and if we expand its determinant by the third row, we get

$$\det(A) = 4\begin{vmatrix} 8 & 2 & -4 \\ 3 & 5 & 11 \\ 2 & 2 & -1 \end{vmatrix}$$

Now, adding 2 times column 3 to column 1 in this determinant gives us

$$\det(A) = 4\begin{vmatrix} 0 & 2 & -4 \\ 25 & 5 & 11 \\ 0 & 2 & -1 \end{vmatrix} \qquad \text{Expand this by column 1}$$

$$= 4(-25)\begin{vmatrix} 2 & -4 \\ 2 & -1 \end{vmatrix}$$

$$= 4(-25)[2(-1) - (-4)2] = -600$$

Since the determinant of A is not zero, A does have an inverse.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **35**

Cramer's Rule

The solutions of linear equations can sometimes be expressed by using determinants. To illustrate, let's solve the following pair of linear equations for the variable x.

$$\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$$

To eliminate the variable y, we multiply the first equation by d and the second by b and subtract:

$$\begin{array}{r}adx + bdy = rd\\ \underline{bcx + bdy = bs}\\ adx - bcx = rd - bs\end{array}$$

Factoring the left-hand side, we get $(ad - bc)x = rd - bs$. Assuming that $ad - bc \neq 0$, we can now solve this equation for x:

$$x = \frac{rd - bs}{ad - bc}$$

Similarly, we find

$$y = \frac{as - cr}{ad - bc}$$

The numerator and denominator of the fractions for x and y are determinants of 2×2 matrices. So we can express the solution of the system using determinants as follows.

CRAMER'S RULE FOR SYSTEMS IN TWO VARIABLES

The linear system

$$\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$$

has the solution

$$x = \frac{\begin{vmatrix} r & b \\ s & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & r \\ c & s \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

provided that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

Using the notation

$$\underbrace{D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{Coefficient matrix}} \qquad \underbrace{D_x = \begin{bmatrix} r & b \\ s & d \end{bmatrix}}_{\text{Replace first column of } D \text{ by } r \text{ and } s} \qquad \underbrace{D_y = \begin{bmatrix} a & r \\ c & s \end{bmatrix}}_{\text{Replace second column of } D \text{ by } r \text{ and } s}$$

we can write the solution of the system as

$$\boxed{x = \frac{|D_x|}{|D|} \qquad \text{and} \qquad y = \frac{|D_y|}{|D|}}$$

EXAMPLE 6 | Using Cramer's Rule to Solve a System with Two Variables

Use Cramer's Rule to solve the system.

$$\begin{cases} 2x + 6y = -1 \\ x + 8y = 2 \end{cases}$$

SOLUTION For this system we have

$$|D| = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = 2 \cdot 8 - 6 \cdot 1 = 10$$

$$|D_x| = \begin{vmatrix} -1 & 6 \\ 2 & 8 \end{vmatrix} = (-1)8 - 6 \cdot 2 = -20$$

$$|D_y| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - (-1)1 = 5$$

The solution is

$$x = \frac{|D_x|}{|D|} = \frac{-20}{10} = -2$$

$$y = \frac{|D_y|}{|D|} = \frac{5}{10} = \frac{1}{2}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41**

Cramer's Rule can be extended to apply to any system of n linear equations in n variables in which the determinant of the coefficient matrix is not zero. As we saw in the preceding section, any such system can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

By analogy with our derivation of Cramer's Rule in the case of two equations in two unknowns, we let D be the coefficient matrix in this system and let D_{x_i} be the matrix obtained by replacing the ith column of D by the numbers $b_1, b_2, \ldots, b_n$ that appear to the right of the equal sign. The solution of the system is then given by the following rule.

CRAMER'S RULE

If a system of n linear equations in the n variables $x_1, x_2, \ldots, x_n$ is equivalent to the matrix equation $DX = B$, and if $|D| \neq 0$, then its solutions are

$$x_1 = \frac{|D_{x_1}|}{|D|} \qquad x_2 = \frac{|D_{x_2}|}{|D|} \qquad \cdots \qquad x_n = \frac{|D_{x_n}|}{|D|}$$

where D_{x_i} is the matrix obtained by replacing the ith column of D by the $n \times 1$ matrix B.

EXAMPLE 7 | Using Cramer's Rule to Solve a System with Three Variables

Use Cramer's Rule to solve the system.

$$\begin{cases} 2x - 3y + 4z = 1 \\ x \qquad\quad + 6z = 0 \\ 3x - 2y \qquad\quad = 5 \end{cases}$$

SOLUTION First, we evaluate the determinants that appear in Cramer's Rule. Note that D is the coefficient matrix and that D_x, D_y, and D_z are obtained by replacing the first, second, and third columns of D by the constant terms.

$$|D| = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & 6 \\ 3 & -2 & 0 \end{vmatrix} = -38 \qquad |D_x| = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 0 & 6 \\ 5 & -2 & 0 \end{vmatrix} = -78$$

$$|D_y| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 6 \\ 3 & 5 & 0 \end{vmatrix} = -22 \qquad |D_z| = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 0 & 0 \\ 3 & -2 & 5 \end{vmatrix} = 13$$

Now we use Cramer's Rule to get the solution:

$$x = \frac{|D_x|}{|D|} = \frac{-78}{-38} = \frac{39}{19} \qquad y = \frac{|D_y|}{|D|} = \frac{-22}{-38} = \frac{11}{19}$$

$$z = \frac{|D_z|}{|D|} = \frac{13}{-38} = -\frac{13}{38}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **47**

Solving the system in Example 7 using Gaussian elimination would involve matrices whose elements are fractions with fairly large denominators. Thus, in cases like Examples 6 and 7, Cramer's Rule gives us an efficient way to solve systems of linear equations. But in systems with more than three equations, evaluating the various determinants that are involved is usually a long and tedious task (unless you are using a graphing calculator). Moreover, the rule doesn't apply if $|D| = 0$ or if D is not a square matrix. So Cramer's Rule is a useful alternative to Gaussian elimination, but only in some situations.

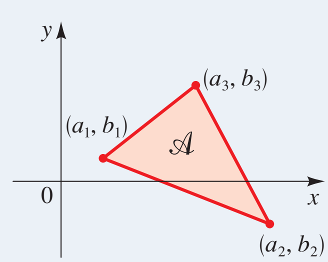
Areas of Triangles Using Determinants

Determinants provide a simple way to calculate the area of a triangle in the coordinate plane.

AREA OF A TRIANGLE

If a triangle in the coordinate plane has vertices (a_1, b_1), (a_2, b_2), and (a_3, b_3), then its area is

$$\mathscr{A} = \pm\tfrac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$



where the sign is chosen to make the area positive.

You are asked to prove this formula in Exercise 71.

EXAMPLE 8 | Area of a Triangle

Find the area of the triangle shown in Figure 2.

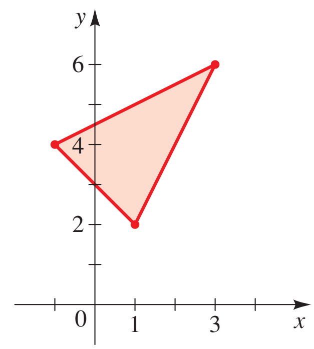


FIGURE 2

We can calculate the determinant by hand or by using a graphing calculator.

```
[A]
        [[-1  4  1]
         [3   6  1]
         [1   2  1]]
det([A])
                -12
```

SOLUTION The vertices are (1, 2), (3, 6), and (−1, 4). Using the formula in the preceding box, we get

$$\mathscr{A} = \pm\tfrac{1}{2}\begin{vmatrix} -1 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \pm\tfrac{1}{2}(-12)$$

To make the area positive, we choose the negative sign in the formula. Thus, the area of the triangle is

$$\mathscr{A} = -\tfrac{1}{2}(-12) = 6$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **63**

6.4 EXERCISES

CONCEPTS

1. *True or false?* det(A) is defined only for a square matrix A.

2. *True or false?* det(A) is a number, not a matrix.

3. *True or false?* If det(A) = 0, then A is not invertible.

4. Fill in the blanks with appropriate numbers to calculate the determinant. Where there is "±", choose the appropriate sign (+ or −).

(a) $\begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} = \blacksquare\blacksquare - \blacksquare\blacksquare = \_\_\_\_\_\_$

(b) $\begin{vmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 0 & -3 & 4 \end{vmatrix} = \pm\blacksquare(\blacksquare\blacksquare - \blacksquare\blacksquare) \pm \blacksquare(\blacksquare\blacksquare - \blacksquare\blacksquare)$

$\pm\blacksquare(\blacksquare\blacksquare - \blacksquare\blacksquare) = \_\_\_\_\_\_$

SKILLS

5–14 ■ Find the determinant of the matrix, if it exists.

5. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

7. $\begin{bmatrix} \frac{3}{2} & 1 \\ -1 & -\frac{2}{3} \end{bmatrix}$

8. $\begin{bmatrix} 0.2 & 0.4 \\ -0.4 & -0.8 \end{bmatrix}$

9. $\begin{bmatrix} 4 & 5 \\ 0 & -1 \end{bmatrix}$

10. $\begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 5 \end{bmatrix}$

12. $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

13. $\begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ 1 & \frac{1}{2} \end{bmatrix}$

14. $\begin{bmatrix} 2.2 & -1.4 \\ 0.5 & 1.0 \end{bmatrix}$

15–20 ■ Evaluate the minor and cofactor using the matrix A.

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -3 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

15. M_{11}, A_{11}

16. M_{33}, A_{33}

17. M_{12}, A_{12}

18. M_{13}, A_{13}

19. M_{23}, A_{23}

20. M_{32}, A_{32}

21–28 ■ Find the determinant of the matrix. Determine whether the matrix has an inverse, but don't calculate the inverse.

21. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 2 & 5 \\ -2 & -3 & 2 \\ 3 & 5 & 3 \end{bmatrix}$

23. $\begin{bmatrix} 30 & 0 & 20 \\ 0 & -10 & -20 \\ 40 & 0 & 10 \end{bmatrix}$

24. $\begin{bmatrix} -2 & -\frac{3}{2} & \frac{1}{2} \\ 2 & 4 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & 8 \\ 0 & 2 & 2 \end{bmatrix}$

26. $\begin{bmatrix} 0 & -1 & 0 \\ 2 & 6 & 4 \\ 1 & 0 & 3 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 0 & 2 \\ 1 & 6 & 4 & 1 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 2 & 0 & 2 \\ 3 & -4 & 0 & 4 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$

29–34 ■ Use a graphing calculator to find the determinant of the matrix. Determine whether the matrix has an inverse, but don't calculate the inverse.

29. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

30. $\begin{bmatrix} 10 & -20 & 31 \\ 10 & -11 & 45 \\ -20 & 40 & -50 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 10 & 2 & 7 \\ 2 & 18 & 18 & 13 \\ -3 & -30 & -4 & -24 \\ 1 & 10 & 2 & 10 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 3 & -2 & 5 \\ -3 & -9 & 11 & 5 \\ 2 & 6 & 0 & 31 \\ 5 & 15 & -10 & 39 \end{bmatrix}$

33. $\begin{bmatrix} 4 & 3 & -2 & 10 \\ -8 & -6 & 24 & -1 \\ 20 & 15 & 3 & 27 \\ 12 & 9 & -6 & -1 \end{bmatrix}$

34. $\begin{bmatrix} 2 & 3 & -5 & 10 \\ -2 & -2 & 26 & 3 \\ 6 & 9 & -16 & 45 \\ -8 & -12 & 20 & -36 \end{bmatrix}$

35–38 ■ Evaluate the determinant, using row or column operations whenever possible to simplify your work.

35. $\begin{vmatrix} 0 & 0 & 4 & 6 \\ 2 & 1 & 1 & 3 \\ 2 & 1 & 2 & 3 \\ 3 & 0 & 1 & 7 \end{vmatrix}$

36. $\begin{vmatrix} -2 & 3 & -1 & 7 \\ 4 & 6 & -2 & 3 \\ 7 & 7 & 0 & 5 \\ 3 & -12 & 4 & 0 \end{vmatrix}$

37. $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix}$

38. $\begin{vmatrix} 2 & -1 & 6 & 4 \\ 7 & 2 & -2 & 5 \\ 4 & -2 & 10 & 8 \\ 6 & 1 & 1 & 4 \end{vmatrix}$

39. Let

$$B = \begin{bmatrix} 4 & 1 & 0 \\ -2 & -1 & 1 \\ 4 & 0 & 3 \end{bmatrix}$$

(a) Evaluate det(B) by expanding by the second row.
(b) Evaluate det(B) by expanding by the third column.
(c) Do your results in parts (a) and (b) agree?

40. Consider the system

$$\begin{cases} x + 2y + 6z = 5 \\ -3x - 6y + 5z = 8 \\ 2x + 6y + 9z = 7 \end{cases}$$

(a) Verify that $x = -1$, $y = 0$, $z = 1$ is a solution of the system.
(b) Find the determinant of the coefficient matrix.
(c) Without solving the system, determine whether there are any other solutions.
(d) Can Cramer's Rule be used to solve this system? Why or why not?

41–56 ■ Use Cramer's Rule to solve the system.

41. $\begin{cases} 2x - y = -9 \\ x + 2y = 8 \end{cases}$

42. $\begin{cases} 6x + 12y = 33 \\ 4x + 7y = 20 \end{cases}$

43. $\begin{cases} x - 6y = 3 \\ 3x + 2y = 1 \end{cases}$

44. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 1 \\ \frac{1}{4}x - \frac{1}{6}y = -\frac{3}{2} \end{cases}$

45. $\begin{cases} 0.4x + 1.2y = 0.4 \\ 1.2x + 1.6y = 3.2 \end{cases}$

46. $\begin{cases} 10x - 17y = 21 \\ 20x - 31y = 39 \end{cases}$

47. $\begin{cases} x - y + 2z = 0 \\ 3x + z = 11 \\ -x + 2y = 0 \end{cases}$

48. $\begin{cases} 5x - 3y + z = 6 \\ 4y - 6z = 22 \\ 7x + 10y = -13 \end{cases}$

49. $\begin{cases} 2x_1 + 3x_2 - 5x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 2x_2 + x_3 = 8 \end{cases}$

50. $\begin{cases} -2a + c = 2 \\ a + 2b - c = 9 \\ 3a + 5b + 2c = 22 \end{cases}$

51. $\begin{cases} \frac{1}{3}x - \frac{1}{5}y + \frac{1}{2}z = \frac{7}{10} \\ -\frac{2}{3}x + \frac{2}{5}y + \frac{3}{2}z = \frac{11}{10} \\ x - \frac{4}{5}y + z = \frac{9}{5} \end{cases}$

52. $\begin{cases} 2x - y = 5 \\ 5x + 3z = 19 \\ 4y + 7z = 17 \end{cases}$

53. $\begin{cases} 3y + 5z = 4 \\ 2x - z = 10 \\ 4x + 7y = 0 \end{cases}$

54. $\begin{cases} 2x - 5y = 4 \\ x + y - z = 8 \\ 3x + 5z = 0 \end{cases}$

55. $\begin{cases} x + y + z + w = 0 \\ 2x + w = 0 \\ y - z = 0 \\ x + 2z = 1 \end{cases}$

56. $\begin{cases} x + y = 1 \\ y + z = 2 \\ z + w = 3 \\ w - x = 4 \end{cases}$

57–58 ■ Evaluate the determinants.

57. $\begin{vmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{vmatrix}$ **58.** $\begin{vmatrix} a & a & a & a & a \\ 0 & a & a & a & a \\ 0 & 0 & a & a & a \\ 0 & 0 & 0 & a & a \\ 0 & 0 & 0 & 0 & a \end{vmatrix}$

59–62 ■ Solve for x.

59. $\begin{vmatrix} x & 12 & 13 \\ 0 & x-1 & 23 \\ 0 & 0 & x-2 \end{vmatrix} = 0$ **60.** $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & x \\ x & 1 & x \end{vmatrix} = 0$

61. $\begin{vmatrix} 1 & 0 & x \\ x^2 & 1 & 0 \\ x & 0 & 1 \end{vmatrix} = 0$ **62.** $\begin{vmatrix} a & b & x-a \\ x & x+b & x \\ 0 & 1 & 1 \end{vmatrix} = 0$

63–66 ■ Sketch the triangle with the given vertices, and use a determinant to find its area.

63. $(0, 0), (6, 2), (3, 8)$ **64.** $(1, 0), (3, 5), (-2, 2)$

65. $(-1, 3), (2, 9), (5, -6)$ **66.** $(-2, 5), (7, 2), (3, -4)$

67. Show that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

APPLICATIONS

68. Buying Fruit A roadside fruit stand sells apples at 75¢ a pound, peaches at 90¢ a pound, and pears at 60¢ a pound. Muriel buys 18 pounds of fruit at a total cost of \$13.80. Her peaches and pears together cost \$1.80 more than her apples.

(a) Set up a linear system for the number of pounds of apples, peaches, and pears that she bought.

(b) Solve the system using Cramer's Rule.

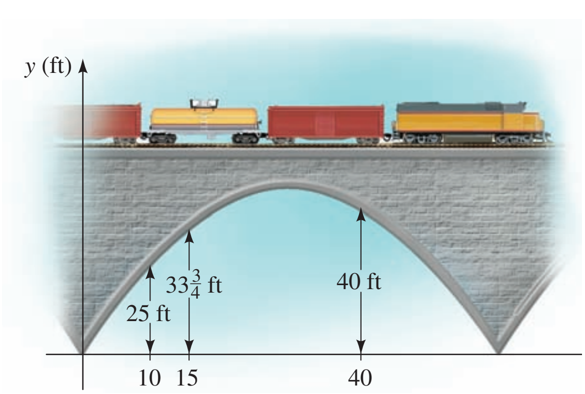
69. The Arch of a Bridge The opening of a railway bridge over a roadway is in the shape of a parabola. A surveyor measures the heights of three points on the bridge, as shown in the figure. He wishes to find an equation of the form

$$y = ax^2 + bx + c$$

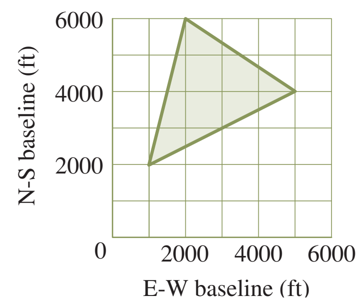
to model the shape of the arch.

(a) Use the surveyed points to set up a system of linear equations for the unknown coefficients a, b, and c.

(b) Solve the system using Cramer's Rule.



70. A Triangular Plot of Land An outdoors club is purchasing land to set up a conservation area. The last remaining piece they need to buy is the triangular plot shown in the figure. Use the determinant formula for the area of a triangle to find the area of the plot.



DISCOVERY ■ DISCUSSION ■ WRITING

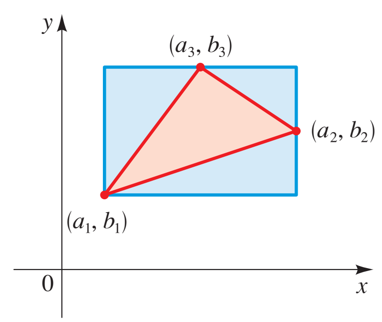
71. Determinant Formula for the Area of a Triangle The figure shows a triangle in the plane with vertices (a_1, b_1), (a_2, b_2), and (a_3, b_3).

(a) Find the coordinates of the vertices of the surrounding rectangle, and find its area.

(b) Find the area of the red triangle by subtracting the areas of the three blue triangles from the area of the rectangle.

(c) Use your answer to part (b) to show that the area of the red triangle is given by

$$\text{area} = \pm\tfrac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$



72. Collinear Points and Determinants

(a) If three points lie on a line, what is the area of the "triangle" that they determine? Use the answer to this question, together with the determinant formula for the area of a triangle, to explain why the points (a_1, b_1), (a_2, b_2), and (a_3, b_3) are collinear if and only if

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$$

(b) Use a determinant to check whether each set of points is collinear. Graph them to verify your answer.

(i) $(-6, 4), (2, 10), (6, 13)$

(ii) $(-5, 10), (2, 6), (15, -2)$

73. Determinant Form for the Equation of a Line

(a) Use the result of Exercise 72(a) to show that the equation of the line containing the points (x_1, y_1) and (x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(b) Use the result of part (a) to find an equation for the line containing the points (20, 50) and (−10, 25).

74. Matrices with Determinant Zero Use the definition of determinant and the elementary row and column operations to explain why matrices of the following types have determinant 0.

(a) A matrix with a row or column consisting entirely of zeros

(b) A matrix with two rows the same or two columns the same

(c) A matrix in which one row is a multiple of another row, or one column is a multiple of another column

75. Solving Linear Systems Suppose you have to solve a linear system with five equations and five variables without the assistance of a calculator or computer. Which method would you prefer: Cramer's Rule or Gaussian elimination? Write a short paragraph explaining the reasons for your answer.

CHAPTER 6 | REVIEW

PROPERTIES AND FORMULAS

Matrices (p. 466)

A **matrix** A of **dimension** $m \times n$ is a rectangular array of numbers with m **rows** and n **columns**:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Augmented Matrix of a System (p. 467)

The **augmented matrix** of a system of linear equations is the matrix consisting of the coefficients and the constant terms. For example, for the two-variable system

$$a_{11}x + a_{12}x = b_1$$
$$a_{21}x + a_{22}x = b_2$$

the augmented matrix is

$$\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix}$$

Elementary Row Operations (p. 467)

To solve a system of linear equations using the augmented matrix of the system, the following operations can be used to transform the rows of the matrix:

1. Add a nonzero multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

Row-Echelon Form of a Matrix (p. 469)

A matrix is in **row-echelon form** if its entries satisfy the following conditions:

1. The first nonzero entry in each row (the **leading entry**) is the number 1.
2. The leading entry of each row is to the right of the leading entry in the row above it.
3. All rows consisting entirely of zeros are at the bottom of the matrix.

If the matrix also satisfies the following condition, it is in **reduced row-echelon** form:

4. If a column contains a leading entry, then every other entry in that column is a 0.

Number of Solutions of a Linear System (p. 472)

If the augmented matrix of a system of linear equations has been reduced to row-echelon form using elementary row operations, then the system has:

1. **No solution** if the row-echelon form contains a row that represents the equation $0 = 1$. In this case the system is **inconsistent**.
2. **One solution** if each variable in the row-echelon form is a **leading variable**.
3. **Infinitely many solutions** if the system is not inconsistent but not every variable is a leading variable. In this case the system is **dependent**.

Operations on Matrices (p. 480)

If A and B are $m \times n$ matrices and c is a scalar (real number), then:

1. The **sum** $A + B$ is the $m \times n$ matrix that is obtained by adding corresponding entries of A and B.
2. The **difference** $A - B$ is the $m \times n$ matrix that is obtained by subtracting corresponding entries of A and B.
3. The **scalar product** cA is the $m \times n$ matrix that is obtained by multiplying each entry of A by c.

Multiplication of Matrices (p. 482)

If A is an $m \times n$ matrix and B is an $n \times k$ matrix (so the number of columns of A is the same as the number of rows of B), then the **matrix product** AB is the $m \times k$ matrix whose ij-entry is the inner product of the ith row of A and the jth column of B.

Properties of Matrix Operations (pp. 481, 484)

If A, B, and C are matrices of compatible dimensions then the following properties hold:

1. **Commutativity of addition:**

$$A + B = B + A$$

2. Associativity:

$$(A + B) + C = A + (B + C)$$
$$(AB)C = A(BC)$$

3. Distributivity:

$$A(B + C) = AB + AC$$
$$(B + C)A = BA + CA$$

(Note that matrix *multiplication* is *not* commutative.)

Identity Matrix (p. 490)

The **identity matrix** I_n is the $n \times n$ matrix whose main diagonal entries are all 1 and whose other entries are all 0:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

If A is an $m \times n$ matrix, then

$$AI_n = A \qquad \text{and} \qquad I_mA = A$$

Inverse of a Matrix (p. 491)

If A is an $n \times n$ matrix, then the inverse of A is the $n \times n$ matrix A^{-1} with the following properties:

$$A^{-1}A = I_n \qquad \text{and} \qquad AA^{-1} = I_n$$

To find the inverse of a matrix, we use a procedure involving elementary row operations (explained on page 467). (Note that *some* square matrices do not have an inverse.)

Inverse of a 2 × 2 Matrix (p. 492)

For 2×2 matrices the following special rule provides a shortcut for finding the inverse:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Writing a Linear System as a Matrix Equation (p. 495)

A system of n linear equations in n variables can be written as a single matrix equation

$$AX = B$$

where A is the $n \times n$ matrix of coefficients, X is the $n \times 1$ matrix of the variables, and B is the $n \times 1$ matrix of the constants. For example, the linear system of two equations in two variables

$$a_{11}x + a_{12}x = b_1$$
$$a_{21}x + a_{22}x = b_2$$

can be expressed as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solving Matrix Equations (p. 496)

If A is an invertible $n \times n$ matrix, X is an $n \times 1$ variable matrix, and B is an $n \times 1$ constant matrix, then the matrix equation

$$AX = B$$

has the unique solution

$$X = A^{-1}B$$

Determinant of a 2 × 2 Matrix (p. 501)

The **determinant** of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the *number*

$$\det(A) = |A| = ad - bc$$

Minors and Cofactors (p. 501)

If $A = |a_{ij}|$ is an $n \times n$ matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and the jth column of A.

The **cofactor** A_{ij} of the entry a_{ij} is

$$A_{ij} = (-1)^{i+j}M_{ij}$$

(Thus, the minor and the cofactor of each entry either are the same or are negatives of each other.)

Determinant of an *n* × *n* Matrix (p. 502)

To find the **determinant** of the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

we choose a row or column to **expand**, and then we calculate the number that is obtained by multiplying each element of that row or column by its cofactor and then adding the resulting products. For example, if we choose to expand about the first row, we get

$$\det(A) = |A| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

Invertibility Criterion (p. 504)

A square matrix has an inverse if and only if its determinant is not 0.

Row and Column Transformations (p. 505)

If we add a nonzero multiple of one row to another row in a square matrix or a nonzero multiple of one column to another column, then the determinant of the matrix is unchanged.

Cramer's Rule (pp. 506–507)

If a system of n linear equations in the n variables $x_1, x_2, \ldots, x_n$ is equivalent to the matrix equation $DX = B$ and if $|D| \neq 0$, then the solutions of the system are

$$x_1 = \frac{|D_{x_1}|}{|D|} \qquad x_2 = \frac{|D_{x_2}|}{|D|} \qquad \cdots \qquad x_n = \frac{|D_{x_n}|}{|D|}$$

where D_{x_i} is the matrix that is obtained from D by replacing its ith column by the constant matrix B.

Area of a Triangle Using Determinants (p. 508)

If a triangle in the coordinate plane has vertices (a_1, b_1), (a_2, b_2), and (a_3, b_3), then the area of the triangle is given by

$$\text{area} = \pm\tfrac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

where the sign is chosen to make the area positive.

LEARNING OBJECTIVES SUMMARY

Section	After completing this chapter, you should be able to . . .	Review Exercises
6.1	▪ Find the augmented matrix of a linear system	1–20
	▪ Solve a linear system using elementary row operations	7–20
	▪ Solve a linear system using the row-echelon form of its matrix	7–12
	▪ Solve a linear system using the reduced row-echelon form of its matrix	13–20
	▪ Determine the number of solutions of a linear system from the row-echelon form of its matrix	7–20
	▪ Model using linear systems	63–64
6.2	▪ Determine whether two matrices are equal	21–22
	▪ Perform addition, subtraction, and scalar multiplication on matrices	23–26
	▪ Perform matrix multiplication	27–34, 35–38
	▪ Express a linear system in matrix form	61–64
6.3	▪ Determine whether two matrices are inverses of each other	45–46
	▪ Find the inverse of a 2×2 matrix	53–55
	▪ Find the inverse of an $n \times n$ matrix	56–60
	▪ Solve a matrix equation	47–52
	▪ Solve a linear system by expressing it as a matrix equation	61–64
	▪ Model using matrix equations	65–66
6.4	▪ Find the determinant of a 2×2 matrix	53–55
	▪ Find the determinant of an $n \times n$ matrix	39–44, 56–60
	▪ Use the Invertibility Criterion	53–60
	▪ Use row and column transformations in finding the determinant of a matrix	57–58, 60
	▪ Use Cramer's Rule to solve a linear system	67–70
	▪ Use determinants to find the area of a triangle in the coordinate plane	71–72

EXERCISES

1–6 ▪ A matrix is given. **(a)** State the dimension of the matrix. **(b)** Is the matrix in row-echelon form? (c) Is the matrix in reduced row-echelon form? **(d)** Write the system of equations for which the given matrix is the augmented matrix.

1. $\begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 3 & 6 & 2 \\ 2 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 & -3 & 4 \\ 1 & 1 & 0 & 7 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 8 & 6 & -4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 2 & -7 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

7–12 ▪ Use Gaussian elimination to find the complete solution of the system, or show that no solution exists.

7. $\begin{cases} x + 2y + 2z = 6 \\ x - y = -1 \\ 2x + y + 3z = 7 \end{cases}$

8. $\begin{cases} x - y + z = 2 \\ x + y + 3z = 6 \\ 2y + 3z = 5 \end{cases}$

9. $\begin{cases} x - 2y + 3z = -2 \\ 2x - y + z = 2 \\ 2x - 7y + 11z = -9 \end{cases}$

10. $\begin{cases} x - y + z = 2 \\ x + y + 3z = 6 \\ 3x - y + 5z = 10 \end{cases}$

11. $\begin{cases} x + y + z + w = 0 \\ x - y - 4z - w = -1 \\ x - 2y + 4w = -7 \\ 2x + 2y + 3z + 4w = -3 \end{cases}$

12. $\begin{cases} x + 3z = -1 \\ y - 4w = 5 \\ 2y + z + w = 0 \\ 2x + y + 5z - 4w = 4 \end{cases}$

13–20 ▪ Use Gauss-Jordan elimination to find the complete solution of the system, or show that no solution exists.

13. $\begin{cases} x - y + 3z = 2 \\ 2x + y + z = 2 \\ 3x + 4z = 4 \end{cases}$

14. $\begin{cases} x - y = 1 \\ x + y + 2z = 3 \\ x - 3y - 2z = -1 \end{cases}$

15. $\begin{cases} x - y + z - w = 0 \\ 3x - y - z - w = 2 \end{cases}$

16. $\begin{cases} x - y = 3 \\ 2x + y = 6 \\ x - 2y = 9 \end{cases}$

17. $\begin{cases} x - y + z = 0 \\ 3x + 2y - z = 6 \\ x + 4y - 3z = 3 \end{cases}$

18. $\begin{cases} x + 2y + 3z = 2 \\ 2x - y - 5z = 1 \\ 4x + 3y + z = 6 \end{cases}$

19. $\begin{cases} x + y - z - w = 2 \\ x - y + z - w = 0 \\ 2x \qquad\qquad + 2w = 2 \\ 2x + 4y - 4z - 2w = 6 \end{cases}$

20. $\begin{cases} x - y - 2z + 3w = 0 \\ y - z + w = 1 \\ 3x - 2y - 7z + 10w = 2 \end{cases}$

21–22 ■ Determine whether the matrices A and B are equal.

21. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \end{bmatrix}$

22. $A = \begin{bmatrix} \sqrt{25} & 1 \\ 0 & 2^{-1} \end{bmatrix} \qquad B = \begin{bmatrix} 5 & e^0 \\ \log 1 & \frac{1}{2} \end{bmatrix}$

23–34 ■ The matrices A, B, C, D, E, F, and G are defined as follows

$$A = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{1}{2} & 3 \\ 2 & \frac{3}{2} \\ -2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 4 \\ 0 & -1 \\ 2 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 4 & 0 & 2 \\ -1 & 1 & 0 \\ 7 & 5 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 5 \end{bmatrix}$$

Carry out the indicated operation, or explain why it cannot be performed.

23. $A + B$ **24.** $C - D$ **25.** $2C + 3D$

26. $5B - 2C$ **27.** GA **28.** AG

29. BC **30.** CB **31.** BF

32. FC **33.** $(C + D)E$ **34.** $F(2C - D)$

35–44 ■ The matrices A and B are defined as follows.

$$A = \begin{bmatrix} 3 & 0 & -3 \\ -2 & 1 & 2 \\ 1 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 4 & -1 \\ 1 & -1 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

Use a graphing calculator to carry out the indicated algebraic operation. State your answer using fractions.

35. AB^2 **36.** A^2B **37.** $A^{-1}BA$

38. BAB^{-1} **39.** $|AB|$ **40.** $|BA|$

41. $|A^{-1}|$ **42.** $\dfrac{1}{|A|}$ **43.** $|A^{-1}BA|$

44. $|A^{-1}||B||A|$

45–46 ■ Verify that the matrices A and B are inverses of each other by calculating the products AB and BA.

45. $A = \begin{bmatrix} 2 & -5 \\ -2 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & \frac{5}{2} \\ 1 & 1 \end{bmatrix}$

46. $A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -\frac{3}{2} & 2 & \frac{5}{2} \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$

47–52 ■ Solve the matrix equation for the unknown matrix, X, or show that no solution exists, where

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 3 \\ -2 & 4 & 0 \end{bmatrix}$$

47. $A + 3X = B$ **48.** $\frac{1}{2}(X - 2B) = A$

49. $2(X - A) = 3B$ **50.** $2X + C = 5A$

51. $AX = C$ **52.** $AX = B$

53–60 ■ Find the determinant and, if possible, the inverse of the matrix.

53. $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$ **54.** $\begin{bmatrix} 2 & 2 \\ 1 & -3 \end{bmatrix}$

55. $\begin{bmatrix} 4 & -12 \\ -2 & 6 \end{bmatrix}$ **56.** $\begin{bmatrix} 2 & 4 & 0 \\ -1 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}$

57. $\begin{bmatrix} 3 & 0 & 1 \\ 2 & -3 & 0 \\ 4 & -2 & 1 \end{bmatrix}$ **58.** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 2 & 5 & 6 \end{bmatrix}$

59. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ **60.** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

61–64 ■ Express the system of linear equations as a matrix equation. Then solve the matrix equation by multiplying each side by the inverse of the coefficient matrix.

61. $\begin{cases} 12x - 5y = 10 \\ 5x - 2y = 17 \end{cases}$ **62.** $\begin{cases} 6x - 5y = 1 \\ 8x - 7y = -1 \end{cases}$

63. $\begin{cases} 2x + y + 5z = \frac{1}{3} \\ x + 2y + 2z = \frac{1}{4} \\ x \qquad + 3z = \frac{1}{6} \end{cases}$ **64.** $\begin{cases} 2x \qquad + 3z = 5 \\ x + y + 6z = 0 \\ 3x - y + z = 5 \end{cases}$

65. Magda and Ivan grow tomatoes, onions, and zucchini in their backyard and sell them at a roadside stand on Saturdays and Sundays. They price tomatoes at \$1.50 per pound, onions at \$1.00 per pound, and zucchini at 50 cents per pound. The following table shows the number of pounds of each type of produce that they sold during the last weekend in July.

	Tomatoes	Onions	Zucchini
Saturday	25	16	30
Sunday	14	12	16

(a) Let

$$A = \begin{bmatrix} 25 & 16 & 30 \\ 14 & 12 & 16 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1.50 \\ 1.00 \\ 0.50 \end{bmatrix}$$

Compare these matrices to the data given in the problem, and describe what their entries represent.

(b) Only one of the products AB or BA is defined. Calculate the product that *is* defined, and describe what its entries represent.

66. An ATM at a bank in Qualicum Beach, British Columbia, dispenses \$20 and \$50 bills. Brodie withdraws \$600 from this machine and receives a total of 18 bills. Let x be the number of \$20 bills and y the number of \$50 bills that he receives.

(a) Find a system of two linear equations in x and y that express the information given in the problem.

(b) Write your linear system as a matrix equation of the form $AX = B$.

(c) Find A^{-1}, and use it to solve your matrix equation in part (b). How many bills of each type did Brodie receive?

67–70 ■ Solve the system using Cramer's Rule.

67. $\begin{cases} 2x + 7y = 13 \\ 6x + 16y = 30 \end{cases}$

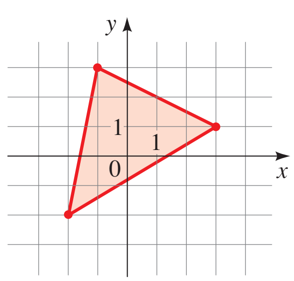
68. $\begin{cases} 12x - 11y = 140 \\ 7x + 9y = 20 \end{cases}$

69. $\begin{cases} 2x - y + 5z = 0 \\ -x + 7y = 9 \\ 5x + 4y + 3z = -9 \end{cases}$

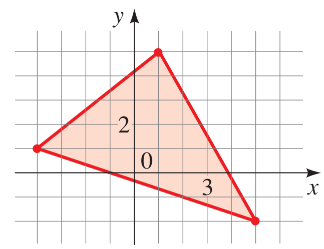
70. $\begin{cases} 3x + 4y - z = 10 \\ x - 4z = 20 \\ 2x + y + 5z = 30 \end{cases}$

71–72 ■ Use the determinant formula for the area of a triangle to find the area of the triangle in the figure.

71.



72.



73–74 ■ Use any of the methods you have learned in this chapter to solve the problem.

73. Clarisse invests \$60,000 in money-market accounts at three different banks. Bank A pays 2% interest per year, bank B pays 2.5%, and bank C pays 3%. She decides to invest twice as much in bank B as in the other two banks. After one year, Clarisse has earned \$1575 in interest. How much did she invest in each bank?

74. A commercial fisherman fishes for haddock, sea bass, and red snapper. He is paid \$1.25 a pound for haddock, \$0.75 a pound for sea bass, and \$2.00 a pound for red snapper. Yesterday he caught 560 lb of fish worth \$575. The haddock and red snapper together are worth \$320. How many pounds of each fish did he catch?

CHAPTER 6 TEST

1–4 ■ Determine whether the matrix is in reduced row-echelon form, row-echelon form, or neither.

1. $\begin{bmatrix} 1 & 8 & 0 & 0 \\ 0 & 1 & 7 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 2 & 5 \\ 0 & 1 & -2 & 7 \\ 1 & 0 & -3 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}$

5–6 ■ Use Gaussian elimination to find the complete solution of the system, or show that no solution exists.

5. $\begin{cases} x - y + 2z = 0 \\ 2x - 4y + 5z = -5 \\ 2y - 3z = 5 \end{cases}$

6. $\begin{cases} 2x - 3y + z = 3 \\ x + 2y + 2z = -1 \\ 4x + y + 5z = 4 \end{cases}$

7. Use Gauss-Jordan elimination to find the complete solution of the system.

$$\begin{cases} x + 3y - z = 0 \\ 3x + 4y - 2z = -1 \\ -x + 2y = 1 \end{cases}$$

8–15 ■ Let

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

Carry out the indicated operation, or explain why it cannot be performed.

8. $A + B$ **9.** AB **10.** $BA - 3B$ **11.** CBA

12. A^{-1} **13.** B^{-1} **14.** $\det(B)$ **15.** $\det(C)$

16. **(a)** Write a matrix equation equivalent to the following system.

$$\begin{cases} 4x - 3y = 10 \\ 3x - 2y = 30 \end{cases}$$

(b) Find the inverse of the coefficient matrix, and use it to solve the system.

17. Only one of the following matrices has an inverse. Find the determinant of each matrix, and use the determinants to identify the one that has an inverse. Then find the inverse.

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

18. Solve using Cramer's Rule:

$$\begin{cases} 2x - z = 14 \\ 3x - y + 5z = 0 \\ 4x + 2y + 3z = -2 \end{cases}$$

19. A shopper buys a mixture of nuts; the almonds cost \$4.75 a pound, and the walnuts cost \$3.45 a pound. Her total purchase weighs 3 lb and costs \$11.91. Use Cramer's Rule to determine how much of each nut she bought.

Computer Graphics

Matrix algebra is the basic tool used in computer graphics to manipulate images on a computer screen. We will see how matrix multiplication can be used to “move” a point in the plane to a prescribed location. Combining such moves enables us to stretch, compress, rotate, and otherwise transform a figure, as we see in the images below.

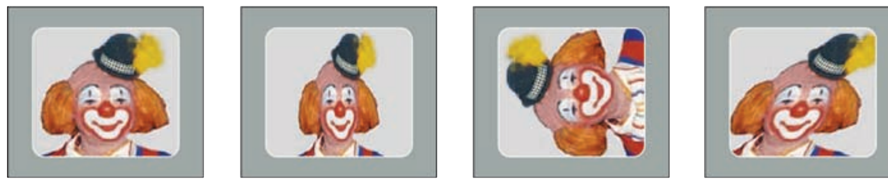


Image | Compressed | Rotated | Sheared

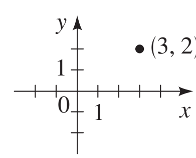
▼ Moving Points in the Plane

Let’s represent the point (x, y) in the plane by a 2×1 matrix:

$$(x, y) \quad \leftrightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix}$$

For example, the point $(3, 2)$ in the figure is represented by the matrix

$$P = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

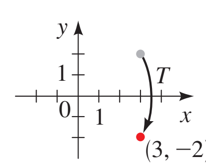


Multiplying by a 2×2 matrix *moves* the point in the plane. For example, if

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

then multiplying P by T, we get

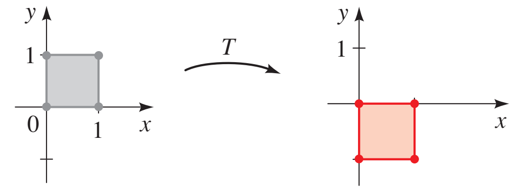
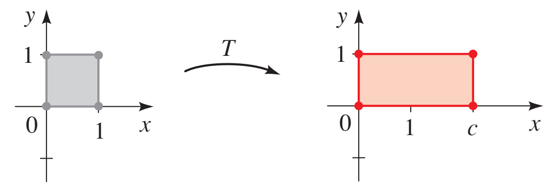
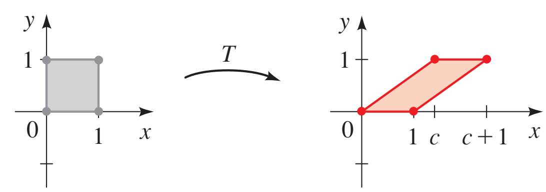
$$TP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$



We see that the point $(3, 2)$ has been moved to the point $(3, -2)$. In general, multiplication by this matrix T reflects points in the x-axis. If every point in an image is multiplied by this matrix, then the entire image will be flipped upside down about the x-axis. Matrix multiplication “transforms” a point to a new point in the plane. For this reason a matrix used in this way is called a **transformation**.

Table 1 gives some standard transformations and their effects on the gray square in the first quadrant.

TABLE 1

Transformation matrix	Effect
$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Reflection in x-axis	
$T = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$ Expansion (or contraction) in the x-direction	
$T = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ Shear in x-direction	

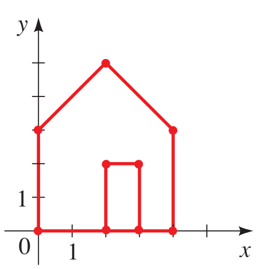


FIGURE 1

▼ Moving Images in the Plane

Simple line drawings such as the house in Figure 1 consist of a collection of vertex points and connecting line segments. The house in Figure 1 can be represented in a computer by the 2×11 **data matrix**

$$D = \begin{bmatrix} 2 & 0 & 0 & 2 & 4 & 4 & 3 & 3 & 2 & 2 & 3 \\ 0 & 0 & 3 & 5 & 3 & 0 & 0 & 2 & 2 & 0 & 0 \end{bmatrix}$$

The columns of D represent the vertex points of the image. To draw the house, we connect successive points (columns) in D by line segments. Now we can transform the whole house by multiplying D by an appropriate transformation matrix. For example, if we apply the shear transformation $T = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$, we get the following matrix.

$$TD = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 2 & 4 & 4 & 3 & 3 & 2 & 2 & 3 \\ 0 & 0 & 3 & 5 & 3 & 0 & 0 & 2 & 2 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1.5 & 4.5 & 5.5 & 4 & 3 & 4 & 3 & 2 & 3 \\ 0 & 0 & 3 & 5 & 3 & 0 & 0 & 2 & 2 & 0 & 0 \end{bmatrix}$$

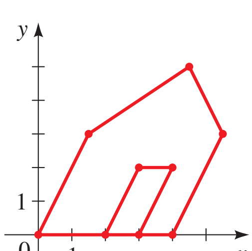


FIGURE 2

To draw the image represented by TD, we start with the point $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, connect it by a line segment to the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then follow that by a line segment to $\begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$, and so on. The resulting tilted house is shown in Figure 2.

```
PROGRAM:IMAGE
:For(N,1,10)
:Line([A](1,N),
 [A](2,N),[A](1,N+1),
 [A](2,N+1))
:End
```

A convenient way to draw an image corresponding to a given data matrix is to use a graphing calculator. The TI-83 program in the margin converts a data matrix stored in `[A]` into the corresponding image, as shown in Figure 3. (To use this program for a data matrix with m columns, store the matrix in `[A]` and change the "10" in the `For` command to $m - 1$.)

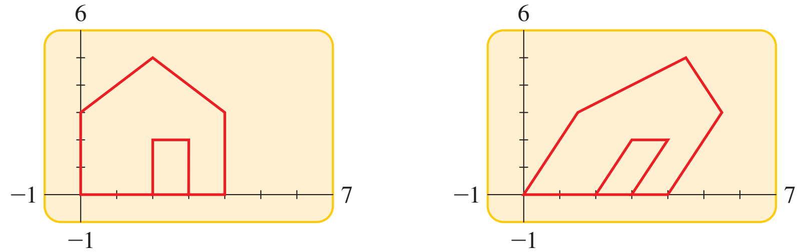


FIGURE 3 (a) House with data matrix D (b) Tilted house with data matrix TD

PROBLEMS

1. The gray square in Table 1 has the following vertices:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Apply each of the three transformations given in Table 1 to these vertices and sketch the result to verify that each transformation has the indicated effect. Use $c = 2$ in the expansion matrix and $c = 1$ in the shear matrix.

2. Verify that multiplication by the given matrix has the indicated effect when applied to the gray square in the table. Use $c = 3$ in the expansion matrix and $c = 1$ in the shear matrix.

$$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \qquad T_3 = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$$

Reflection in y-axis Expansion (or contraction) in y-direction Shear in y-direction

3. Let $T = \begin{bmatrix} 1 & 1.5 \\ 0 & 1 \end{bmatrix}$.

(a) What effect does T have on the gray square in the Table 1?
(b) Find T^{-1}.
(c) What effect does T^{-1} have on the gray square?
(d) What happens to the square if we first apply T, then T^{-1}?

4. **(a)** Let $T = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. What effect does T have on the gray square in Table 1?

(b) Let $S = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. What effect does S have on the gray square in Table 1?

(c) Apply S to the vertices of the square, and then apply T to the result. What is the effect of the combined transformation?

(d) Find the product matrix $W = TS$.

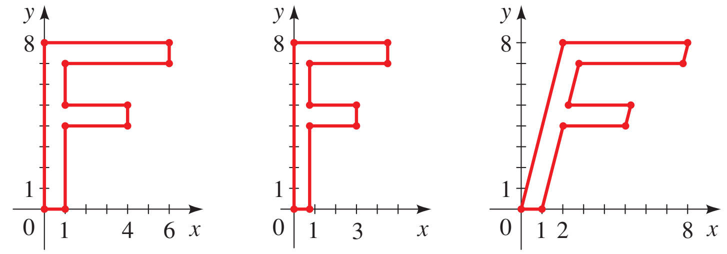
(e) Apply the transformation W to the square. Compare to your final result in part (c). What do you notice?

5. The figure shows three outline versions of the letter **F**. The second one is obtained from the first by shrinking horizontally by a factor of 0.75, and the third is obtained from the first by shearing horizontally by a factor of 0.25.

(a) Find a data matrix D for the first letter **F**.

(b) Find the transformation matrix T that transforms the first **F** into the second. Calculate TD, and verify that this is a data matrix for the second **F**.

(c) Find the transformation matrix S that transforms the first **F** into the third. Calculate SD, and verify that this is a data matrix for the third **F**.



6. Here is a data matrix for a line drawing:

$$D = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 4 & 4 & 0 \end{bmatrix}$$

(a) Draw the image represented by D.

(b) Let $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. Calculate the matrix product TD, and draw the image represented by this product. What is the effect of the transformation T?

(c) Express T as a product of a shear matrix and a reflection matrix. (See Problem 2.)

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7

CHAPTER

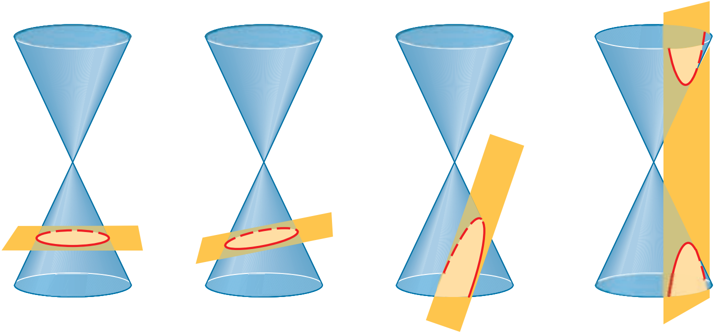
Conic Sections

Equations for Geometric Curves Conic sections are the curves that are formed when a plane cuts a cone, as shown in the figure. For example, if a cone is cut horizontally, the cross section is a circle. So a circle is a conic section. Other ways of cutting a cone by a plane produce parabolas, ellipses, and hyperbolas.

Some of the most common curves in the real world are conic sections. For example, when a volleyball is hit, it travels through the air in a parabolic path. The path of a planet around the sun is an ellipse. Hyperbolas occur when a light from a lamp falls on a wall. These and many other uses of conics make them important curves for modeling real-world phenomena.

In Section 1.2 we found the equation of a circle in a coordinate plane by using geometric properties of the circle. Similarly, to find equations for the other conics, we begin by placing the conic in a coordinate plane. We then use geometric properties of the conic to derive its equation.

In *Focus on Modeling* at the end of the chapter we explore how these curves are used in architecture.



7.1 PARABOLAS

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find geometric properties of a parabola from its equation ▶ Find the equation of a parabola from some of its geometric properties

▼ Geometric Definition of a Parabola

We saw in Section 3.1 that the graph of the equation

$$y = ax^2 + bx + c$$

is a U-shaped curve called a *parabola* that opens either upward or downward, depending on whether the sign of a is positive or negative.

In this section we study parabolas from a geometric, rather than an algebraic, point of view. We begin with the geometric definition of a parabola and show how this leads to the algebraic formula that we are already familiar with.

GEOMETRIC DEFINITION OF A PARABOLA

A **parabola** is the set of points in the plane that are equidistant from a fixed point F (called the **focus**) and a fixed line l (called the **directrix**).

This definition is illustrated in Figure 1. The **vertex** V of the parabola lies halfway between the focus and the directrix, and the **axis of symmetry** is the line that runs through the focus perpendicular to the directrix.

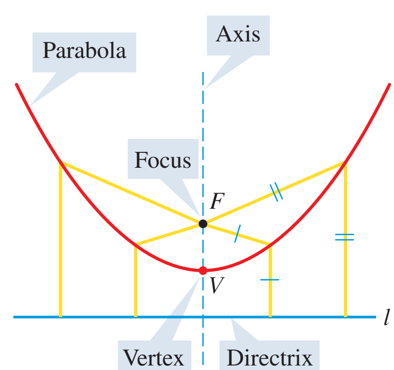


FIGURE 1

In this section we restrict our attention to parabolas that are situated with the vertex at the origin and that have a vertical or horizontal axis of symmetry. (Parabolas in more general positions will be considered in Section 7.4.) If the focus of such a parabola is the point $F(0, p)$, then the axis of symmetry must be vertical, and the directrix has the equation $y = -p$. Figure 2 illustrates the case $p > 0$.

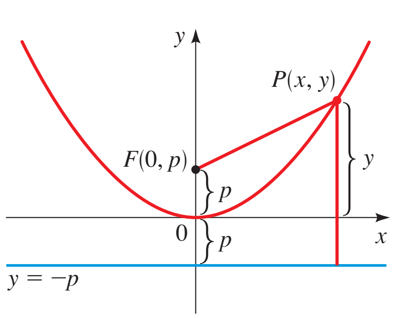


FIGURE 2

Deriving the Equation of a Parabola If $P(x, y)$ is any point on the parabola, then the distance from P to the focus F (using the Distance Formula) is

$$\sqrt{x^2 + (y - p)^2}$$

The distance from P to the directrix is

$$|y - (-p)| = |y + p|$$

By the definition of a parabola these two distances must be equal:

$$\sqrt{x^2 + (y - p)^2} = |y + p|$$

$$x^2 + (y - p)^2 = |y + p|^2 = (y + p)^2 \qquad \text{Square both sides}$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 \qquad \text{Expand}$$

$$x^2 - 2py = 2py \qquad \text{Simplify}$$

$$x^2 = 4py$$

If $p > 0$, then the parabola opens upward; but if $p < 0$, it opens downward. When x is replaced by $-x$, the equation remains unchanged, so the graph is symmetric about the y-axis.

Equations and Graphs of Parabolas

The following box summarizes what we have just proved about the equation and features of a parabola with a vertical axis.

PARABOLA WITH VERTICAL AXIS

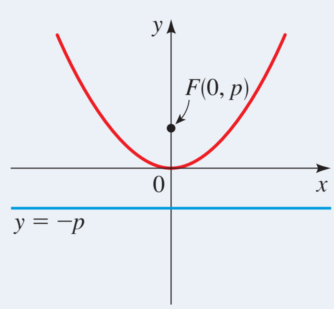
The graph of the equation

$$x^2 = 4py$$

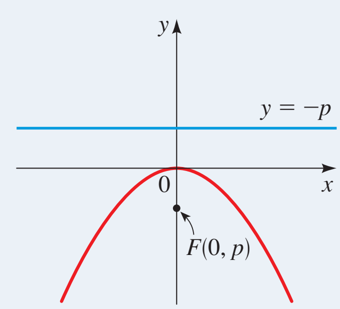
is a parabola with the following properties.

VERTEX	$V(0, 0)$
FOCUS	$F(0, p)$
DIRECTRIX	$y = -p$

The parabola opens upward if $p > 0$ or downward if $p < 0$.



$x^2 = 4py$ with $p > 0$



$x^2 = 4py$ with $p < 0$

EXAMPLE 1 | Finding the Equation of a Parabola

Find an equation for the parabola with vertex $V(0, 0)$ and focus $F(0, 2)$, and sketch its graph.

SOLUTION Since the focus is $F(0, 2)$, we conclude that $p = 2$ (so the directrix is $y = -2$). Thus the equation of the parabola is

$$x^2 = 4(2)y \qquad x^2 = 4py \text{ with } p = 2$$

$$x^2 = 8y$$

Since $p = 2 > 0$, the parabola opens upward. See Figure 3.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **31** AND **47**

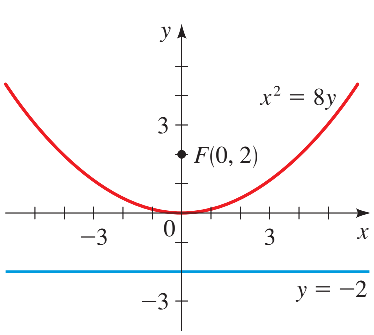
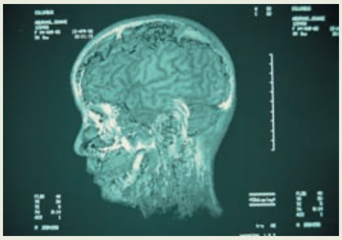


FIGURE 3

MATHEMATICS IN THE MODERN WORLD



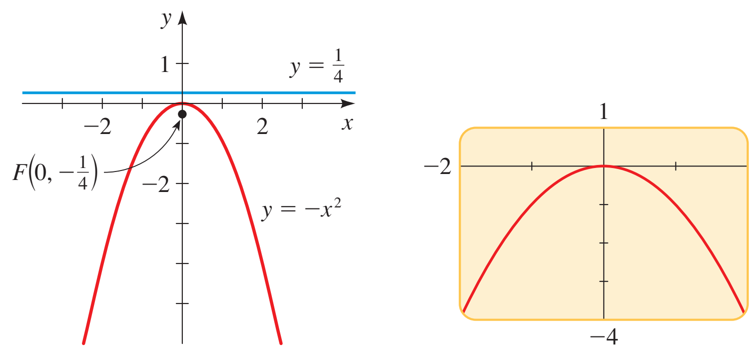
Looking Inside Your Head

How would you like to look inside your head? The idea isn't particularly appealing to most of us, but doctors often need to do just that. If they can look without invasive surgery, all the better. An X-ray doesn't really give a look inside, it simply gives a "graph" of the density of tissue the X-rays must pass through. So an X-ray is a "flattened" view in one direction. Suppose you get an X-ray view from many different directions. Can these "graphs" be used to reconstruct the three-dimensional inside view? This is a purely mathematical problem and was solved by mathematicians a long time ago. However, reconstructing the inside view requires thousands of tedious computations. Today, mathematics and high-speed computers make it possible to "look inside" by a process called computer-aided tomography (or CAT scan). Mathematicians continue to search for better ways of using mathematics to reconstruct images. One of the latest techniques, called magnetic resonance imaging (MRI), combines molecular biology and mathematics for a clear "look inside."

EXAMPLE 2 | Finding the Focus and Directrix of a Parabola from Its Equation

Find the focus and directrix of the parabola $y = -x^2$, and sketch the graph.

SOLUTION To find the focus and directrix, we put the given equation in the standard form $x^2 = -y$. Comparing this to the general equation $x^2 = 4py$, we see that $4p = -1$, so $p = -\frac{1}{4}$. Thus the focus is $F\left(0, -\frac{1}{4}\right)$, and the directrix is $y = \frac{1}{4}$. The graph of the parabola, together with the focus and the directrix, is shown in Figure 4(a). We can also draw the graph using a graphing calculator as shown in Figure 4(b).



(a) (b)

FIGURE 4

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **11**

Reflecting the graph in Figure 2 about the diagonal line $y = x$ has the effect of interchanging the roles of x and y. This results in a parabola with horizontal axis. By the same method as before, we can prove the following properties.

PARABOLA WITH HORIZONTAL AXIS

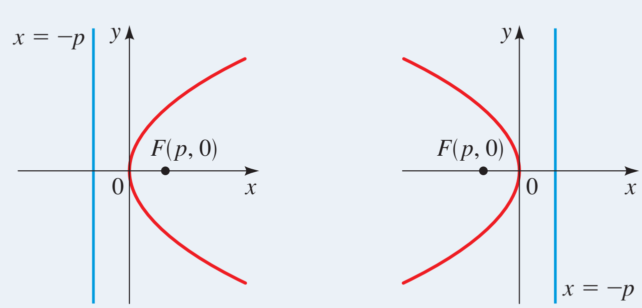
The graph of the equation

$$y^2 = 4px$$

is a parabola with the following properties.

VERTEX	$V(0, 0)$
FOCUS	$F(p, 0)$
DIRECTRIX	$x = -p$

The parabola opens to the right if $p > 0$ or to the left if $p < 0$.



$y^2 = 4px$ with $p > 0$ $y^2 = 4px$ with $p < 0$

EXAMPLE 3 | A Parabola with Horizontal Axis

A parabola has the equation $6x + y^2 = 0$.

(a) Find the focus and directrix of the parabola, and sketch the graph.

(b) Use a graphing calculator to draw the graph.

SOLUTION

(a) To find the focus and directrix, we put the given equation in the standard form $y^2 = -6x$. Comparing this to the general equation $y^2 = 4px$, we see that $4p = -6$, so $p = -\frac{3}{2}$. Thus the focus is $F\left(-\frac{3}{2}, 0\right)$ and the directrix is $x = \frac{3}{2}$. Since $p < 0$, the parabola opens to the left. The graph of the parabola, together with the focus and the directrix, is shown in Figure 5(a) below.

(b) To draw the graph using a graphing calculator, we need to solve for y:

$$6x + y^2 = 0$$

$$y^2 = -6x \qquad \text{Subtract } 6x$$

$$y = \pm\sqrt{-6x} \qquad \text{Take square roots}$$

To obtain the graph of the parabola, we graph both functions

$$y = \sqrt{-6x} \qquad \text{and} \qquad y = -\sqrt{-6x}$$

as shown in Figure 5(b).

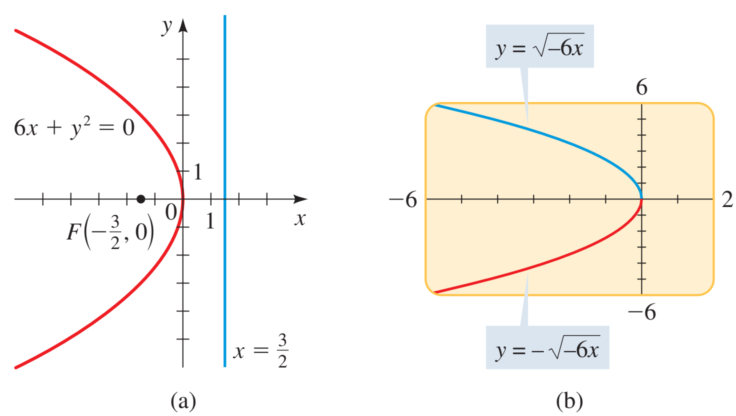


FIGURE 5

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **13** AND **25**

Graphing Calculator Note The equation $y^2 = 4px$, does not define y as a function of x (see page 190). So to use a graphing calculator to graph a parabola with a horizontal axis, we must first solve for y. This leads to two functions: $y = \sqrt{4px}$ and $y = -\sqrt{4px}$. We need to graph both functions to get the complete graph of the parabola. For example, in Figure 5(b) we had to graph both $y = \sqrt{-6x}$ and $y = -\sqrt{-6x}$ to graph the parabola $y^2 = -6x$.

We can use the coordinates of the focus to estimate the "width" of a parabola when sketching its graph. The line segment that runs through the focus perpendicular to the axis, with endpoints on the parabola, is called the **latus rectum**, and its length is the **focal diameter** of the parabola. From Figure 6 we can see that the distance from an endpoint Q of the latus rectum to the directrix is $|2p|$. Thus the distance from Q to the focus must be $|2p|$ as well (by the definition of a parabola), so the focal diameter is $|4p|$. In the next example we use the focal diameter to determine the "width" of a parabola when graphing it.

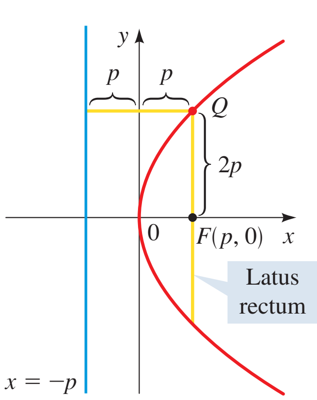


FIGURE 6

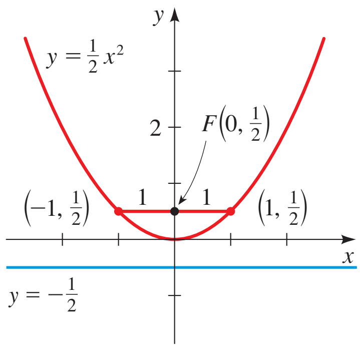


FIGURE 7

EXAMPLE 4 | The Focal Diameter of a Parabola

Find the focus, directrix, and focal diameter of the parabola $y = \frac{1}{2}x^2$, and sketch its graph.

SOLUTION We first put the equation in the form $x^2 = 4py$:

$$y = \tfrac{1}{2}x^2$$

$$x^2 = 2y \qquad \text{Multiply by 2, switch sides}$$

From this equation we see that $4p = 2$, so the focal diameter is 2. Solving for p gives $p = \frac{1}{2}$, so the focus is $\left(0, \frac{1}{2}\right)$ and the directrix is $y = -\frac{1}{2}$. Since the focal diameter is 2, the latus rectum extends 1 unit to the left and 1 unit to the right of the focus. The graph is sketched in Figure 7.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **15**

In the next example we graph a family of parabolas to show how changing the distance between the focus and the vertex affects the "width" of a parabola.

EXAMPLE 5 | A Family of Parabolas

(a) Find equations for the parabolas with vertex at the origin and foci $F_1\left(0, \frac{1}{8}\right)$, $F_2\left(0, \frac{1}{2}\right)$, $F_3(0, 1)$, and $F_4(0, 4)$.

(b) Draw the graphs of the parabolas in part (a). What do you conclude?

SOLUTION

(a) Since the foci are on the positive y-axis, the parabolas open upward and have equations of the form $x^2 = 4py$. This leads to the following equations.

Focus	p	Equation $x^2 = 4py$	Form of the equation for graphing calculator
$F_1\left(0, \frac{1}{8}\right)$	$p = \frac{1}{8}$	$x^2 = \frac{1}{2}y$	$y = 2x^2$
$F_2\left(0, \frac{1}{2}\right)$	$p = \frac{1}{2}$	$x^2 = 2y$	$y = 0.5x^2$
$F_3(0, 1)$	$p = 1$	$x^2 = 4y$	$y = 0.25x^2$
$F_4(0, 4)$	$p = 4$	$x^2 = 16y$	$y = 0.0625x^2$

(b) The graphs are drawn in Figure 8. We see that the closer the focus is to the vertex, the narrower the parabola.

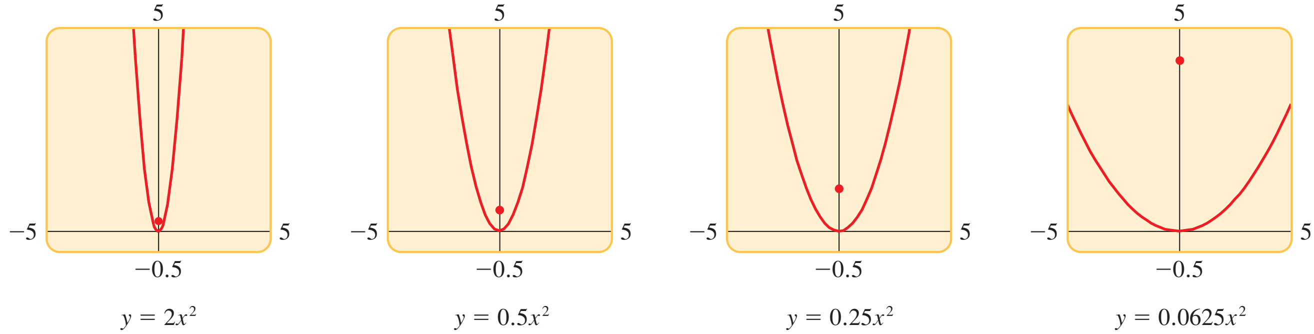


FIGURE 8 A family of parabolas

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **57**

Applications

Parabolas have an important property that makes them useful as reflectors for lamps and telescopes. Light from a source placed at the focus of a surface with parabolic cross section will be reflected in such a way that it travels parallel to the axis of the parabola (see Figure 9). Thus a parabolic mirror reflects the light into a beam of parallel rays. Conversely, light approaching the reflector in rays parallel to its axis of symmetry is concentrated to the focus. This *reflection property*, which can be proved by using calculus, is used in the construction of reflecting telescopes.

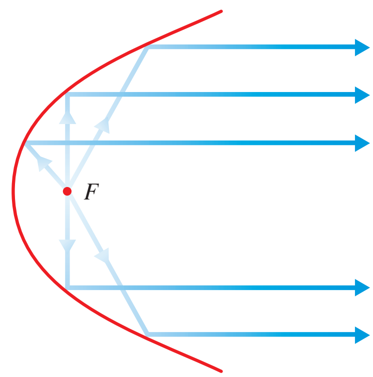


FIGURE 9 Parabolic reflector

EXAMPLE 6 | Finding the Focal Point of a Searchlight Reflector

A searchlight has a parabolic reflector that forms a "bowl," which is 12 in. wide from rim to rim and 8 in. deep, as shown in Figure 10. If the filament of the light bulb is located at the focus, how far from the vertex of the reflector is it?

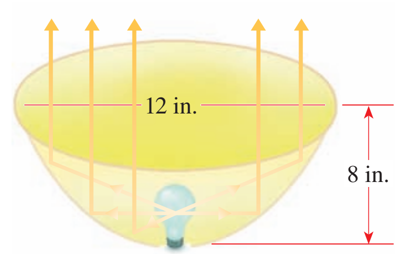


FIGURE 10 A parabolic reflector

ARCHIMEDES (287–212 B.C.) was the greatest mathematician of the ancient world. He was born in Syracuse, a Greek colony on Sicily, a generation after Euclid (see page 57). One of his many discoveries is the Law of the Lever (see page 120). He famously said, "Give me a place to stand and a fulcrum for my lever, and I can lift the earth."

Renowned as a mechanical genius for his many engineering inventions, he designed pulleys for lifting heavy ships and the spiral screw for transporting water to higher levels. He is said to have used parabolic mirrors to concentrate the rays of the sun to set fire to Roman ships attacking Syracuse.

King Hieron II of Syracuse once suspected a goldsmith of keeping part of the gold intended for the king's crown and replacing it with an equal amount of silver. The king asked Archimedes for advice. While in deep thought at a public bath, Archimedes discovered the solution to the king's problem when he noticed that his body's volume was the same as the volume of water it displaced from the tub. Using this insight he was able to measure the volume of each crown, and so determine which was the denser, all-gold crown. As the story is told, he ran home naked, shouting "Eureka, eureka!" ("I have found it, I have found it!") This incident attests to his enormous powers of concentration.

In spite of his engineering prowess, Archimedes was most proud of his mathematical discoveries. These include the formulas for the volume of a sphere, $\left(V = \frac{4}{3}\pi r^3\right)$ and the surface area of a sphere $\left(S = 4\pi r^2\right)$ and a careful analysis of the properties of parabolas and other conics.

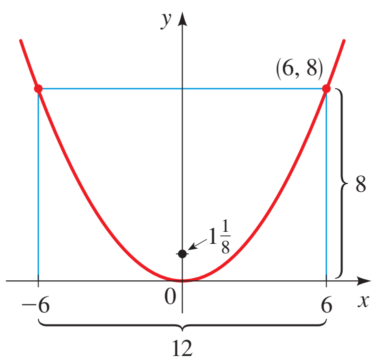


FIGURE 11

SOLUTION We introduce a coordinate system and place a parabolic cross section of the reflector so that its vertex is at the origin and its axis is vertical (see Figure 11). Then the equation of this parabola has the form $x^2 = 4py$. From Figure 11 we see that the point $(6, 8)$ lies on the parabola. We use this to find p.

$$6^2 = 4p(8) \qquad \text{The point (6, 8) satisfies the equation } x^2 = 4py$$
$$36 = 32p$$
$$p = \tfrac{9}{8}$$

The focus is $F\left(0, \frac{9}{8}\right)$, so the distance between the vertex and the focus is $\frac{9}{8} = 1\frac{1}{8}$ in. Because the filament is positioned at the focus, it is located $1\frac{1}{8}$ in. from the vertex of the reflector.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **59**

7.1 EXERCISES

CONCEPTS

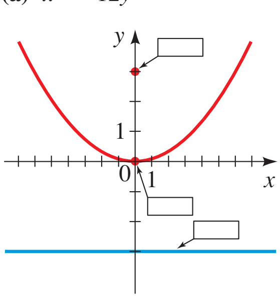
1. A parabola is the set of all points in the plane that are equidistant from a fixed point called the ________ and a fixed line called the ________ of the parabola.

2. The graph of the equation $x^2 = 4py$ is a parabola with focus $F(\_\_, \_\_)$ and directrix $y =$ ________. So the graph of $x^2 = 12y$ is a parabola with focus $F(\_\_, \_\_)$ and directrix $y =$ ________.

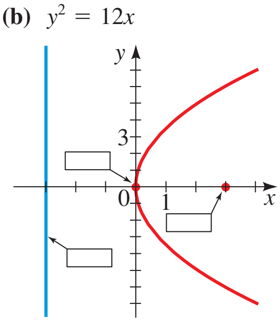
3. The graph of the equation $y^2 = 4px$ is a parabola with focus $F(\_\_, \_\_)$ and directrix $x =$ ________. So the graph of $y^2 = 12x$ is a parabola with focus $F(\_\_, \_\_)$ and directrix $x =$ ________.

4. Label the focus, directrix, and vertex on the graphs given for the parabolas in Exercises 2 and 3.

(a) $x^2 = 12y$

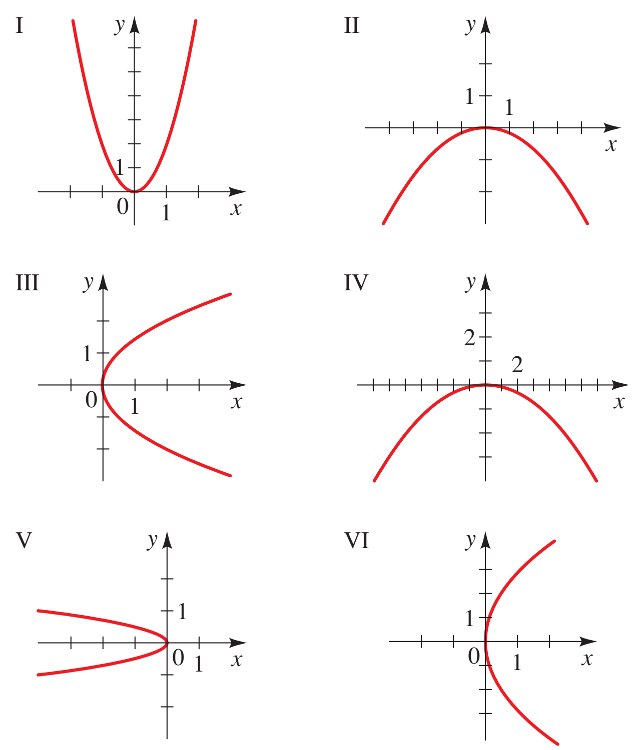


(b) $y^2 = 12x$



SKILLS

5–10 ■ Match the equation with the graphs labeled I–VI. Give reasons for your answers.



5. $y^2 = 2x$ **6.** $y^2 = -\frac{1}{4}x$

7. $x^2 = -6y$ **8.** $2x^2 = y$

9. $y^2 - 8x = 0$ **10.** $12y + x^2 = 0$

11–24 ■ An equation of a parabola is given. **(a)** Find the focus, directrix, and focal diameter of the parabola. **(b)** Sketch a graph of the parabola and its directrix.

11. $x^2 = 8y$ **12.** $x^2 = -4y$

13. $y^2 = -24x$ **14.** $y^2 = 16x$

15. $y = -\frac{1}{8}x^2$ **16.** $x = 2y^2$

17. $x = -2y^2$ **18.** $y = \frac{1}{4}x^2$

19. $5y = x^2$ **20.** $9x = y^2$

21. $x^2 + 12y = 0$ **22.** $x + \frac{1}{5}y^2 = 0$

23. $5x + 3y^2 = 0$ **24.** $8x^2 + 12y = 0$

25–30 ■ Use a graphing device to graph the parabola.

25. $x^2 = 16y$ **26.** $x^2 = -8y$

27. $y^2 = -\frac{1}{3}x$ **28.** $8y^2 = x$

29. $4x + y^2 = 0$ **30.** $x - 2y^2 = 0$

31–46 ■ Find an equation for the parabola that has its vertex at the origin and satisfies the given condition(s).

31. Focus: $F(0, 2)$ **32.** Focus: $F\left(0, -\frac{1}{2}\right)$

33. Focus: $F(-8, 0)$ **34.** Focus: $F(5, 0)$

35. Focus: $F\left(0, -\frac{3}{4}\right)$ **36.** Focus: $F\left(-\frac{1}{12}, 0\right)$

37. Directrix: $x = 2$ **38.** Directrix: $y = 6$

39. Directrix: $y = -10$ **40.** Directrix: $x = -\frac{1}{8}$

41. Directrix: $x = \frac{1}{20}$ **42.** Directrix: $y = -5$

43. Focus on the positive x-axis, 2 units away from the directrix

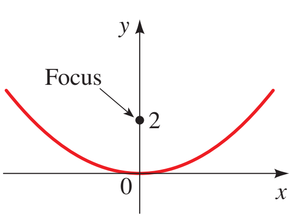
44. Directrix has y-intercept 6

45. Opens upward with focus 5 units from the vertex

46. Focal diameter 8 and focus on the negative y-axis

47–56 ■ Find an equation of the parabola whose graph is shown.

47.



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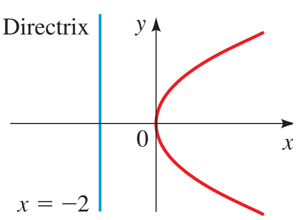

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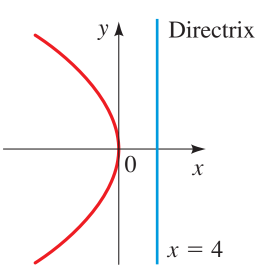

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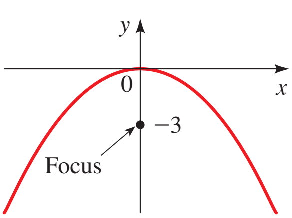

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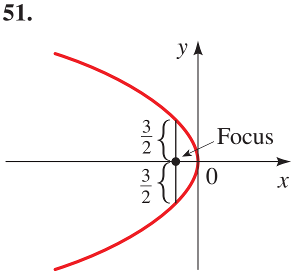

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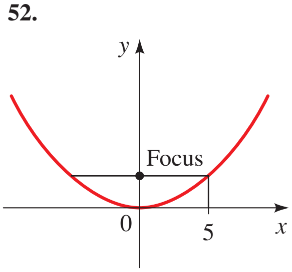

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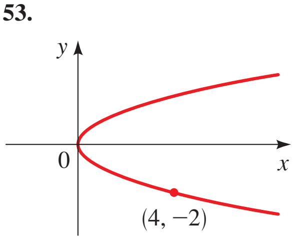

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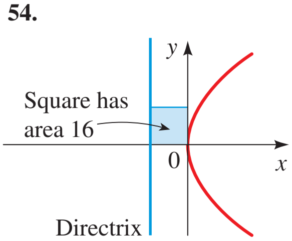

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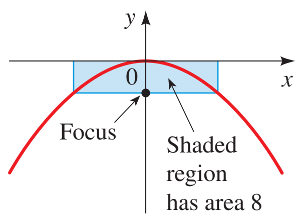

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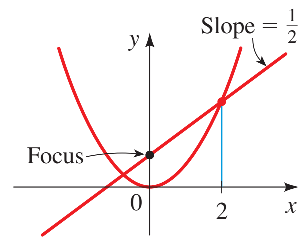


57. **(a)** Find equations for the family of parabolas with vertex at the origin and with directrixes $y = \frac{1}{2}$, $y = 1$, $y = 4$, and $y = 8$.

(b) Draw the graphs. What do you conclude?

58. **(a)** Find equations for the family of parabolas with vertex at the origin, focus on the positive y-axis, and with focal diameters 1, 2, 4, and 8.

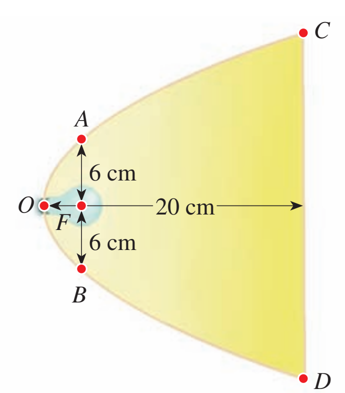
(b) Draw the graphs. What do you conclude?

APPLICATIONS

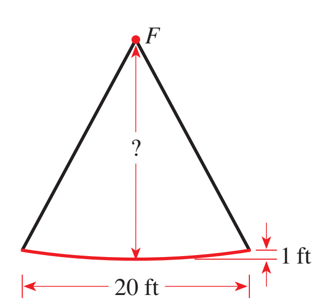
59. Parabolic Reflector A lamp with a parabolic reflector is shown in the figure. The bulb is placed at the focus, and the focal diameter is 12 cm.

(a) Find an equation of the parabola.

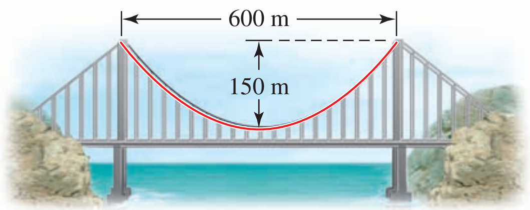
(b) Find the diameter $d(C, D)$ of the opening, 20 cm from the vertex.



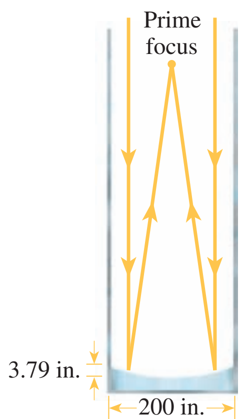
60. Satellite Dish A reflector for a satellite dish is parabolic in cross section, with the receiver at the focus F. The reflector is 1 ft deep and 20 ft wide from rim to rim (see the figure). How far is the receiver from the vertex of the parabolic reflector?



61. Suspension Bridge In a suspension bridge the shape of the suspension cables is parabolic. The bridge shown in the figure has towers that are 600 m apart, and the lowest point of the suspension cables is 150 m below the top of the towers. Find the equation of the parabolic part of the cables, placing the origin of the coordinate system at the vertex. [*Note*: This equation is used to find the length of cable needed in the construction of the bridge.]



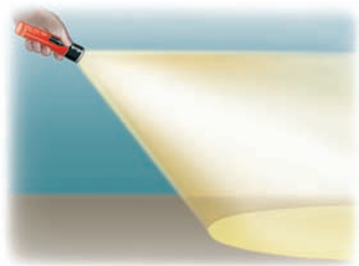
62. Reflecting Telescope The Hale telescope at the Mount Palomar Observatory has a 200-in. mirror, as shown in the figure. The mirror is constructed in a parabolic shape that collects light from the stars and focuses it at the **prime focus**, that is, the focus of the parabola. The mirror is 3.79 in. deep at its center. Find the **focal length** of this parabolic mirror, that is, the distance from the vertex to the focus.



DISCOVERY ■ DISCUSSION ■ WRITING

63. Parabolas in the Real World Several examples of the uses of parabolas are given in the text. Find other situations in real life in which parabolas occur. Consult a scientific encyclopedia in the reference section of your library, or search the Internet.

64. Light Cone from a Flashlight A flashlight is held to form a lighted area on the ground, as shown in the figure. Is it possible to angle the flashlight in such a way that the boundary of the lighted area is a parabola? Explain your answer.



DISCOVERY PROJECT **Rolling Down a Ramp**

In this project we investigate the process of modeling the motion of falling objects using a calculator-based motion detector. You can find the project at the book companion website: **www.stewartmath.com**

7.2 ELLIPSES

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find geometric properties of an ellipse from its equation ▶ Find the equation of an ellipse from some of its geometric properties

▼ Geometric Definition of an Ellipse

An ellipse is an oval curve that looks like an elongated circle. More precisely, we have the following definition.

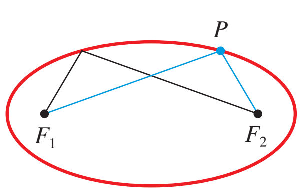


FIGURE 1

GEOMETRIC DEFINITION OF AN ELLIPSE

An **ellipse** is the set of all points in the plane the sum of whose distances from two fixed points F_1 and F_2 is a constant. (See Figure 1.) These two fixed points are the **foci** (plural of **focus**) of the ellipse.

The geometric definition suggests a simple method for drawing an ellipse. Place a sheet of paper on a drawing board, and insert thumbtacks at the two points that are to be the foci of the ellipse. Attach the ends of a string to the tacks, as shown in Figure 2(a). With the point of a pencil, hold the string taut. Then carefully move the pencil around the foci, keeping the string taut at all times. The pencil will trace out an ellipse, because the sum of the distances from the point of the pencil to the foci will always equal the length of the string, which is constant.

If the string is only slightly longer than the distance between the foci, then the ellipse that is traced out will be elongated in shape, as in Figure 2(a), but if the foci are close together relative to the length of the string, the ellipse will be almost circular, as shown in Figure 2(b).

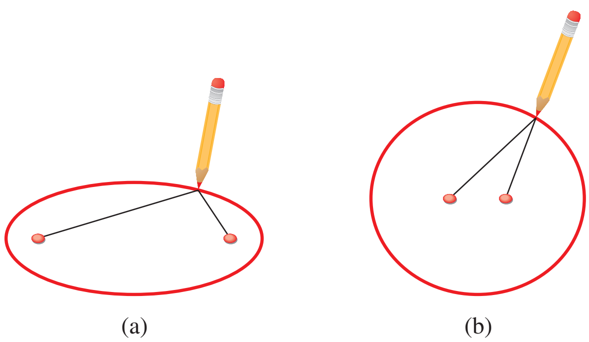


FIGURE 2

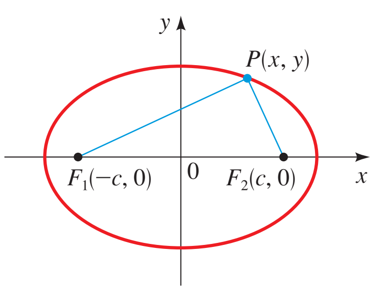


FIGURE 3

Deriving the Equation of an Ellipse To obtain the simplest equation for an ellipse, we place the foci on the x-axis at $F_1(-c, 0)$ and $F_2(c, 0)$ so that the origin is halfway between them (see Figure 3).

For later convenience we let the sum of the distances from a point on the ellipse to the foci be $2a$. Then if $P(x, y)$ is any point on the ellipse, we have

$$d(P, F_1) + d(P, F_2) = 2a$$

So from the Distance Formula we have

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

or

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

Squaring each side and expanding, we get

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x^2 + 2cx + c^2 + y^2)$$

which simplifies to

$$4a\sqrt{(x+c)^2 + y^2} = 4a^2 + 4cx$$

Dividing each side by 4 and squaring again, we get

$$a^2[(x+c)^2 + y^2] = (a^2 + cx)^2$$

$$a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

Since the sum of the distances from P to the foci must be larger than the distance between the foci, we have that $2a > 2c$, or $a > c$. Thus $a^2 - c^2 > 0$, and we can divide each side of the preceding equation by $a^2(a^2 - c^2)$ to get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

For convenience let $b^2 = a^2 - c^2$ (with $b > 0$). Since $b^2 < a^2$, it follows that $b < a$. The preceding equation then becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{with } a > b$$

This is the equation of the ellipse. To graph it, we need to know the x- and y-intercepts. Setting $y = 0$, we get

$$\frac{x^2}{a^2} = 1$$

so $x^2 = a^2$, or $x = \pm a$. Thus, the ellipse crosses the x-axis at $(a, 0)$ and $(-a, 0)$, as in Figure 4. These points are called the **vertices** of the ellipse, and the segment that joins them is called the **major axis**. Its length is $2a$.

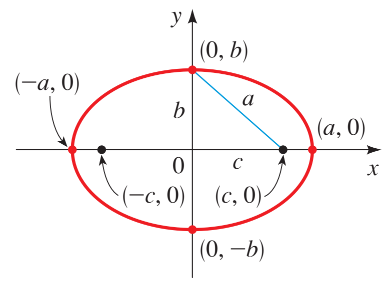


FIGURE 4
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $a > b$

Similarly, if we set $x = 0$, we get $y = \pm b$, so the ellipse crosses the y-axis at $(0, b)$ and $(0, -b)$. The segment that joins these points is called the **minor axis**, and it has length $2b$. Note that $2a > 2b$, so the major axis is longer than the minor axis. The origin is the **center** of the ellipse.

If the foci of the ellipse are placed on the y-axis at $(0, \pm c)$ rather than on the x-axis, then the roles of x and y are reversed in the preceding discussion, and we get a vertical ellipse.

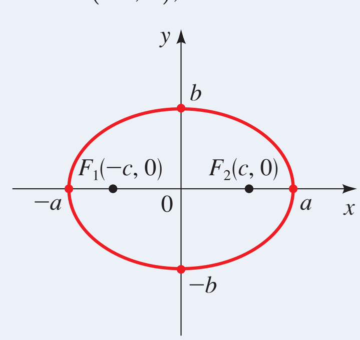
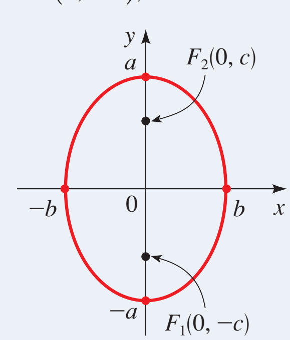
Equations and Graphs of Ellipses

The following box summarizes what we have just proved about the equation and features of an ellipse centered at the origin.

In the standard equation for an ellipse, a^2 is the *larger* denominator and b^2 is the *smaller*. To find c^2, we subtract: larger denominator minus smaller denominator.

ELLIPSE WITH CENTER AT THE ORIGIN

The graph of each of the following equations is an ellipse with center at the origin and having the given properties.

EQUATION	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$
	$a > b > 0$	$a > b > 0$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
MAJOR AXIS	Horizontal, length $2a$	Vertical, length $2a$
MINOR AXIS	Vertical, length $2b$	Horizontal, length $2b$
FOCI	$(\pm c, 0),\ c^2 = a^2 - b^2$	$(0, \pm c),\ c^2 = a^2 - b^2$
GRAPH		

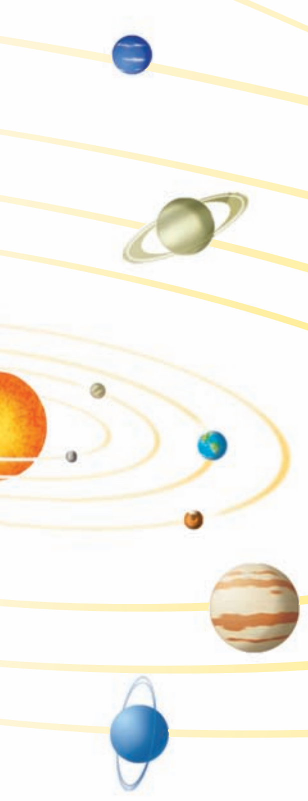
EXAMPLE 1 | Sketching an Ellipse

An ellipse has the equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

(a) Find the foci, the vertices, and the lengths of the major and minor axes, and sketch the graph.

(b) Draw the graph using a graphing calculator.



The orbits of the planets are ellipses, with the sun at one focus.

SOLUTION

(a) Since the denominator of x^2 is larger, the ellipse has a horizontal major axis. This gives $a^2 = 9$ and $b^2 = 4$, so $c^2 = a^2 - b^2 = 9 - 4 = 5$. Thus $a = 3$, $b = 2$, and $c = \sqrt{5}$.

FOCI	$(\pm\sqrt{5}, 0)$
VERTICES	$(\pm 3, 0)$
LENGTH OF MAJOR AXIS	6
LENGTH OF MINOR AXIS	4

The graph is shown in Figure 5(a).

(b) To draw the graph using a graphing calculator, we need to solve for y:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$\frac{y^2}{4} = 1 - \frac{x^2}{9} \qquad \text{Subtract } \frac{x^2}{9}$$

$$y^2 = 4\left(1 - \frac{x^2}{9}\right) \qquad \text{Multiply by 4}$$

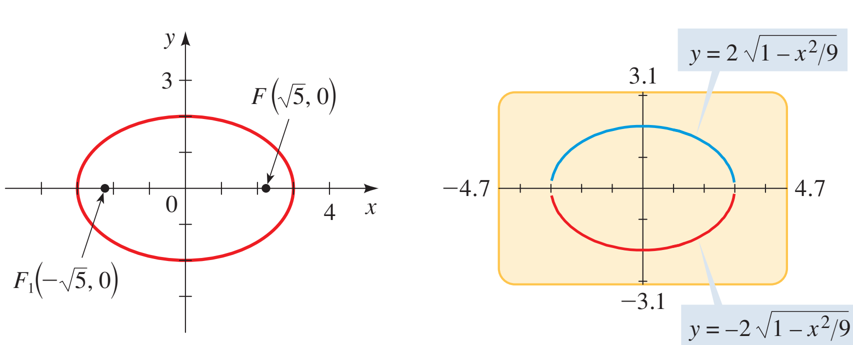
$$y = \pm 2\sqrt{1 - \frac{x^2}{9}} \qquad \text{Take square roots}$$

To obtain the graph of the ellipse, we graph both functions

$$y = 2\sqrt{1 - x^2/9} \qquad \text{and} \qquad y = -2\sqrt{1 - x^2/9}$$

as shown in Figure 5(b).

Note that the equation of an ellipse does not define y as a function of x (see page 190). That's why we need to graph two functions to graph an ellipse.



(a) (b)

FIGURE 5 $\frac{x^2}{9} + \frac{y^2}{4} = 1$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **9** AND **35**

EXAMPLE 2 | Finding the Foci of an Ellipse

Find the foci of the ellipse $16x^2 + 9y^2 = 144$, and sketch its graph.

SOLUTION First we put the equation in standard form. Dividing by 144, we get

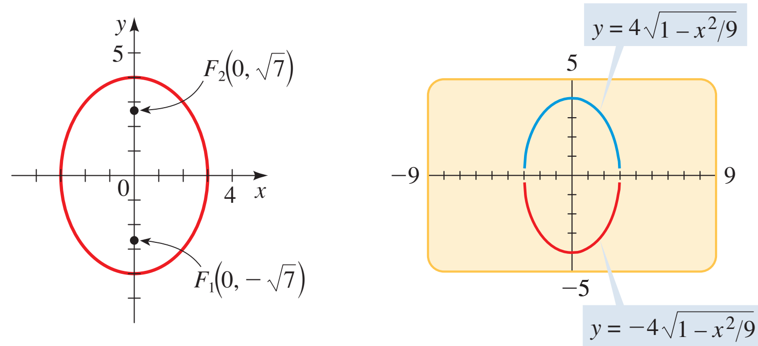
$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Since $16 > 9$, this is an ellipse with its foci on the y-axis and with $a = 4$ and $b = 3$. We have

$$c^2 = a^2 - b^2 = 16 - 9 = 7$$
$$c = \sqrt{7}$$

Thus the foci are $(0, \pm\sqrt{7})$. The graph is shown in Figure 6(a).

We can also draw the graph using a graphing calculator as shown in Figure 6(b).



(a) (b)

FIGURE 6
$16x^2 + 9y^2 = 144$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **15**

EXAMPLE 3 | Finding the Equation of an Ellipse

The vertices of an ellipse are $(\pm 4, 0)$, and the foci are $(\pm 2, 0)$. Find its equation, and sketch the graph.

SOLUTION Since the vertices are $(\pm 4, 0)$, we have $a = 4$ and the major axis is horizontal. The foci are $(\pm 2, 0)$, so $c = 2$. To write the equation, we need to find b. Since $c^2 = a^2 - b^2$, we have

$$2^2 = 4^2 - b^2$$
$$b^2 = 16 - 4 = 12$$

Thus the equation of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

The graph is shown in Figure 7.

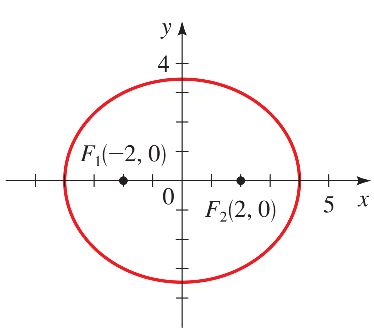


FIGURE 7
$\frac{x^2}{16} + \frac{y^2}{12} = 1$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **31** AND **39**

Eccentricity of an Ellipse

We saw earlier in this section (Figure 2) that if $2a$ is only slightly greater than $2c$, the ellipse is long and thin, whereas if $2a$ is much greater than $2c$, the ellipse is almost circular. We measure the deviation of an ellipse from being circular by the ratio of a and c.

DEFINITION OF ECCENTRICITY

For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ or $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ (with $a > b > 0$), the **eccentricity** $\boldsymbol{e}$ is the number

$$e = \frac{c}{a}$$

where $c = \sqrt{a^2 - b^2}$. The eccentricity of every ellipse satisfies $0 < e < 1$.

Thus if e is close to 1, then c is almost equal to a, and the ellipse is elongated in shape, but if e is close to 0, then the ellipse is close to a circle in shape. The eccentricity is a measure of how "stretched" the ellipse is.

In Figure 8 we show a number of ellipses to demonstrate the effect of varying the eccentricity e.

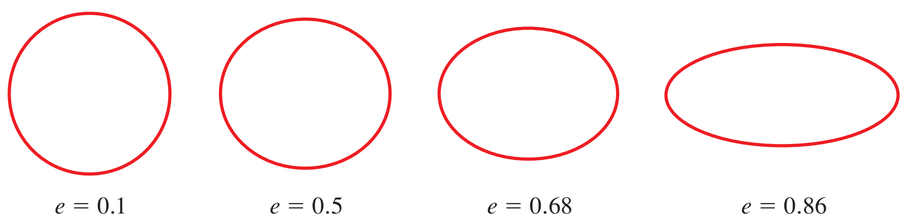


FIGURE 8 Ellipses with various eccentricities

EXAMPLE 4 | Finding the Equation of an Ellipse from Its Eccentricity and Foci

Find the equation of the ellipse with foci $(0, \pm 8)$ and eccentricity $e = \frac{4}{5}$, and sketch its graph.

SOLUTION We are given $e = \frac{4}{5}$ and $c = 8$. Thus

$$\frac{4}{5} = \frac{8}{a} \qquad \text{Eccentricity } e = \frac{c}{a}$$

$$4a = 40 \qquad \text{Cross-multiply}$$

$$a = 10$$

To find b, we use the fact that $c^2 = a^2 - b^2$:

$$8^2 = 10^2 - b^2$$

$$b^2 = 10^2 - 8^2 = 36$$

$$b = 6$$

Thus the equation of the ellipse is

$$\frac{x^2}{36} + \frac{y^2}{100} = 1$$

Because the foci are on the y-axis, the ellipse is oriented vertically. To sketch the ellipse, we find the intercepts: The x-intercepts are ± 6, and the y-intercepts are ± 10. The graph is sketched in Figure 9.

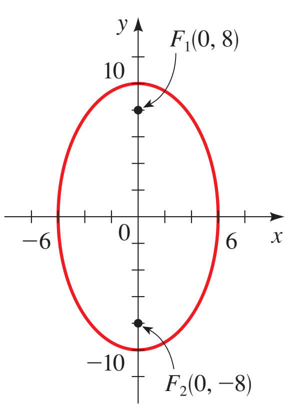


FIGURE 9 $\dfrac{x^2}{36} + \dfrac{y^2}{100} = 1$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **53**

Eccentricities of the Orbits of the Planets

The orbits of the planets are ellipses with the sun at one focus. For most planets these ellipses have very small eccentricity, so they are nearly circular. However, Mercury and Pluto, the innermost and outermost known planets, have visibly elliptical orbits.

Planet	Eccentricity
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.056
Uranus	0.046
Neptune	0.010
Pluto	0.248

Gravitational attraction causes the planets to move in elliptical orbits around the sun with the sun at one focus. This remarkable property was first observed by Johannes Kepler and was later deduced by Isaac Newton from his inverse square Law of Gravity, using calculus. The orbits of the planets have different eccentricities, but most are nearly circular (see the margin).

Ellipses, like parabolas, have an interesting *reflection property* that leads to a number of practical applications. If a light source is placed at one focus of a reflecting surface with elliptical cross sections, then all the light will be reflected off the surface to the other focus, as shown in Figure 10. This principle, which works for sound waves as well as for light, is used in *lithotripsy*, a treatment for kidney stones. The patient is placed in a tub of water with elliptical cross sections in such a way that the kidney stone is accurately located at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it with minimal damage to surrounding tissue. The patient is spared the trauma of surgery and recovers within days instead of weeks.

The reflection property of ellipses is also used in the construction of *whispering galleries*. Sound coming from one focus bounces off the walls and ceiling of an elliptical room and passes through the other focus. In these rooms even quiet whispers spoken at one focus can be heard clearly at the other. Famous whispering galleries include the National Statuary Hall of the U.S. Capitol in Washington, D.C. (see page 563), and the Mormon Tabernacle in Salt Lake City, Utah.

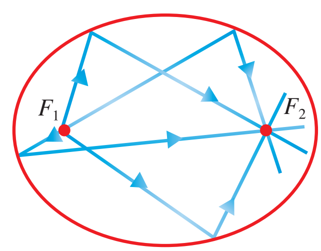


FIGURE 10

7.2 EXERCISES

CONCEPTS

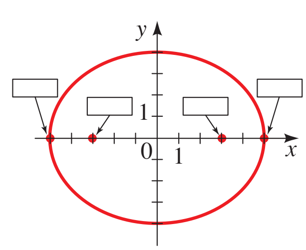
1. An ellipse is the set of all points in the plane for which the ________ of the distances from two fixed points F_1 and F_2 is constant. The points F_1 and F_2 are called the ________ of the ellipse.

2. The graph of the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $a > b > 0$ is an ellipse with vertices (___, ___) and (___, ___) and foci $(\pm c, 0)$, where $c =$ ________. So the graph of $\dfrac{x^2}{5^2} + \dfrac{y^2}{4^2} = 1$ is an ellipse with vertices (___, ___) and (___, ___) and foci (___, ___) and (___, ___).

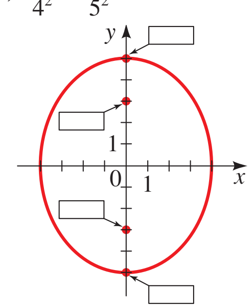
3. The graph of the equation $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ with $a > b > 0$ is an ellipse with vertices (___, ___) and (___, ___) and foci $(0, \pm c)$, where $c =$ ________. So the graph of $\dfrac{x^2}{4^2} + \dfrac{y^2}{5^2} = 1$ is an ellipse with vertices (___, ___) and (___, ___) and foci (___, ___) and (___, ___).

4. Label the vertices and foci on the graphs given for the ellipses in Exercises 2 and 3.

(a) $\dfrac{x^2}{5^2} + \dfrac{y^2}{4^2} = 1$



(b) $\dfrac{x^2}{4^2} + \dfrac{y^2}{5^2} = 1$



SKILLS

5–8 ■ Match the equation with the graphs labeled I–IV. Give reasons for your answers.

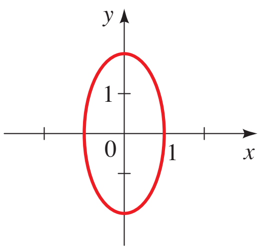
5. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

6. $x^2 + \dfrac{y^2}{9} = 1$

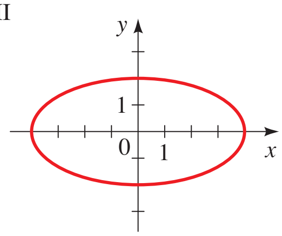
7. $4x^2 + y^2 = 4$

8. $16x^2 + 25y^2 = 400$

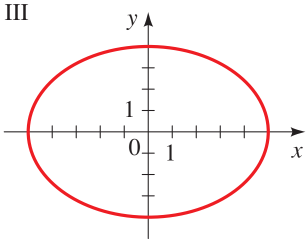
I



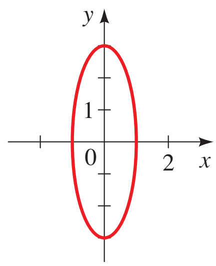
II



III



IV



9–28 ■ An equation of an ellipse is given. **(a)** Find the vertices, foci, and eccentricity of the ellipse. **(b)** Determine the lengths of the major and minor axes. **(c)** Sketch a graph of the ellipse.

9. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

10. $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

11. $\dfrac{x^2}{36} + \dfrac{y^2}{81} = 1$

12. $\dfrac{x^2}{4} + y^2 = 1$

13. $\dfrac{x^2}{49} + \dfrac{y^2}{25} = 1$

14. $\dfrac{x^2}{9} + \dfrac{y^2}{64} = 1$

15. $9x^2 + 4y^2 = 36$

16. $4x^2 + 25y^2 = 100$

17. $x^2 + 4y^2 = 16$

18. $4x^2 + y^2 = 16$

19. $16x^2 + 25y^2 = 1600$

20. $2x^2 + 49y^2 = 98$

21. $3x^2 + y^2 = 9$

22. $x^2 + 3y^2 = 9$

23. $2x^2 + y^2 = 4$

24. $3x^2 + 4y^2 = 12$

25. $x^2 + 4y^2 = 1$

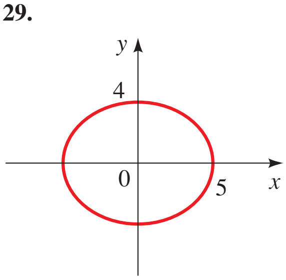
26. $9x^2 + 4y^2 = 1$

27. $x^2 = 4 - 2y^2$

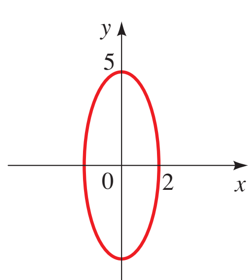
28. $y^2 = 1 - 2x^2$

29–34 ■ Find an equation for the ellipse whose graph is shown.

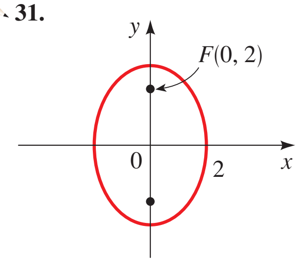
29.



30.



31.



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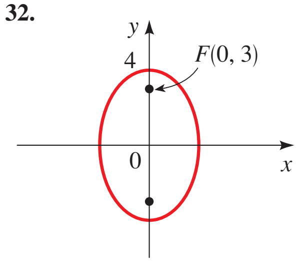

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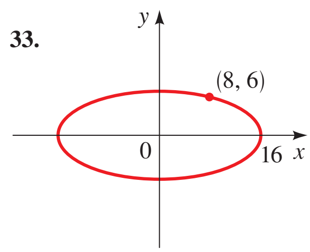

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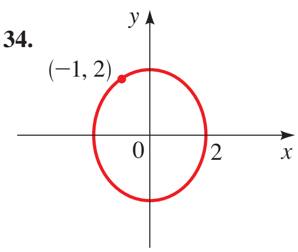


35–38 ■ Use a graphing device to graph the ellipse.

35. $\dfrac{x^2}{25} + \dfrac{y^2}{20} = 1$

36. $x^2 + \dfrac{y^2}{12} = 1$

37. $6x^2 + y^2 = 36$

38. $x^2 + 2y^2 = 8$

39–56 ■ Find an equation for the ellipse that satisfies the given conditions.

39. Foci: $(\pm 4, 0)$, vertices: $(\pm 5, 0)$

40. Foci: $(0, \pm 3)$, vertices: $(0, \pm 5)$

41. Foci: $F(\pm 1, 0)$, vertices $(\pm 2, 0)$

42. Foci: $F(0, \pm 2)$, vertices $(0, \pm 3)$

43. Foci: $F(0, \pm\sqrt{10})$, vertices $(0, \pm 7)$

44. Foci: $F(\pm\sqrt{15}, 0)$, vertices $(\pm 6, 0)$

45. Length of major axis: 4, length of minor axis: 2, foci on y-axis

46. Length of major axis: 6, length of minor axis: 4, foci on x-axis

47. Foci: $(0, \pm 2)$, length of minor axis: 6

48. Foci: $(\pm 5, 0)$, length of major axis: 12

49. Endpoints of major axis: $(\pm 10, 0)$, distance between foci: 6

50. Endpoints of minor axis: $(0, \pm 3)$, distance between foci: 8

51. Length of major axis: 10, foci on x-axis, ellipse passes through the point $(\sqrt{5}, 2)$

52. Length of minor axis: 10, foci on y-axis, ellipse passes through the point $(\sqrt{5}, \sqrt{40})$

53. Eccentricity: $\frac{1}{3}$, foci: $(0, \pm 2)$

54. Eccentricity: 0.75, foci: $(\pm 1.5, 0)$

55. Eccentricity: $\sqrt{3}/2$, foci on y-axis, length of major axis: 4

56. Eccentricity: $\sqrt{5}/3$, foci on x-axis, length of major axis: 12

57–60 ■ Find the intersection points of the pair of ellipses. Sketch the graphs of each pair of equations on the same coordinate axes, and label the points of intersection.

57. $\begin{cases} 4x^2 + y^2 = 4 \\ 4x^2 + 9y^2 = 36 \end{cases}$

58. $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1 \\ \dfrac{x^2}{9} + \dfrac{y^2}{16} = 1 \end{cases}$

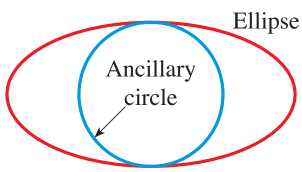
59. $\begin{cases} 100x^2 + 25y^2 = 100 \\ x^2 + \dfrac{y^2}{9} = 1 \end{cases}$

60. $\begin{cases} 25x^2 + 144y^2 = 3600 \\ 144x^2 + 25y^2 = 3600 \end{cases}$

61. The **ancillary circle** of an ellipse is the circle with radius equal to half the length of the minor axis and center the same as the ellipse (see the figure). The ancillary circle is thus the largest circle that can fit within an ellipse.

(a) Find an equation for the ancillary circle of the ellipse $x^2 + 4y^2 = 16$.

(b) For the ellipse and ancillary circle of part (a), show that if (s, t) is a point on the ancillary circle, then $(2s, t)$ is a point on the ellipse.



62. (a) Use a graphing device to sketch the top half (the portion in the first and second quadrants) of the family of ellipses $x^2 + ky^2 = 100$ for $k = 4, 10, 25$, and 50.

(b) What do the members of this family of ellipses have in common? How do they differ?

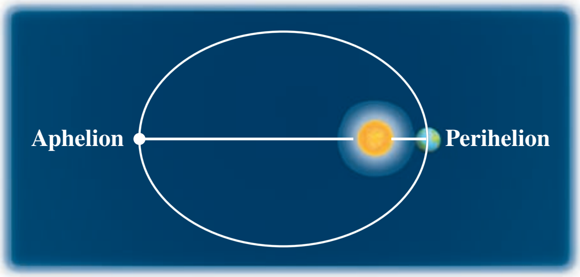
63. If $k > 0$, the following equation represents an ellipse:

$$\frac{x^2}{k} + \frac{y^2}{4 + k} = 1$$

Show that all the ellipses represented by this equation have the same foci, no matter what the value of k.

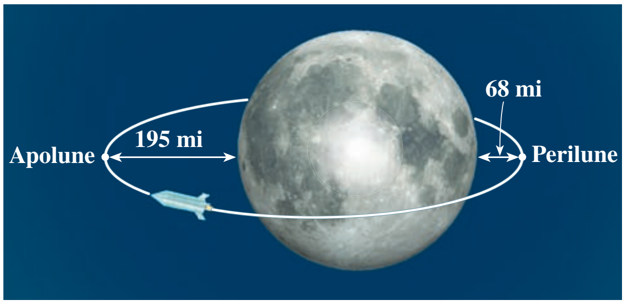
APPLICATIONS

64. Perihelion and Aphelion The planets move around the sun in elliptical orbits with the sun at one focus. The point in the orbit at which the planet is closest to the sun is called **perihelion**, and the point at which it is farthest is called **aphelion**. These points are the vertices of the orbit. The earth's distance from the sun is 147,000,000 km at perihelion and 153,000,000 km at aphelion. Find an equation for the earth's orbit. (Place the origin at the center of the orbit with the sun on the x-axis.)



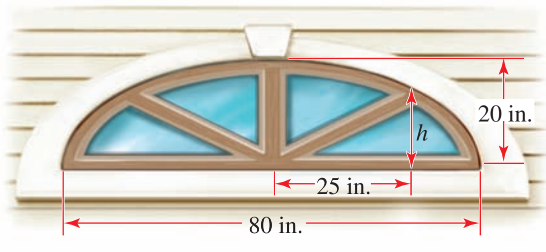
65. The Orbit of Pluto With an eccentricity of 0.25, Pluto's orbit is the most eccentric in the solar system. The length of the minor axis of its orbit is approximately 10,000,000,000 km. Find the distance between Pluto and the sun at perihelion and at aphelion. (See Exercise 64.)

66. Lunar Orbit For an object in an elliptical orbit around the moon, the points in the orbit that are closest to and farthest from the center of the moon are called **perilune** and **apolune**, respectively. These are the vertices of the orbit. The center of the moon is at one focus of the orbit. The *Apollo 11* spacecraft was placed in a lunar orbit with perilune at 68 mi and apolune at 195 mi above the surface of the moon. Assuming that the moon is a sphere of radius 1075 mi, find an equation for the orbit of *Apollo 11*. (Place the coordinate axes so that the origin is at the center of the orbit and the foci are located on the x-axis.)



67. Plywood Ellipse A carpenter wishes to construct an elliptical table top from a sheet of plywood, 4 ft by 8 ft. He will trace out the ellipse using the "thumbtack and string" method illustrated in Figures 2 and 3. What length of string should he use, and how far apart should the tacks be located, if the ellipse is to be the largest possible that can be cut out of the plywood sheet?

68. Sunburst Window A "sunburst" window above a doorway is constructed in the shape of the top half of an ellipse, as shown in the figure. The window is 20 in. tall at its highest point and 80 in. wide at the bottom. Find the height of the window 25 in. from the center of the base.



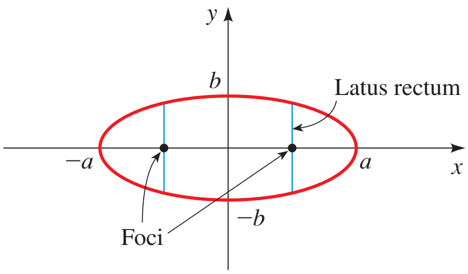
DISCOVERY ■ DISCUSSION ■ WRITING

69. Drawing an Ellipse on a Blackboard Try drawing an ellipse as accurately as possible on a blackboard. How would a piece of string and two friends help this process?

70. Light Cone from a Flashlight A flashlight shines on a wall, as shown in the figure. What is the shape of the boundary of the lighted area? Explain your answer.

71. How Wide Is an Ellipse at Its Foci? A *latus rectum* for an ellipse is a line segment perpendicular to the major axis at a focus, with endpoints on the ellipse, as shown in the figure at the top of the next column. Show that the length of a latus rectum is $2b^2/a$ for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{with } a > b$$



72. Is It an Ellipse? A piece of paper is wrapped around a cylindrical bottle, and then a compass is used to draw a circle on the paper, as shown in the figure. When the paper is laid flat, is the shape drawn on the paper an ellipse? (You don't need to prove your answer, but you might want to do the experiment and see what you get.)

7.3 HYPERBOLAS

LEARNING OBJECTIVES After completing this section, you will be able to:

Find geometric properties of a hyperbola from its equation ▶ Find the equation of a hyperbola from some of its geometric properties

▼ Geometric Definition of a Hyperbola

Although ellipses and hyperbolas have completely different shapes, their definitions and equations are similar. Instead of using the *sum* of distances from two fixed foci, as in the case of an ellipse, we use the *difference* to define a hyperbola.

GEOMETRIC DEFINITION OF A HYPERBOLA

A **hyperbola** is the set of all points in the plane, the difference of whose distances from two fixed points F_1 and F_2 is a constant. (See Figure 1.) These two fixed points are the **foci** of the hyperbola.

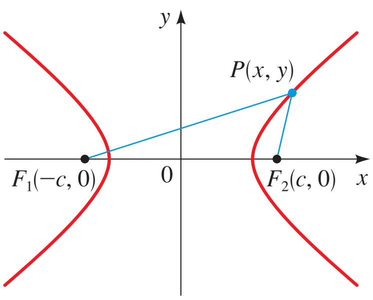


FIGURE 1 P is on the hyperbola if $|d(P, F_1) - d(P, F_2)| = 2a$.

Deriving the Equation of a Hyperbola As in the case of the ellipse, we get the simplest equation for the hyperbola by placing the foci on the x-axis at $(\pm c, 0)$, as shown in Figure 1. By definition, if $P(x, y)$ lies on the hyperbola, then either $d(P, F_1) - d(P, F_2)$ or $d(P, F_2) - d(P, F_1)$ must equal some positive constant, which we call $2a$. Thus we have

$$d(P, F_1) - d(P, F_2) = \pm 2a$$

or

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a$$

Proceeding as we did in the case of the ellipse (Section 7.2), we simplify this to

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$

From triangle PF_1F_2 in Figure 1 we see that $|d(P, F_1) - d(P, F_2)| < 2c$. It follows that $2a < 2c$, or $a < c$. Thus $c^2 - a^2 > 0$, so we can set $b^2 = c^2 - a^2$. We then simplify the last displayed equation to get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is the *equation of the hyperbola*. If we replace x by $-x$ or y by $-y$ in this equation, it remains unchanged, so the hyperbola is symmetric about both the x- and y-axes and about the origin. The x-intercepts are $\pm a$, and the points $(a, 0)$ and $(-a, 0)$ are the **vertices** of the hyperbola. There is no y-intercept, because setting $x = 0$ in the equation of the hyperbola leads to $-y^2 = b^2$, which has no real solution. Furthermore, the equation of the hyperbola implies that

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} + 1 \geq 1$$

so $x^2/a^2 \geq 1$; thus $x^2 \geq a^2$, and hence $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two parts, called its **branches**. The segment joining the two vertices on the separate branches is the **transverse axis** of the hyperbola, and the origin is called its **center**.

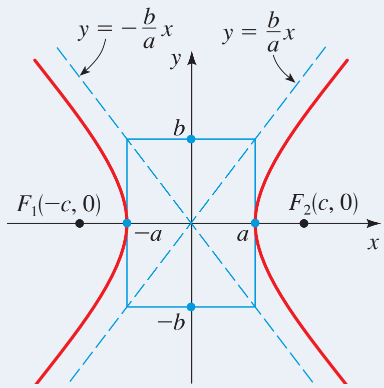
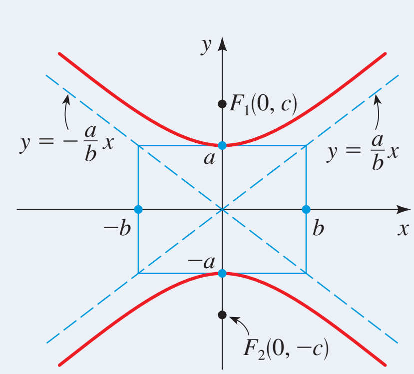
If we place the foci of the hyperbola on the y-axis rather than on the x-axis, this has the effect of reversing the roles of x and y in the derivation of the equation of the hyperbola. This leads to a hyperbola with a vertical transverse axis.

▼ Equations and Graphs of Hyperbolas

The main properties of hyperbolas are listed in the following box.

HYPERBOLA WITH CENTER AT THE ORIGIN

The graph of each of the following equations is a hyperbola with center at the origin and having the given properties.

EQUATION	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$	$\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1 \quad (a > 0, b > 0)$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
TRANSVERSE AXIS	Horizontal, length $2a$	Vertical, length $2a$
ASYMPTOTES	$y = \pm\dfrac{b}{a}x$	$y = \pm\dfrac{a}{b}x$
FOCI	$(\pm c, 0), \quad c^2 = a^2 + b^2$	$(0, \pm c), \quad c^2 = a^2 + b^2$
GRAPH		

Asymptotes of rational functions are discussed in Section 3.7.

The *asymptotes* mentioned in this box are lines that the hyperbola approaches for large values of x and y. To find the asymptotes in the first case in the box, we solve the equation for y to get

$$\begin{aligned} y &= \pm\frac{b}{a}\sqrt{x^2 - a^2} \\ &= \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}} \end{aligned}$$

As x gets large, a^2/x^2 gets closer to zero. In other words, as $x \to \infty$, we have $a^2/x^2 \to 0$. So for large x the value of y can be approximated as $y = \pm(b/a)x$. This shows that these lines are asymptotes of the hyperbola.

Asymptotes are an essential aid for graphing a hyperbola; they help us to determine its shape. A convenient way to find the asymptotes, for a hyperbola with horizontal transverse axis, is to first plot the points $(a, 0)$, $(-a, 0)$, $(0, b)$, and $(0, -b)$. Then sketch horizontal and vertical segments through these points to construct a rectangle, as shown in Figure 2(a). We call this rectangle the **central box** of the hyperbola. The slopes of the diagonals of the central box are $\pm b/a$, so by extending them, we obtain the asymptotes $y = \pm(b/a)x$, as sketched in Figure 2(b). Finally, we plot the vertices and use the asymptotes as a guide in sketching the hyperbola shown in Figure 2(c). (A similar procedure applies to graphing a hyperbola that has a vertical transverse axis.)

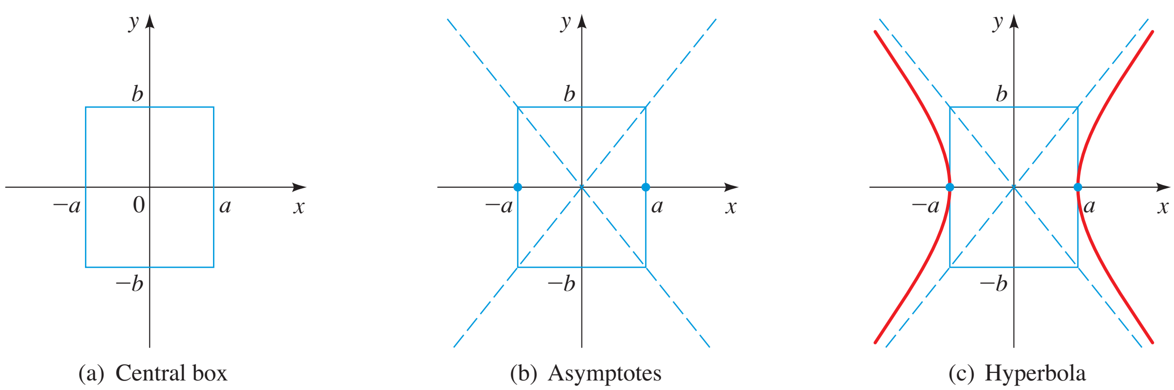


FIGURE 2 Steps in graphing the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

HOW TO SKETCH A HYPERBOLA

1. **Sketch the Central Box.** This is the rectangle centered at the origin, with sides parallel to the axes, that crosses one axis at $\pm a$, the other at $\pm b$.
2. **Sketch the Asymptotes.** These are the lines obtained by extending the diagonals of the central box.
3. **Plot the Vertices.** These are the two x-intercepts or the two y-intercepts.
4. **Sketch the Hyperbola.** Start at a vertex, and sketch a branch of the hyperbola, approaching the asymptotes. Sketch the other branch in the same way.

EXAMPLE 1 | A Hyperbola with Horizontal Transverse Axis

A hyperbola has the equation

$$9x^2 - 16y^2 = 144$$

(a) Find the vertices, foci, length of the transverse axis, and asymptotes, and sketch the graph.

(b) Draw the graph using a graphing calculator.

SOLUTION

(a) First we divide both sides of the equation by 144 to put it into standard form:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Because the x^2-term is positive, the hyperbola has a horizontal transverse axis; its vertices and foci are on the x-axis. Since $a^2 = 16$ and $b^2 = 9$, we get $a = 4$, $b = 3$, and $c = \sqrt{16 + 9} = 5$. Thus we have

VERTICES	$(\pm 4, 0)$
FOCI	$(\pm 5, 0)$
ASYMPTOTES	$y = \pm \frac{3}{4}x$

The length of the transverse axis is $2a = 8$. After sketching the central box and asymptotes, we complete the sketch of the hyperbola as in Figure 3(a).

Note that the equation of a hyperbola does not define y as a function of x (see page 190). That's why we need to graph two functions to graph a hyperbola.

(b) To draw the graph using a graphing calculator, we need to solve for y:

$$9x^2 - 16y^2 = 144$$

$$-16y^2 = -9x^2 + 144 \qquad \text{Subtract } 9x^2$$

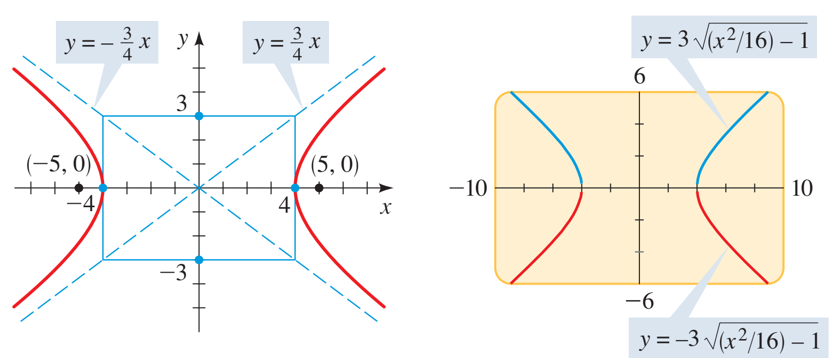
$$y^2 = 9\left(\frac{x^2}{16} - 1\right) \qquad \text{Divide by } -16 \text{ and factor } 9$$

$$y = \pm 3\sqrt{\frac{x^2}{16} - 1} \qquad \text{Take square roots}$$

To obtain the graph of the hyperbola, we graph the functions

$$y = 3\sqrt{(x^2/16) - 1} \qquad \text{and} \qquad y = -3\sqrt{(x^2/16) - 1}$$

as shown in Figure 3(b).



(a) (b)

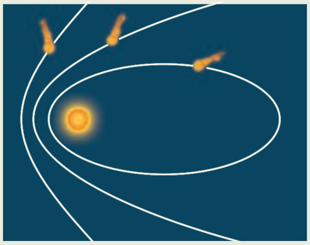
FIGURE 3
$9x^2 - 16y^2 = 144$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **9** AND **33**

EXAMPLE 2 | A Hyperbola with Vertical Transverse Axis

Find the vertices, foci, length of the transverse axis, and asymptotes of the hyperbola, and sketch its graph.

$$x^2 - 9y^2 + 9 = 0$$



Paths of Comets

The path of a comet is an ellipse, a parabola, or a hyperbola with the sun at a focus. This fact can be proved by using calculus and Newton's laws of motion.* If the path is a parabola or a hyperbola, the comet will never return. If the path is an ellipse, it can be determined precisely when and where the comet can be seen again. Halley's comet has an elliptical path and returns every 75 years; it was last seen in 1987. The brightest comet of the 20th century was comet Hale-Bopp, seen in 1997. Its orbit is a very eccentric ellipse; it is expected to return to the inner solar system around the year 4377.

*James Stewart, *Calculus,* 7th ed. (Belmont, CA: Brooks/Cole, 2012), pages 892 and 896.

SOLUTION We begin by writing the equation in the standard form for a hyperbola:

$$x^2 - 9y^2 = -9$$

$$y^2 - \frac{x^2}{9} = 1 \qquad \text{Divide by } -9$$

Because the y^2-term is positive, the hyperbola has a vertical transverse axis; its foci and vertices are on the y-axis. Since $a^2 = 1$ and $b^2 = 9$, we get $a = 1$, $b = 3$, and $c = \sqrt{1 + 9} = \sqrt{10}$. Thus we have

VERTICES	$(0, \pm 1)$
FOCI	$(0, \pm\sqrt{10})$
ASYMPTOTES	$y = \pm\frac{1}{3}x$

The length of the transverse axis is $2a = 2$. We sketch the central box and asymptotes, then complete the graph, as shown in Figure 4(a). We can also draw the graph using a graphing calculator, as shown in Figure 4(b).

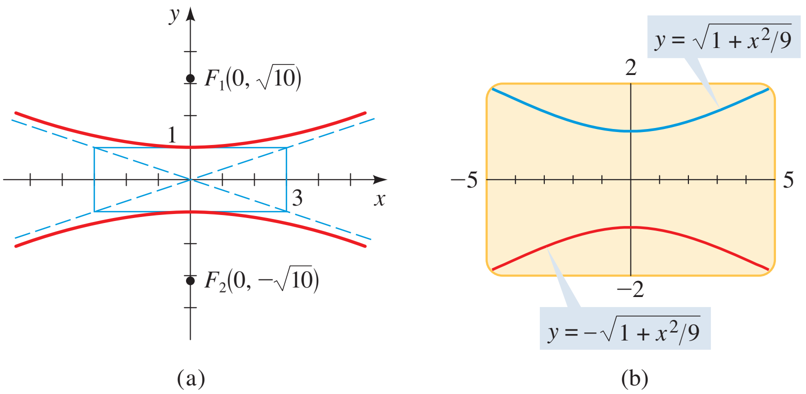


FIGURE 4
$x^2 - 9y^2 + 9 = 0$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **21** AND **35** ■

EXAMPLE 3 | Finding the Equation of a Hyperbola from Its Vertices and Foci

Find the equation of the hyperbola with vertices $(\pm 3, 0)$ and foci $(\pm 4, 0)$. Sketch the graph.

SOLUTION Since the vertices are on the x-axis, the hyperbola has a horizontal transverse axis. Its equation is of the form

$$\frac{x^2}{3^2} - \frac{y^2}{b^2} = 1$$

We have $a = 3$ and $c = 4$. To find b, we use the relation $a^2 + b^2 = c^2$:

$$3^2 + b^2 = 4^2$$

$$b^2 = 4^2 - 3^2 = 7$$

$$b = \sqrt{7}$$

Thus the equation of the hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{7} = 1$$

The graph is shown in Figure 5.

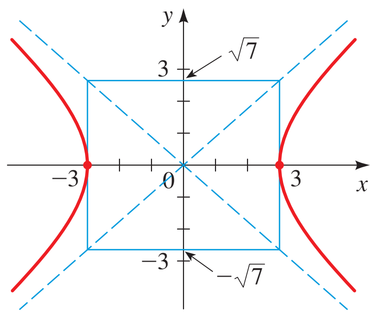


FIGURE 5
$\frac{x^2}{9} - \frac{y^2}{7} = 1$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **27** AND **37** ■

EXAMPLE 4 | Finding the Equation of a Hyperbola from Its Vertices and Asymptotes

Find the equation and the foci of the hyperbola with vertices $(0, \pm 2)$ and asymptotes $y = \pm 2x$. Sketch the graph.

SOLUTION Since the vertices are on the y-axis, the hyperbola has a vertical transverse axis with $a = 2$. From the asymptote equation we see that $a/b = 2$. Since $a = 2$, we get $2/b = 2$, so $b = 1$. Thus the equation of the hyperbola is

$$\frac{y^2}{4} - x^2 = 1$$

To find the foci, we calculate $c^2 = a^2 + b^2 = 2^2 + 1^2 = 5$, so $c = \sqrt{5}$. Thus the foci are $(0, \pm\sqrt{5})$. The graph is shown in Figure 6.

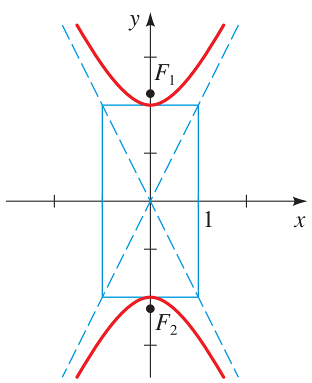


FIGURE 6
$\frac{y^2}{4} - x^2 = 1$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **31** AND **41** ■

Like parabolas and ellipses, hyperbolas have an interesting *reflection property*. Light aimed at one focus of a hyperbolic mirror is reflected toward the other focus, as shown in Figure 7. This property is used in the construction of Cassegrain-type telescopes. A hyperbolic mirror is placed in the telescope tube so that light reflected from the primary parabolic reflector is aimed at one focus of the hyperbolic mirror. The light is then refocused at a more accessible point below the primary reflector (Figure 8).

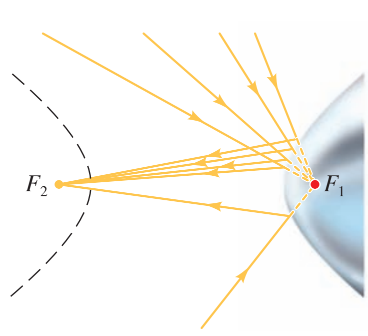


FIGURE 7 Reflection property of hyperbolas

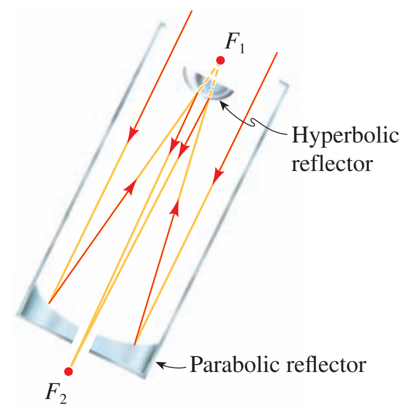


FIGURE 8 Cassegrain-type telescope

The LORAN (LOng RAnge Navigation) system was used until the early 1990s; it has now been superseded by the GPS system (see page 442). In the LORAN system, hyperbolas are used onboard a ship to determine its location. In Figure 9, radio stations at A and B transmit signals simultaneously for reception by the ship at P. The onboard computer converts the time difference in reception of these signals into a distance difference $d(P, A) - d(P, B)$. From the definition of a hyperbola this locates the ship on one branch of a hyperbola with foci at A and B (sketched in black in the figure). The same procedure is carried out with two other radio stations at C and D, and this locates the ship on a second hyperbola (shown in red in the figure). (In practice, only three stations are needed because one station can be used as a focus for both hyperbolas.) The coordinates of the intersection point of these two hyperbolas, which can be calculated precisely by the computer, give the location of P.

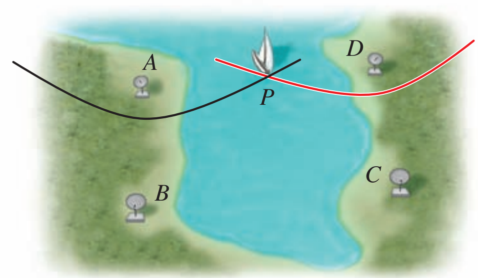


FIGURE 9 LORAN system for finding the location of a ship

7.3 EXERCISES

CONCEPTS

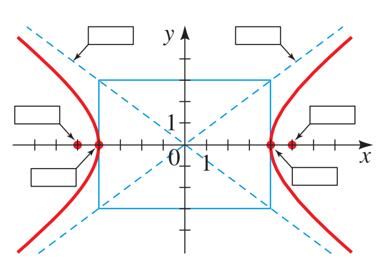
1. A hyperbola is the set of all points in the plane for which the ________ of the distances from two fixed points F_1 and F_2 is constant. The points F_1 and F_2 are called the ________ of the hyperbola.

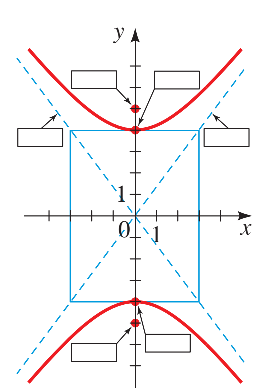
2. The graph of the equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with $a > 0, b > 0$ is a hyperbola with ________ (horizontal/vertical) transverse axis, vertices (___, ___) and (___, ___) and foci $(\pm c, 0)$, where $c =$ ________. So the graph of $\dfrac{x^2}{4^2} - \dfrac{y^2}{3^2} = 1$ is a hyperbola with vertices (___, ___) and (___, ___) and foci (___, ___) and (___, ___).

3. The graph of the equation $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ with $a > 0, b > 0$ is a hyperbola with ________ (horizontal/vertical) transverse axis, vertices (___, ___) and (___, ___) and foci $(0, \pm c)$, where $c =$ ________. So the graph of $\dfrac{y^2}{4^2} - \dfrac{x^2}{3^2} = 1$ is a hyperbola with vertices (___, ___) and (___, ___) and foci (___, ___) and (___, ___).

4. Label the vertices, foci, and asymptotes on the graphs given for the hyperbolas in Exercises 2 and 3.

(a) $\dfrac{x^2}{4^2} - \dfrac{y^2}{3^2} = 1$ **(b)** $\dfrac{y^2}{4^2} - \dfrac{x^2}{3^2} = 1$





SKILLS

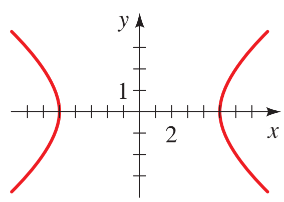
5–8 ■ Match the equation with the graphs labeled I–IV. Give reasons for your answers.

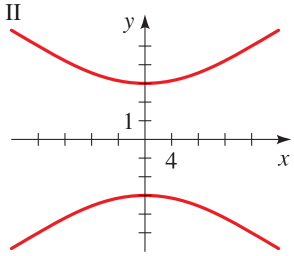
5. $\dfrac{x^2}{4} - y^2 = 1$

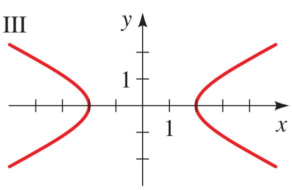
6. $y^2 - \dfrac{x^2}{9} = 1$

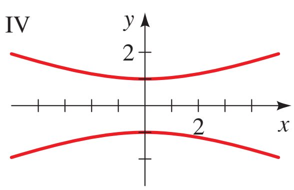
7. $16y^2 - x^2 = 144$

8. $9x^2 - 25y^2 = 225$

I


II


III


IV


9–26 ■ An equation of a hyperbola is given. **(a)** Find the vertices, foci, and asymptotes of the hyperbola. **(b)** Determine the length of the transverse axis. **(c)** Sketch a graph of the hyperbola.

9. $\dfrac{x^2}{4} - \dfrac{y^2}{16} = 1$

10. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$

11. $\dfrac{y^2}{36} - \dfrac{x^2}{4} = 1$

12. $\dfrac{x^2}{9} - \dfrac{y^2}{64} = 1$

13. $y^2 - \dfrac{x^2}{25} = 1$

14. $\dfrac{x^2}{2} - y^2 = 1$

15. $x^2 - y^2 = 1$

16. $\dfrac{x^2}{16} - \dfrac{y^2}{12} = 1$

17. $9x^2 - 4y^2 = 36$

18. $25y^2 - 9x^2 = 225$

19. $4y^2 - 9x^2 = 144$

20. $y^2 - 25x^2 = 100$

21. $x^2 - 4y^2 - 8 = 0$

22. $3y^2 - x^2 - 9 = 0$

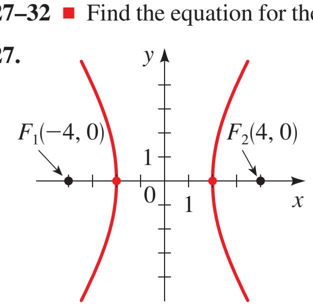
23. $x^2 - y^2 + 4 = 0$

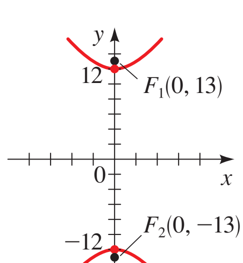
24. $x^2 - 3y^2 + 12 = 0$

25. $4y^2 - x^2 = 1$

26. $9x^2 - 16y^2 = 1$

27–32 ■ Find the equation for the hyperbola whose graph is shown.

27.


28.


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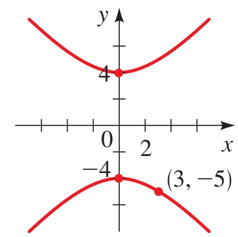

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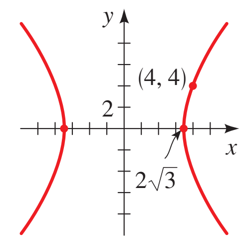
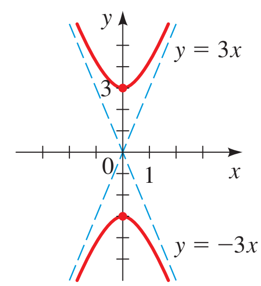
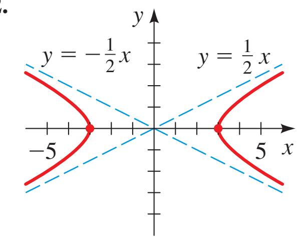


31.


32.


33–36 ■ Use a graphing device to graph the hyperbola.

33. $x^2 - 2y^2 = 8$

34. $3y^2 - 4x^2 = 24$

35. $\dfrac{y^2}{2} - \dfrac{x^2}{6} = 1$

36. $\dfrac{x^2}{100} - \dfrac{y^2}{64} = 1$

37–48 ■ Find an equation for the hyperbola that satisfies the given conditions.

37. Foci: $(\pm 5, 0)$, vertices: $(\pm 3, 0)$

38. Foci: $(0, \pm 10)$, vertices: $(0, \pm 8)$

39. Foci: $(0, \pm 2)$, vertices: $(0, \pm 1)$

40. Foci: $(\pm 6, 0)$, vertices: $(\pm 2, 0)$

41. Vertices: $(\pm 1, 0)$, asymptotes: $y = \pm 5x$

42. Vertices: $(0, \pm 6)$, asymptotes: $y = \pm\frac{1}{3}x$

43. Vertices: $(0, \pm 6)$, hyperbola passes through $(-5, 9)$

44. Vertices: $(\pm 2, 0)$, hyperbola passes through $(3, \sqrt{30})$

45. Asymptotes: $y = \pm x$, hyperbola passes through $(5, 3)$

46. Asymptotes: $y = \pm x$, hyperbola passes through $(1, 2)$

47. Foci: $(\pm 5, 0)$, length of transverse axis: 6

48. Foci: $(0, \pm 1)$, length of transverse axis: 1

49. **(a)** Show that the asymptotes of the hyperbola $x^2 - y^2 = 5$ are perpendicular to each other.
(b) Find an equation for the hyperbola with foci $(\pm c, 0)$ and with asymptotes perpendicular to each other.

50. The hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

are said to be **conjugate** to each other.

(a) Show that the hyperbolas

$$x^2 - 4y^2 + 16 = 0 \qquad \text{and} \qquad 4y^2 - x^2 + 16 = 0$$

are conjugate to each other, and sketch their graphs on the same coordinate axes.

(b) What do the hyperbolas of part (a) have in common?

(c) Show that any pair of conjugate hyperbolas have the relationship you discovered in part (b).

51. In the derivation of the equation of the hyperbola at the beginning of this section we said that the equation

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a$$

simplifies to

$$(c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2)$$

Supply the steps needed to show this.

52. **(a)** For the hyperbola

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

determine the values of a, b, and c, and find the coordinates of the foci F_1 and F_2.

(b) Show that the point $P(5, \frac{16}{3})$ lies on this hyperbola.

(c) Find $d(P, F_1)$ and $d(P, F_2)$.

(d) Verify that the difference between $d(P, F_1)$ and $d(P, F_2)$ is $2a$.

53. Hyperbolas are called **confocal** if they have the same foci.

(a) Show that the hyperbolas

$$\frac{y^2}{k}-\frac{x^2}{16-k}=1 \quad \text{with } 0<k<16$$

are confocal.

(b) Use a graphing device to draw the top branches of the family of hyperbolas in part (a) for $k = 1, 4, 8$, and 12. How does the shape of the graph change as k increases?

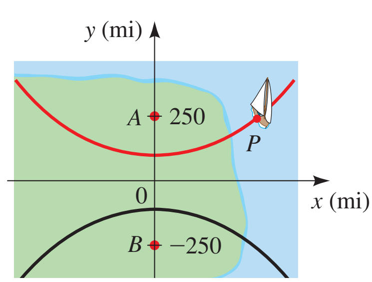
APPLICATIONS

54. Navigation In the figure, the LORAN stations at A and B are 500 mi apart, and the ship at P receives station A's signal 2640 microseconds (μs) before it receives the signal from station B.

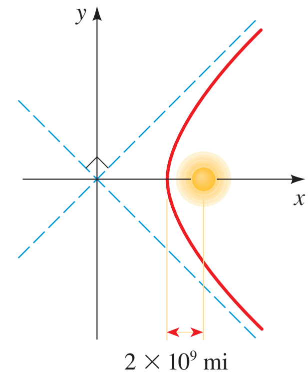
(a) Assuming that radio signals travel at 980 ft/μs, find $d(P, A) - d(P, B)$.

(b) Find an equation for the branch of the hyperbola indicated in red in the figure. (Use miles as the unit of distance.)

(c) If A is due north of B and if P is due east of A, how far is P from A?



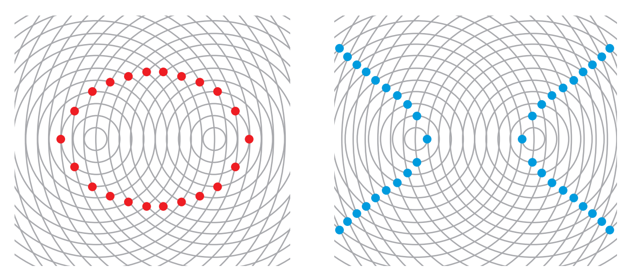
55. Comet Trajectories Some comets, such as Halley's comet, are a permanent part of the solar system, traveling in elliptical orbits around the sun. Other comets pass through the solar system only once, following a hyperbolic path with the sun at a focus. The figure at the top of the next column shows the path of such a comet. Find an equation for the path, assuming that the closest the comet comes to the sun is 2×10^9 mi and that the path the comet was taking before it neared the solar system is at a right angle to the path it continues on after leaving the solar system.



56. Ripples in Pool Two stones are dropped simultaneously into a calm pool of water. The crests of the resulting waves form equally spaced concentric circles, as shown in the figures. The waves interact with each other to create certain interference patterns.

(a) Explain why the red dots lie on an ellipse.

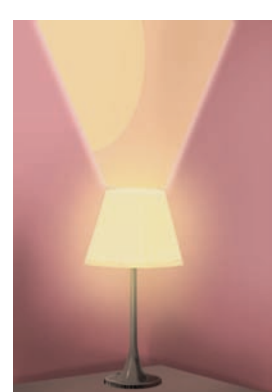
(b) Explain why the blue dots lie on a hyperbola.



DISCOVERY ▪ DISCUSSION ▪ WRITING

57. Hyperbolas in the Real World Several examples of the uses of hyperbolas are given in the text. Find other situations in real life in which hyperbolas occur. Consult a scientific encyclopedia in the reference section of your library, or search the Internet.

58. Light from a Lamp The light from a lamp forms a lighted area on a wall, as shown in the figure. Why is the boundary of this lighted area a hyperbola? How can one hold a flashlight so that its beam forms a hyperbola on the ground?



7.4 Shifted Conics

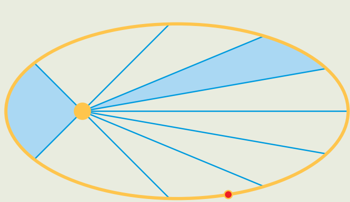
LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find geometric properties of a shifted conic from its equation ▶ Find the equation of a shifted conic from some of its geometric properties

JOHANNES KEPLER (1571–1630) was the first to give a correct description of the motion of the planets. The cosmology of his time postulated complicated systems of circles moving on circles to describe these motions. Kepler sought a simpler and more harmonious description. As the official astronomer at the imperial court in Prague, he studied the astronomical observations of the Danish astronomer Tycho Brahe, whose data were the most accurate available at the time. After numerous attempts to find a theory, Kepler made the momentous discovery that the orbits of the planets are elliptical. His three great laws of planetary motion are

1. The orbit of each planet is an ellipse with the sun at one focus.
2. The line segment that joins the sun to a planet sweeps out equal areas in equal time (see the figure).
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

His formulation of these laws is perhaps the most impressive deduction from empirical data in the history of science.



In the preceding sections we studied parabolas with vertices at the origin and ellipses and hyperbolas with centers at the origin. We restricted ourselves to these cases because these equations have the simplest form. In this section we consider conics whose vertices and centers are not necessarily at the origin, and we determine how this affects their equations.

▼ Shifting Graphs of Equations

In Section 2.5 we studied transformations of functions that have the effect of shifting their graphs. In general, for any equation in x and y, if we replace x by $x - h$ or by $x + h$, the graph of the new equation is simply the old graph shifted horizontally; if y is replaced by $y - k$ or by $y + k$, the graph is shifted vertically. The following box gives the details.

SHIFTING GRAPHS OF EQUATIONS

If h and k are positive real numbers, then replacing x by $x - h$ or by $x + h$ and replacing y by $y - k$ or by $y + k$ has the following effect(s) on the graph of any equation in x and y.

Replacement	How the graph is shifted
1. x replaced by $x - h$	Right h units
2. x replaced by $x + h$	Left h units
3. y replaced by $y - k$	Upward k units
4. y replaced by $y + k$	Downward k units

▼ Shifted Ellipses

Let's apply horizontal and vertical shifting to the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

whose graph is shown in Figure 1. If we shift it so that its center is at the point (h, k) instead of at the origin, then its equation becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

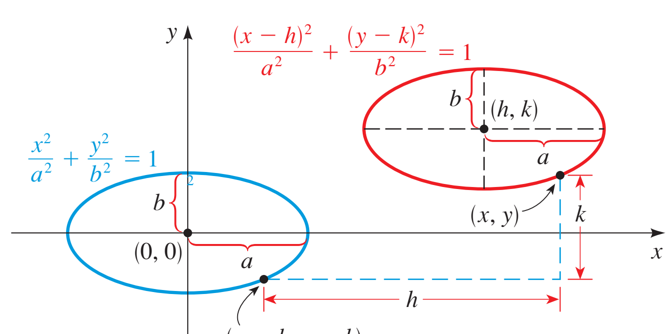


FIGURE 1 Shifted ellipse

EXAMPLE 1 | Sketching the Graph of a Shifted Ellipse

Sketch a graph of the ellipse

$$\frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1$$

and determine the coordinates of the foci.

SOLUTION The ellipse

$$\frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1 \qquad \text{Shifted ellipse}$$

is shifted so that its center is at $(-1, 2)$. It is obtained from the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad \text{Ellipse with center at origin}$$

by shifting it left 1 unit and upward 2 units. The endpoints of the minor and major axes of the ellipse with center at the origin are $(2, 0)$, $(-2, 0)$, $(0, 3)$, $(0, -3)$. We apply the required shifts to these points to obtain the corresponding points on the shifted ellipse:

$$\begin{aligned}
(2, 0) &\rightarrow (2 - 1, 0 + 2) = (1, 2) \\
(-2, 0) &\rightarrow (-2 - 1, 0 + 2) = (-3, 2) \\
(0, 3) &\rightarrow (0 - 1, 3 + 2) = (-1, 5) \\
(0, -3) &\rightarrow (0 - 1, -3 + 2) = (-1, -1)
\end{aligned}$$

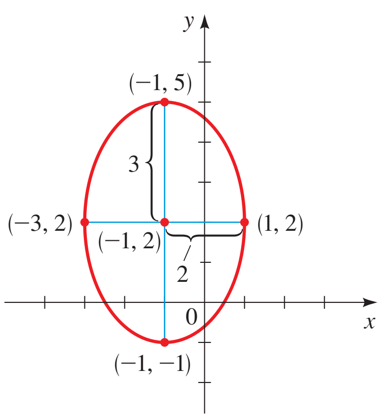


FIGURE 2
$\frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1$

This helps us sketch the graph in Figure 2.

To find the foci of the shifted ellipse, we first find the foci of the ellipse with center at the origin. Since $a^2 = 9$ and $b^2 = 4$, we have $c^2 = 9 - 4 = 5$, so $c = \sqrt{5}$. So the foci are $(0, \pm\sqrt{5})$. Shifting left 1 unit and upward 2 units, we get

$$\begin{aligned}
(0, \sqrt{5}) &\rightarrow (0 - 1, \sqrt{5} + 2) = (-1, 2 + \sqrt{5}) \\
(0, -\sqrt{5}) &\rightarrow (0 - 1, -\sqrt{5} + 2) = (-1, 2 - \sqrt{5})
\end{aligned}$$

Thus the foci of the shifted ellipse are

$$(-1, 2 + \sqrt{5}) \qquad \text{and} \qquad (-1, 2 - \sqrt{5})$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **7**

EXAMPLE 2 | Finding the Equation of a Shifted Ellipse

The vertices of an ellipse are $(-7, 3)$ and $(3, 3)$, and the foci are $(-6, 3)$ and $(2, 3)$. Find the equation for the ellipse, and sketch its graph.

SOLUTION The center of the ellipse is the midpoint of the line segment between the vertices. By the Midpoint Formula the center is

The Midpoint Formula is given on page 76.

$$\left(\frac{-7+3}{2}, \frac{3+3}{2}\right) = (-2, 3) \qquad \text{Center}$$

Since the vertices lie on a horizontal line, the major axis is horizontal. The length of the major axis is $3 - (-7) = 10$, so $a = 5$. The distance between the foci is $2 - (-6) = 8$, so $c = 4$. Since $c^2 = a^2 - b^2$, we have

$$\begin{aligned}
4^2 &= 5^2 - b^2 && c = 4, a = 5 \\
b^2 &= 25 - 16 = 9 && \text{Solve for } b^2
\end{aligned}$$

Thus the equation of the ellipse is

$$\frac{(x+2)^2}{25} + \frac{(y-3)^2}{9} = 1 \qquad \text{Equation of shifted ellipse}$$

The graph is shown in Figure 3.

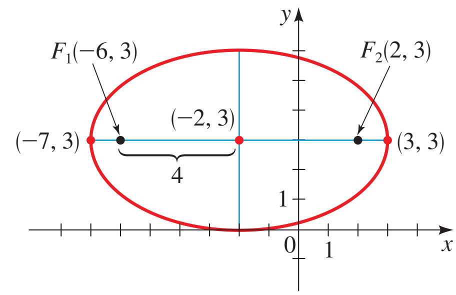
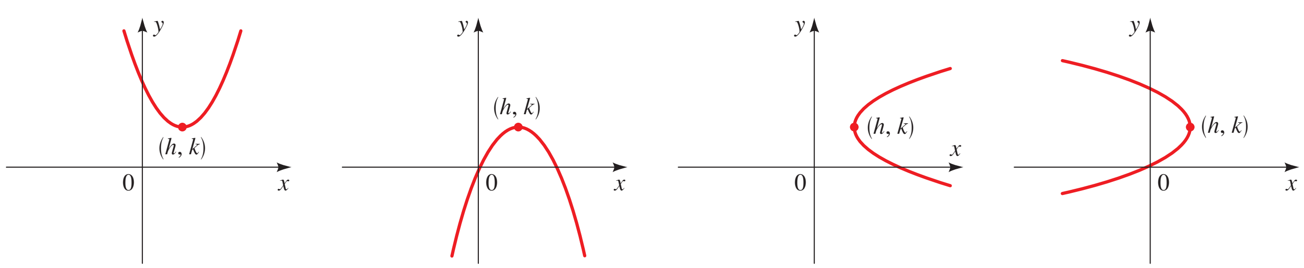


FIGURE 3 Graph of $\frac{(x+2)^2}{25} + \frac{(y-3)^2}{9} = 1$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **35**

Shifted Parabolas

Applying shifts to parabolas leads to the equations and graphs shown in Figure 4.



(a) $(x-h)^2 = 4p(y-k)$
$p > 0$

(b) $(x-h)^2 = 4p(y-k)$
$p < 0$

(c) $(y-k)^2 = 4p(x-h)$
$p > 0$

(d) $(y-k)^2 = 4p(x-h)$
$p < 0$

FIGURE 4 Shifted parabolas

EXAMPLE 3 | Graphing a Shifted Parabola

Determine the vertex, focus, and directrix, and sketch a graph of the parabola.

$$x^2 - 4x = 8y - 28$$

SOLUTION We complete the square in x to put this equation into one of the forms in Figure 4.

$$x^2 - 4x + 4 = 8y - 28 + 4 \qquad \text{Add 4 to complete the square}$$

$$(x-2)^2 = 8y - 24 \qquad \text{Perfect square}$$

$$(x-2)^2 = 8(y-3) \qquad \text{Shifted parabola}$$

This parabola opens upward with vertex at $(2, 3)$. It is obtained from the parabola

$$x^2 = 8y \qquad \text{Parabola with vertex at origin}$$

by shifting right 2 units and upward 3 units. Since $4p = 8$, we have $p = 2$, so the focus is 2 units above the vertex and the directrix is 2 units below the vertex. Thus the focus is $(2, 5)$, and the directrix is $y = 1$. The graph is shown in Figure 5.

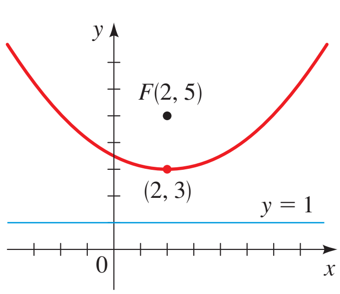
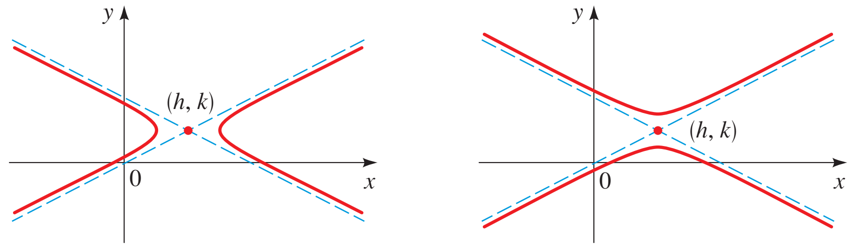


FIGURE 5
$x^2 - 4x = 8y - 28$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **13** AND **19**

Shifted Hyperbolas

Applying shifts to hyperbolas leads to the equations and graphs shown in Figure 6.



(a) $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ (b) $-\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$

FIGURE 6 Shifted hyperbolas

EXAMPLE 4 | Graphing a Shifted Hyperbola

A shifted conic has the equation

$$9x^2 - 72x - 16y^2 - 32y = 16$$

(a) Complete the square in x and y to show that the equation represents a hyperbola.

(b) Find the center, vertices, foci, and asymptotes of the hyperbola, and sketch its graph.

(c) Draw the graph using a graphing calculator.

SOLUTION

(a) We complete the squares in both x and y:

$$9(x^2 - 8x \qquad) - 16(y^2 + 2y \qquad) = 16 \qquad \text{Group terms and factor}$$

$$9(x^2 - 8x + 16) - 16(y^2 + 2y + 1) = 16 + 9 \cdot 16 - 16 \cdot 1 \qquad \text{Complete the squares}$$

$$9(x-4)^2 - 16(y+1)^2 = 144 \qquad \text{Divide this by 144}$$

$$\frac{(x-4)^2}{16} - \frac{(y+1)^2}{9} = 1 \qquad \text{Shifted hyperbola}$$

Comparing this to Figure 6(a), we see that this is the equation of a shifted hyperbola.

(b) The shifted hyperbola has center $(4, -1)$ and a horizontal transverse axis.

$$\text{CENTER} \qquad (4, -1)$$

Its graph will have the same shape as the unshifted hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \qquad \text{Hyperbola with center at origin}$$

Since $a^2 = 16$ and $b^2 = 9$, we have $a = 4$, $b = 3$, and $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$. Thus the foci lie 5 units to the left and to the right of the center, and the vertices lie 4 units to either side of the center.

$$\text{FOCI} \qquad (-1, -1) \quad \text{and} \quad (9, -1)$$

$$\text{VERTICES} \qquad (0, -1) \quad \text{and} \quad (8, -1)$$

The asymptotes of the unshifted hyperbola are $y = \pm\frac{3}{4}x$, so the asymptotes of the shifted hyperbola are found as follows.

$$\text{ASYMPTOTES} \qquad y + 1 = \pm\tfrac{3}{4}(x - 4)$$

$$y + 1 = \pm\tfrac{3}{4}x \mp 3$$

$$y = \tfrac{3}{4}x - 4 \qquad \text{and} \qquad y = -\tfrac{3}{4}x + 2$$

To help us sketch the hyperbola, we draw the central box; it extends 4 units left and right from the center and 3 units upward and downward from the center. We then draw the asymptotes and complete the graph of the shifted hyperbola as shown in Figure 7(a).

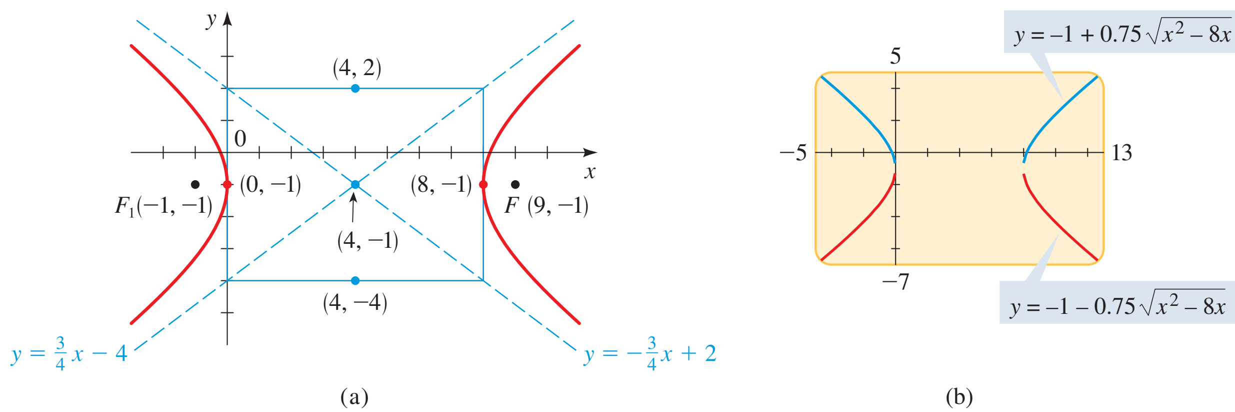


FIGURE 7 $9x^2 - 72x - 16y^2 - 32y = 16$

(c) To draw the graph using a graphing calculator, we need to solve for y. The given equation is a quadratic equation in y, so we use the Quadratic Formula to solve for y. Writing the equation in the form

$$16y^2 + 32y - 9x^2 + 72x + 16 = 0$$

Note that the equation of a hyperbola does not define y as a function of x (see page 190). That's why we need to graph two functions to graph a hyperbola.

we get

$$\begin{aligned} y &= \frac{-32 \pm \sqrt{32^2 - 4(16)(-9x^2 + 72x + 16)}}{2(16)} && \text{Quadratic Formula} \\ &= \frac{-32 \pm \sqrt{576x^2 - 4608x}}{32} && \text{Expand} \\ &= \frac{-32 \pm 24\sqrt{x^2 - 8x}}{32} && \text{Factor 576 from under the radical} \\ &= -1 \pm \tfrac{3}{4}\sqrt{x^2 - 8x} && \text{Simplify} \end{aligned}$$

To obtain the graph of the hyperbola, we graph the functions

$$y = -1 + 0.75\sqrt{x^2 - 8x}$$

and

$$y = -1 - 0.75\sqrt{x^2 - 8x}$$

as shown in Figure 7(b).

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **21, 27** AND **59**

The General Equation of a Shifted Conic

If we expand and simplify the equations of any of the shifted conics illustrated in Figures 1, 4, and 6, then we will always obtain an equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where A and C are not both 0. Conversely, if we begin with an equation of this form, then we can complete the square in x and y to see which type of conic section the equation represents. In some cases the graph of the equation turns out to be just a pair of lines or a single point, or there might be no graph at all. These cases are called **degenerate conics**. If the equation is not degenerate, then we can tell whether it represents a parabola, an ellipse, or a hyperbola simply by examining the signs of A and C, as described in the following box.

GENERAL EQUATION OF A SHIFTED CONIC

The graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where A and C are not both 0, is a conic or a degenerate conic. In the nondegenerate cases the graph is

1. a parabola if A or C is 0,
2. an ellipse if A and C have the same sign (or a circle if $A = C$),
3. a hyperbola if A and C have opposite signs.

EXAMPLE 5 | An Equation That Leads to a Degenerate Conic

Sketch the graph of the equation

$$9x^2 - y^2 + 18x + 6y = 0$$

SOLUTION Because the coefficients of x^2 and y^2 are of opposite sign, this equation looks as if it should represent a hyperbola (like the equation of Example 4). To see whether this is in fact the case, we complete the squares:

$$9(x^2 + 2x \qquad) - (y^2 - 6y \qquad) = 0 \qquad \text{Group terms and factor 9}$$

$$9(x^2 + 2x + 1) - (y^2 - 6y + 9) = 0 + 9 \cdot 1 - 9 \qquad \text{Complete the squares}$$

$$9(x + 1)^2 - (y - 3)^2 = 0 \qquad \text{Factor}$$

$$(x + 1)^2 - \frac{(y - 3)^2}{9} = 0 \qquad \text{Divide by 9}$$

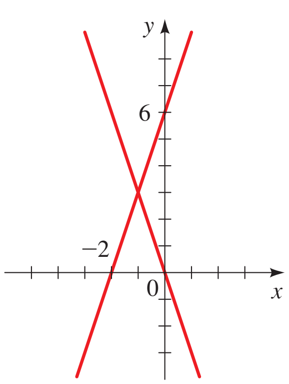


FIGURE 8
$9x^2 - y^2 + 18x + 6y = 0$

For this to fit the form of the equation of a hyperbola, we would need a nonzero constant to the right of the equal sign. In fact, further analysis shows that this is the equation of a pair of intersecting lines:

$$(y - 3)^2 = 9(x + 1)^2$$

$$y - 3 = \pm 3(x + 1) \qquad \text{Take square roots}$$

$$y = 3(x + 1) + 3 \qquad \text{or} \qquad y = -3(x + 1) + 3$$

$$y = 3x + 6 \qquad\qquad y = -3x$$

These lines are graphed in Figure 8.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **53**

Because the equation in Example 5 looked at first glance like the equation of a hyperbola but, in fact, turned out to represent simply a pair of lines, we refer to its graph as a **degenerate hyperbola**. Degenerate ellipses and parabolas can also arise when we complete the square(s) in an equation that seems to represent a conic. For example, the equation

$$4x^2 + y^2 - 8x + 2y + 6 = 0$$

looks as if it should represent an ellipse, because the coefficients of x^2 and y^2 have the same sign. But completing the squares leads to

$$(x - 1)^2 + \frac{(y + 1)^2}{4} = -\frac{1}{4}$$

which has no solution at all (since the sum of two squares cannot be negative). This equation is therefore degenerate.

7.4 EXERCISES

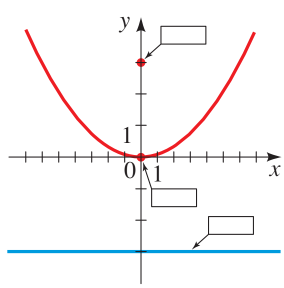
CONCEPTS

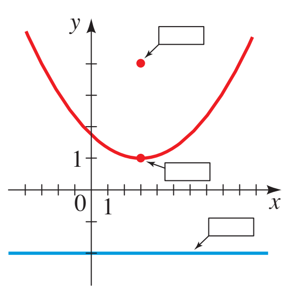
1. Suppose we want to graph an equation in x and y.

(a) If we replace x by $x - 3$, the graph of the equation is shifted to the ________ by 3 units. If we replace x by $x + 3$, the graph of the equation is shifted to the ________ by 3 units.

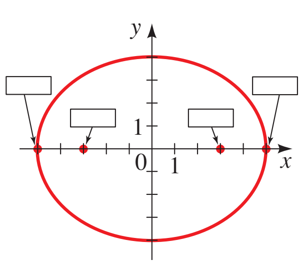
(b) If we replace y by $y - 1$, the graph of the equation is shifted ________ by 1 unit. If we replace y by $y + 1$, the graph of the equation is shifted ________ by 1 unit.

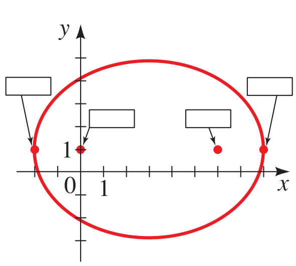
2. The graphs of $x^2 = 12y$ and $(x - 3)^2 = 12(y - 1)$ are given. Label the focus, directrix, and vertex on each parabola.



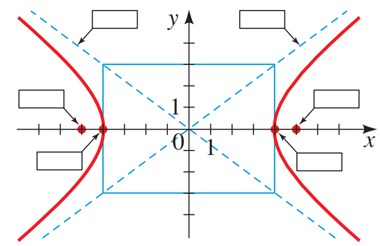


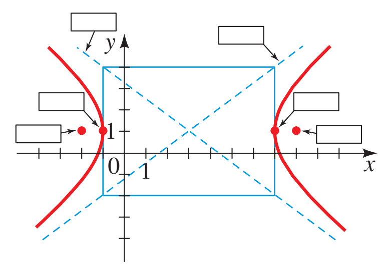
3. The graphs of $\dfrac{x^2}{5^2} + \dfrac{y^2}{4^2} = 1$ and $\dfrac{(x - 3)^2}{5^2} + \dfrac{(y - 1)^2}{4^2} = 1$ are given. Label the vertices and foci on each ellipse.





4. The graphs of $\dfrac{x^2}{4^2} - \dfrac{y^2}{3^2} = 1$ and $\dfrac{(x - 3)^2}{4^2} - \dfrac{(y - 1)^2}{3^2} = 1$ are given. Label the vertices, foci, and asymptotes on each hyperbola.





SKILLS

5–12 ■ An equation of an ellipse is given. **(a)** Find the center, vertices, and foci of the ellipse. **(b)** Determine the lengths of the major and minor axes. **(c)** Sketch a graph of the ellipse.

5. $\dfrac{(x - 2)^2}{9} + \dfrac{(y - 1)^2}{4} = 1$

6. $\dfrac{(x - 3)^2}{16} + (y + 3)^2 = 1$

7. $\dfrac{x^2}{9} + \dfrac{(y + 5)^2}{25} = 1$

8. $x^2 + \dfrac{(y + 2)^2}{4} = 1$

9. $\dfrac{(x + 5)^2}{16} + \dfrac{(y - 1)^2}{4} = 1$

10. $\dfrac{(x + 1)^2}{36} + \dfrac{(y + 1)^2}{64} = 1$

11. $4x^2 + 25y^2 - 50y = 75$

12. $9x^2 - 54x + y^2 + 2y + 46 = 0$

13–20 ■ An equation of a parabola is given. **(a)** Find the vertex, focus, and directrix of the parabola. **(b)** Sketch a graph showing the parabola and its directrix.

13. $(x - 3)^2 = 8(y + 1)$

14. $(y + 1)^2 = 16(x - 3)$

15. $(y + 5)^2 = -6x + 12$

16. $y^2 = 16x - 8$

17. $2(x - 1)^2 = y$

18. $-4(x + \frac{1}{2})^2 = y$

19. $y^2 - 6y - 12x + 33 = 0$

20. $x^2 + 2x - 20y + 41 = 0$

21–28 ■ An equation of a hyperbola is given. **(a)** Find the center, vertices, foci, and asymptotes of the hyperbola. **(b)** Sketch a graph showing the hyperbola and its asymptotes.

21. $\dfrac{(x + 1)^2}{9} - \dfrac{(y - 3)^2}{16} = 1$

22. $(x - 8)^2 - (y + 6)^2 = 1$

23. $y^2 - \dfrac{(x+1)^2}{4} = 1$ **24.** $\dfrac{(y-1)^2}{25} - (x+3)^2 = 1$

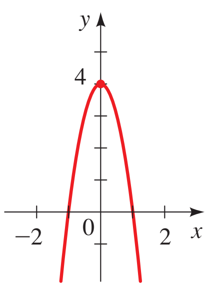
25. $\dfrac{(x+1)^2}{9} - \dfrac{(y+1)^2}{4} = 1$ **26.** $\dfrac{(y+2)^2}{36} - \dfrac{x^2}{64} = 1$

27. $36x^2 + 72x - 4y^2 + 32y + 116 = 0$

28. $25x^2 - 9y^2 - 54y = 306$

29–34 ■ Find an equation for the conic whose graph is shown.

29.



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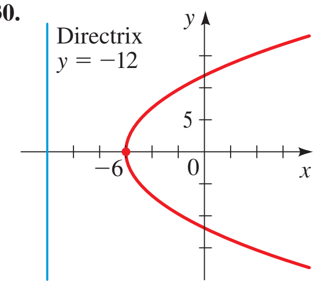

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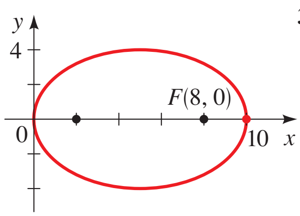

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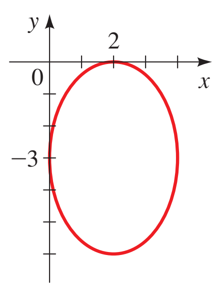

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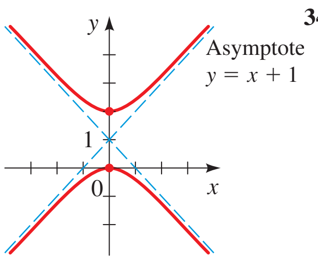

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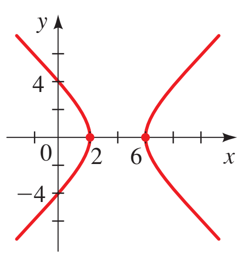


35–44 ■ Find an equation for the conic section with the given properties.

35. The ellipse with center $C(2, -3)$, vertices $V_1(-8, -3)$ and $V_2(12, -3)$, and foci $F_1(-4, -3)$ and $F_2(8, -3)$

36. The ellipse with vertices $V_1(-1, -4)$ and $V_2(-1, 6)$ and foci $F_1(-1, -3)$ and $F_2(-1, 5)$

37. The hyperbola with center $C(-1, 4)$, vertices $V_1(-1, -3)$ and $V_2(-1, 11)$, and foci $F_1(-1, -5)$ and $F_2(-1, 13)$

38. The hyperbola with vertices $V_1(-1, -1)$ and $V_2(5, -1)$ and foci $F_1(-4, -1)$ and $F_2(8, -1)$

39. The parabola with vertex $V(-3, 5)$ and directrix $y = 2$

40. The parabola with focus $F(1, 3)$ and directrix $x = 3$

41. The hyperbola with foci $F_1(1, -5)$ and $F_2(1, 5)$ that passes through the point $(1, 4)$

42. The ellipse with foci $F_1(1, -4)$ and $F_2(5, -4)$ that passes through the point $(3, 1)$

43. The ellipse with foci $F_1(3, -4)$ and $F_2(3, 4)$, and x-intercepts 0 and 6

44. The parabola that passes through the point $(6, 1)$, with vertex $V(-1, 2)$ and horizontal axis of symmetry

45–56 ■ Complete the square to determine whether the equation represents an ellipse, a parabola, a hyperbola, or a degenerate conic. If the graph is an ellipse, find the center, foci, vertices, and lengths of the major and minor axes. If it is a parabola, find the vertex, focus, and directrix. If it is a hyperbola, find the center, foci, vertices, and asymptotes. Then sketch the graph of the equation. If the equation has no graph, explain why.

45. $y^2 = 4(x + 2y)$

46. $9x^2 - 36x + 4y^2 = 0$

47. $x^2 - 5y^2 - 2x + 20y = 44$

48. $x^2 + 6x + 12y + 9 = 0$

49. $4x^2 + 25y^2 - 24x + 250y + 561 = 0$

50. $2x^2 + y^2 = 2y + 1$

51. $16x^2 - 9y^2 - 96x + 288 = 0$

52. $4x^2 - 4x - 8y + 9 = 0$

53. $x^2 + 16 = 4(y^2 + 2x)$

54. $x^2 - y^2 = 10(x - y) + 1$

55. $3x^2 + 4y^2 - 6x - 24y + 39 = 0$

56. $x^2 + 4y^2 + 20x - 40y + 300 = 0$

57–60 ■ Use a graphing device to graph the conic.

57. $2x^2 - 4x + y + 5 = 0$

58. $4x^2 + 9y^2 - 36y = 0$

59. $9x^2 + 36 = y^2 + 36x + 6y$

60. $x^2 - 4y^2 + 4x + 8y = 0$

61. Determine what the value of F must be if the graph of the equation

$$4x^2 + y^2 + 4(x - 2y) + F = 0$$

is **(a)** an ellipse, **(b)** a single point, or **(c)** the empty set.

62. Find an equation for the ellipse that shares a vertex and a focus with the parabola $x^2 + y = 100$ and has its other focus at the origin.

63. This exercise deals with **confocal parabolas**, that is, families of parabolas that have the same focus.

(a) Draw graphs of the family of parabolas

$$x^2 = 4p(y + p)$$

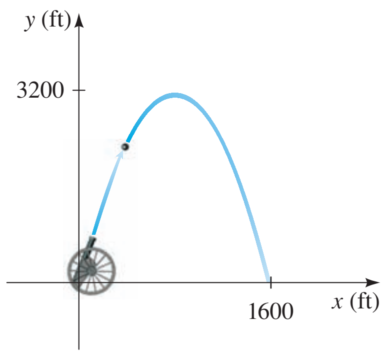
for $p = -2, -\frac{3}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 2$.

(b) Show that each parabola in this family has its focus at the origin.

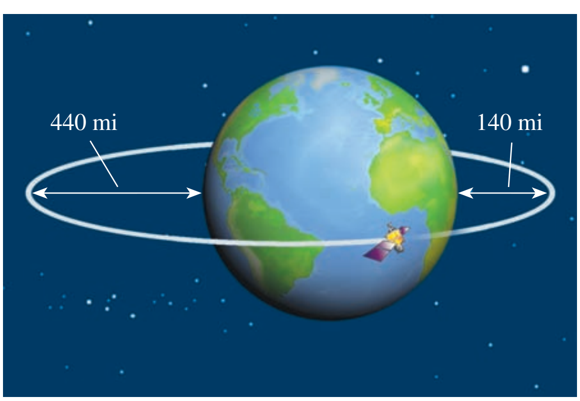
(c) Describe the effect on the graph of moving the vertex closer to the origin.

APPLICATIONS

64. Path of a Cannonball A cannon fires a cannonball as shown in the figure. The path of the cannonball is a parabola with vertex at the highest point of the path. If the cannonball lands 1600 ft from the cannon and the highest point it reaches is 3200 ft above the ground, find an equation for the path of the cannonball. Place the origin at the location of the cannon.



65. Orbit of a Satellite A satellite is in an elliptical orbit around the earth with the center of the earth at one focus, as shown in the figure. The height of the satellite above the earth varies between 140 mi and 440 mi. Assume that the earth is a sphere with radius 3960 mi. Find an equation for the path of the satellite with the origin at the center of the earth.



DISCOVERY ▪ DISCUSSION ▪ WRITING

66. A Family of Confocal Conics Conics that share a focus are called **confocal**. Consider the family of conics that have a focus at $(0, 1)$ and a vertex at the origin, as shown in the figure.

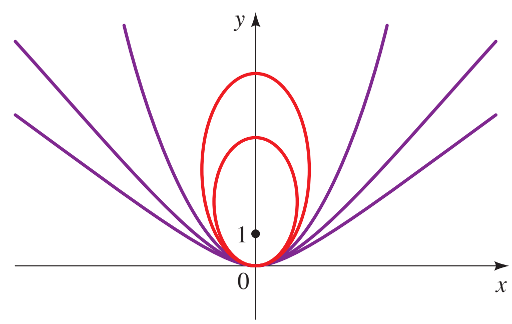
(a) Find equations of two different ellipses that have these properties.

(b) Find equations of two different hyperbolas that have these properties.

(c) Explain why only one parabola satisfies these properties. Find its equation.

(d) Sketch the conics you found in parts (a), (b), and (c) on the same coordinate axes (for the hyperbolas, sketch the top branches only).

(e) How are the ellipses and hyperbolas related to the parabola?



CHAPTER 7 | REVIEW

PROPERTIES AND FORMULAS

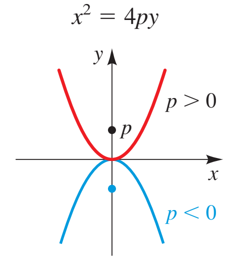
Geometric Definition of a Parabola (p. 524)

A **parabola** is the set of points in the plane that are equidistant from a fixed point F (the **focus**) and a fixed line l (the **directrix**).

Graphs of Parabolas with Vertex at the Origin (pp. 525, 526)

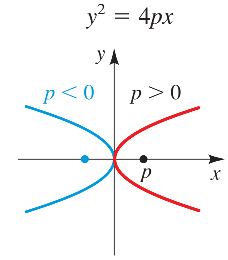
A parabola with vertex at the origin has an equation of the form $x^2 = 4py$ if its axis is vertical and an equation of the form $y^2 = 4px$ if its axis is horizontal.

$x^2 = 4py$



Focus $(0, p)$, directrix $y = -p$

$y^2 = 4px$



Focus $(p, 0)$, directrix $x = -p$

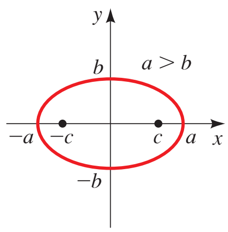
Geometric Definition of an Ellipse (p. 532)

An **ellipse** is the set of all the points in the plane for which the sum of the distances to each of two given points F_1 and F_2 (the **foci**) is a fixed constant.

Graphs of Ellipses with Center at the Origin (p. 534)

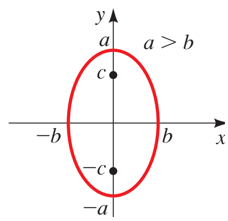
An ellipse with center at the origin has an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if its axis is horizontal and an equation of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ if its axis is vertical (where in each case $a > b > 0$).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Foci $(\pm c, 0)$, $c^2 = a^2 - b^2$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$



Foci $(0, \pm c)$, $c^2 = a^2 - b^2$

Eccentricity of an Ellipse (p. 537)

The **eccentricity** of an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (where $a > b > 0$) is the number

$$e = \frac{c}{a}$$

where $c = \sqrt{a^2 - b^2}$. The eccentricity e of any ellipse is a number between 0 and 1. If e is close to 0, then the ellipse is nearly circular; the closer e gets to 1, the more elongated it becomes.

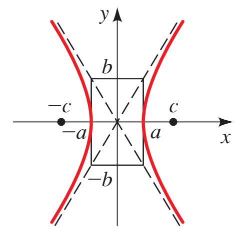
Geometric Definition of a Hyperbola (p. 541)

A **hyperbola** is the set of all those points in the plane for which the absolute value of the difference of the distances to each of two given points F_1 and F_2 (the **foci**) is a fixed constant.

Graphs of Hyperbolas with Center at the Origin (p. 542)

A **hyperbola** with center at the origin has an equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if its axis is horizontal and an equation of the form $-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ if its axis is vertical.

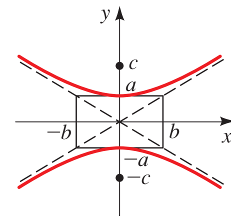
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Foci $(\pm c, 0)$, $c^2 = a^2 + b^2$

Asymptotes: $y = \pm\frac{b}{a}x$

$$-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$



Foci $(0, \pm c)$, $c^2 = a^2 + b^2$

Asymptotes: $y = \pm\frac{a}{b}x$

Shifted Conics (pp. 550–554)

If the vertex of a parabola or the center of an ellipse or a hyperbola does not lie at the origin but rather at the point (h, k), then we refer to the curve as a **shifted conic**. To find the equation of the shifted conic, we use the "unshifted" form for the appropriate curve and replace x by $x - h$ and y by $y - k$.

General Equation of a Shifted Conic (p. 555)

The graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

(where A and C are not both 0) is either a conic or a degenerate conic. In the nondegenerate cases the graph is:

1. A **parabola** if $A = 0$ or $C = 0$.
2. An **ellipse** if A and C have the same sign (or a circle if $A = C$).
3. A **hyperbola** if A and C have opposite sign.

To graph a conic whose equation is given in general form, **complete the square** in x and y to put the equation in standard form for a parabola, an ellipse, or a hyperbola.

■ LEARNING OBJECTIVES SUMMARY

Section	After completing this chapter, you should be able to . . .	Review Exercises
7.1	▪ Find geometric properties of a parabola from its equation	1–6
	▪ Find the equation of a parabola from some of its geometric properties	37, 55–56
7.2	▪ Find geometric properties of an ellipse from its equation	13–18
	▪ Find the equation of an ellipse from some of its geometric properties	38, 57
7.3	▪ Find geometric properties of a hyperbola from its equation	25–30
	▪ Find the equation of a hyperbola from some of its geometric properties	39, 58
7.4	▪ Find geometric properties of a shifted conic from its equation	7–12, 19–24, 31–36, 43–54
	▪ Find the equation of a shifted conic from some of its geometric properties	40–42, 59–66

EXERCISES

1–12 ■ An equation of a parabola is given. **(a)** Find the vertex, focus, and directrix of the parabola. **(b)** Sketch a graph of the parabola and its directrix.

1. $y^2 = 4x$

2. $x = \frac{1}{12}y^2$

3. $\frac{1}{8}x^2 = y$

4. $x^2 = -8y$

5. $x^2 + 8y = 0$

6. $2x - y^2 = 0$

7. $(y - 2)^2 = 4(x + 2)$

8. $(x + 3)^2 = -20(y + 2)$

9. $\frac{1}{2}(y - 3)^2 + x = 0$

10. $2(x + 1)^2 = y$

11. $\frac{1}{2}x^2 + 2x = 2y + 4$

12. $x^2 = 3(x + y)$

13–24 ■ An equation of an ellipse is given. **(a)** Find the center, vertices, and foci of the ellipse. **(b)** Determine the lengths of the major and minor axes. **(c)** Sketch a graph of the ellipse.

13. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

14. $\dfrac{x^2}{49} + \dfrac{y^2}{9} = 1$

15. $\dfrac{x^2}{49} + \dfrac{y^2}{4} = 1$

16. $\dfrac{x^2}{4} + \dfrac{y^2}{36} = 1$

17. $x^2 + 4y^2 = 16$

18. $9x^2 + 4y^2 = 1$

19. $\dfrac{(x - 3)^2}{9} + \dfrac{y^2}{16} = 1$

20. $\dfrac{(x - 2)^2}{25} + \dfrac{(y + 3)^2}{16} = 1$

21. $\dfrac{(x - 2)^2}{9} + \dfrac{(y + 3)^2}{36} = 1$

22. $\dfrac{x^2}{3} + \dfrac{(y + 5)^2}{25} = 1$

23. $4x^2 + 9y^2 = 36y$

24. $2x^2 + y^2 = 2 + 4(x - y)$

25–36 ■ An equation of a hyperbola is given. **(a)** Find the center, vertices, foci, and asymptotes of the hyperbola. **(b)** Sketch a graph of the hyperbola.

25. $-\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

26. $\dfrac{x^2}{49} - \dfrac{y^2}{32} = 1$

27. $\dfrac{x^2}{4} - \dfrac{y^2}{49} = 1$

28. $\dfrac{y^2}{25} - \dfrac{x^2}{4} = 1$

29. $x^2 - 2y^2 = 16$

30. $x^2 - 4y^2 + 16 = 0$

31. $\dfrac{(x + 4)^2}{16} - \dfrac{y^2}{16} = 1$

32. $\dfrac{(x - 2)^2}{8} - \dfrac{(y + 2)^2}{8} = 1$

33. $\dfrac{(y - 3)^2}{4} - \dfrac{(x + 1)^2}{36} = 1$

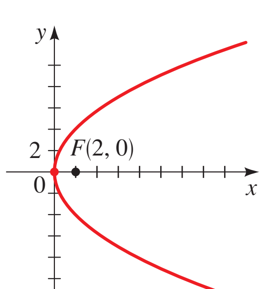
34. $\dfrac{(y - 3)^2}{3} - \dfrac{x^2}{16} = 1$

35. $9y^2 + 18y = x^2 + 6x + 18$

36. $y^2 = x^2 + 6y$

37–42 ■ Find an equation for the conic whose graph is shown.

37.



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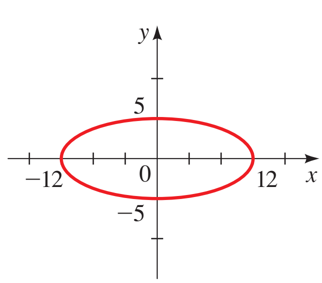

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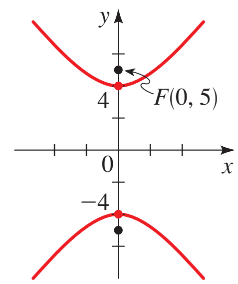

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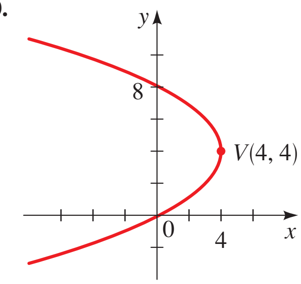

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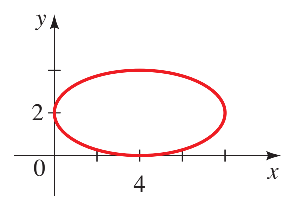

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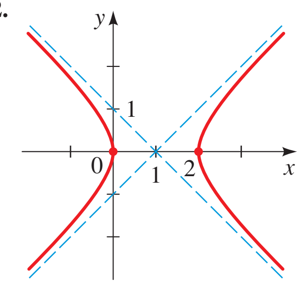


43–54 ■ Determine whether the equation represents an ellipse, a parabola, a hyperbola, or a degenerate conic. If the graph is an ellipse, find the center, foci, and vertices. If it is a parabola, find the vertex, focus, and directrix. If it is a hyperbola, find the center, foci, vertices, and asymptotes. Then sketch the graph of the equation. If the equation has no graph, explain why.

43. $\dfrac{x^2}{12} + y = 1$

44. $\dfrac{x^2}{12} + \dfrac{y^2}{144} = \dfrac{y}{12}$

45. $x^2 - y^2 + 144 = 0$

46. $x^2 + 6x = 9y^2$

47. $4x^2 + y^2 = 8(x + y)$

48. $3x^2 - 6(x + y) = 10$

49. $x = y^2 - 16y$

50. $2x^2 + 4 = 4x + y^2$

51. $2x^2 - 12x + y^2 + 6y + 26 = 0$

52. $36x^2 - 4y^2 - 36x - 8y = 31$

53. $9x^2 + 8y^2 - 15x + 8y + 27 = 0$

54. $x^2 + 4y^2 = 4x + 8$

55–64 ■ Find an equation for the conic section with the given properties.

55. The parabola with focus $F(0, 1)$ and directrix $y = -1$

56. The parabola with vertex at the origin and focus $F(5, 0)$

57. The ellipse with center at the origin and with x-intercepts ± 2 and y-intercepts ± 5

58. The hyperbola with vertices $V(0, \pm 2)$ and asymptotes $y = \pm \frac{1}{2}x$

59. The ellipse with center $C(0, 4)$, foci $F_1(0, 0)$ and $F_2(0, 8)$, and major axis of length 10

60. The hyperbola with center $C(2, 4)$, foci $F_1(2, 1)$ and $F_2(2, 7)$, and vertices $V_1(2, 6)$ and $V_2(2, 2)$

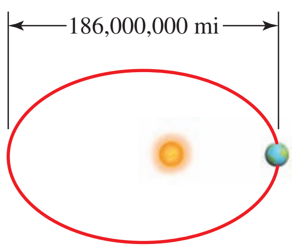
61. The ellipse with foci $F_1(1, 1)$ and $F_2(1, 3)$, and with one vertex on the x-axis

62. The parabola with vertex $V(5, 5)$ and directrix the y-axis

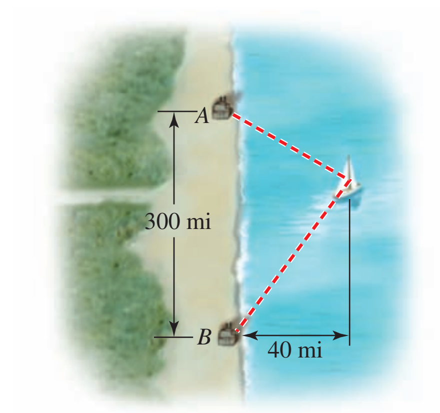
63. The ellipse with vertices $V_1(7, 12)$ and $V_2(7, -8)$, and passing through the point $P(1, 8)$

64. The parabola with vertex $V(-1, 0)$ and horizontal axis of symmetry, and crossing the y-axis at $y = 2$

65. The path of the earth around the sun is an ellipse with the sun at one focus. The ellipse has major axis 186,000,000 mi and eccentricity 0.017. Find the distance between the earth and the sun when the earth is **(a)** closest to the sun and **(b)** farthest from the sun.



66. A ship is located 40 mi from a straight shoreline. LORAN stations A and B are located on the shoreline, 300 mi apart. From the LORAN signals, the captain determines that his ship is 80 mi closer to A than to B. Find the location of the ship. (Place A and B on the y-axis with the x-axis halfway between them. Find the x- and y-coordinates of the ship.)



67. **(a)** Draw graphs of the following family of ellipses for $k = 1, 2, 4$, and 8.

$$\frac{x^2}{16 + k^2} + \frac{y^2}{k^2} = 1$$

(b) Prove that all the ellipses in part (a) have the same foci.

68. **(a)** Draw graphs of the following family of parabolas for $k = \frac{1}{2}, 1, 2$, and 4.

$$y = kx^2$$

(b) Find the foci of the parabolas in part (a).

(c) How does the location of the focus change as k increases?

CHAPTER 10 TEST

1. Find the focus and directrix of the parabola $x^2 = -12y$, and sketch its graph.

2. Find the vertices, foci, and the lengths of the major and minor axes for the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$. Then sketch its graph.

3. Find the vertices, foci, and asymptotes of the hyperbola $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$. Then sketch its graph.

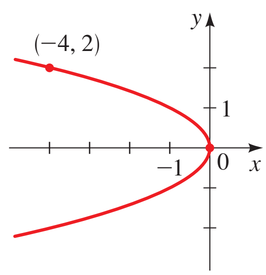
4. Find an equation for the parabola with vertex $(0, 0)$ and focus $(4, 0)$.

5. Find an equation for the ellipse with foci $(\pm 3, 0)$ and vertices $(\pm 4, 0)$.

6. Find an equation for the hyperbola with foci $(0, \pm 5)$ and with asymptotes $y = \pm\frac{3}{4}x$.

7–9 ■ Find an equation for the conic whose graph is shown.

7.



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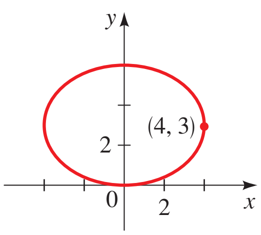

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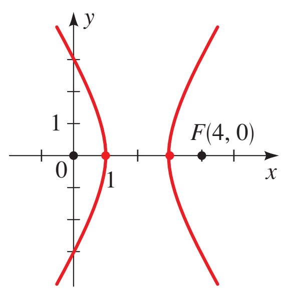


10–12 ■ Determine whether the equation represents an ellipse, a parabola, or a hyperbola. If the graph is an ellipse, find the center, foci, and vertices. If it is a parabola, find the vertex, focus, and directrix. If it is a hyperbola, find the center, foci, vertices, and asymptotes. Then sketch the graph of the equation.

10. $16x^2 + 36y^2 - 96x + 36y + 9 = 0$

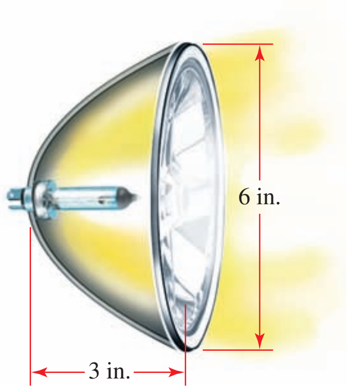
11. $9x^2 - 8y^2 + 36x + 64y = 164$

12. $2x + y^2 + 8y + 8 = 0$

13. Find an equation for the ellipse with center $(2, 0)$, foci $(2, \pm 3)$ and major axis of length 8.

14. Find an equation for the parabola with focus $(2, 4)$ and directrix the x-axis.

15. A parabolic reflector for a car headlight forms a bowl shape that is 6 in. wide at its opening and 3 in. deep, as shown in the figure at the left. How far from the vertex should the filament of the bulb be placed if it is to be located at the focus?



Conics in Architecture

Many buildings employ conic sections in their design. Architects have various reasons for using these curves, ranging from structural stability to simple beauty. But how can a huge parabola, ellipse, or hyperbola be accurately constructed in concrete and steel? In this *Focus on Modeling,* we will see how the geometric properties of the conics can be used to construct these shapes.

▼ Conics in Buildings

In ancient times architecture was part of mathematics, so architects had to be mathematicians. Many of the structures they built—pyramids, temples, amphitheaters, and irrigation projects—still stand. In modern times architects employ even more sophisticated mathematical principles. The photographs below show some structures that employ conic sections in their design.

Roman Amphitheater in Alexandria, Egypt (circle)
© Nik Wheeler/CORBIS

Ceiling of Statuary Hall in the U.S. Capitol (ellipse)
Architect of the Capitol

Roof of the Skydome in Toronto, Canada (parabola)
Walter Schmid/Stone/Getty Images

Roof of Washington Dulles Airport (hyperbola and parabola)
© Richard T. Nowitz/CORBIS

McDonnell Planetarium, St. Louis, MO (hyperbola)
VisionsofAmerica/Joe Sohm/Jupiter Images

Attic in La Pedrera, Barcelona, Spain (parabola)
© O. Alamany & E. Vincens/CORBIS

Architects have different reasons for using conics in their designs. For example, the Spanish architect Antoni Gaudí used parabolas in the attic of La Pedrera (see photo above). He reasoned that since a rope suspended between two points with an equally distributed load (as in a suspension bridge) has the shape of a parabola, an inverted parabola would provide the best support for a flat roof.

▼ Constructing Conics

The equations of the conics are helpful in manufacturing small objects, because a computer-controlled cutting tool can accurately trace a curve given by an equation. But in a building project, how can we construct a portion of a parabola, ellipse, or hyperbola that spans the ceiling or walls of a building? The geometric properties of the conics provide practical ways of constructing them. For example, if you were building a circular tower, you would choose a center point, then make sure that the walls of the tower were a fixed

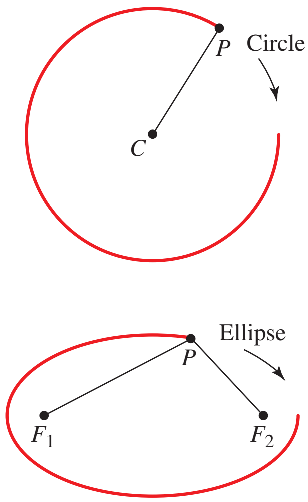


FIGURE 1 Constructing a circle and an ellipse

distance from that point. Elliptical walls can be constructed using a string anchored at two points, as shown in Figure 1.

To construct a parabola, we can use the apparatus shown in Figure 2. A piece of string of length a is anchored at F and A. The T-square, also of length a, slides along the straight bar L. A pencil at P holds the string taut against the T-square. As the T-square slides to the right the pencil traces out a curve.

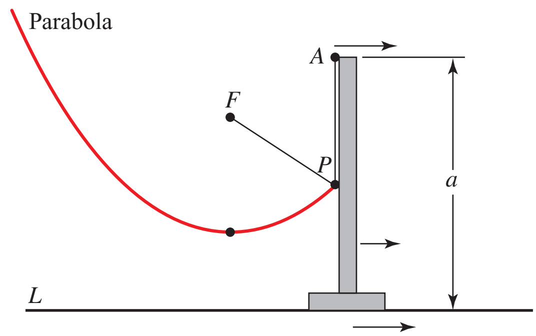


FIGURE 2 Constructing a parabola

From the figure we see that

$$d(F, P) + d(P, A) = a \qquad \text{The string is of length } a$$

$$d(L, P) + d(P, A) = a \qquad \text{The T-square is of length } a$$

It follows that $d(F, P) + d(P, A) = d(L, P) + d(P, A)$. Subtracting $d(P, A)$ from each side, we get

$$d(F, P) = d(L, P)$$

The last equation says that the distance from F to P is equal to the distance from P to the line L. Thus the curve is a parabola with focus F and directrix L.

In building projects it is easier to construct a straight line than a curve. So in some buildings, such as in the Kobe Tower (see Problem 4), a curved surface is produced by using many straight lines. We can also produce a curve using straight lines, such as the parabola shown in Figure 3.

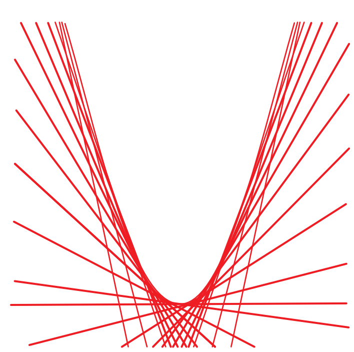


FIGURE 3 Tangent lines to a parabola

Each line is **tangent** to the parabola; that is, the line meets the parabola at exactly one point and does not cross the parabola. The line tangent to the parabola $y = x^2$ at the point (a, a^2) is

$$y = 2ax - a^2$$

You are asked to show this in Problem 6. The parabola is called the **envelope** of all such lines.

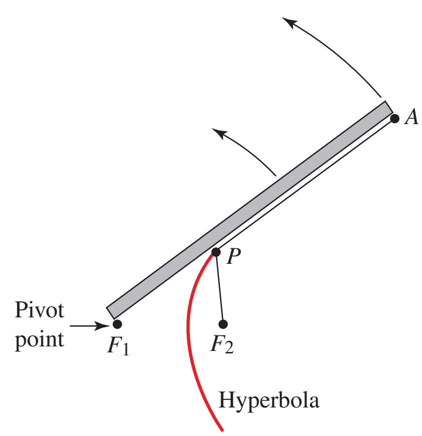
PROBLEMS

1. **Conics in Architecture** The photographs on page 563 show six examples of buildings that contain conic sections. Search the Internet to find other examples of structures that employ parabolas, ellipses, or hyperbolas in their design. Find at least one example for each type of conic.

2. **Constructing a Hyperbola** In this problem we construct a hyperbola. The wooden bar in the figure can pivot at F_1. A string that is shorter than the bar is anchored at F_2 and at A, the other end of the bar. A pencil at P holds the string taut against the bar as it moves counterclockwise around F_1.

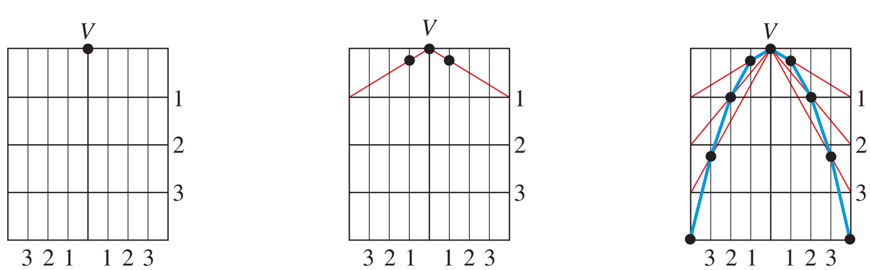
   (a) Show that the curve traced out by the pencil is one branch of a hyperbola with foci at F_1 and F_2.

   (b) How should the apparatus be reconfigured to draw the other branch of the hyperbola?



3. **A Parabola in a Rectangle** The following method can be used to construct a parabola that fits in a given rectangle. The parabola will be approximated by many short line segments.

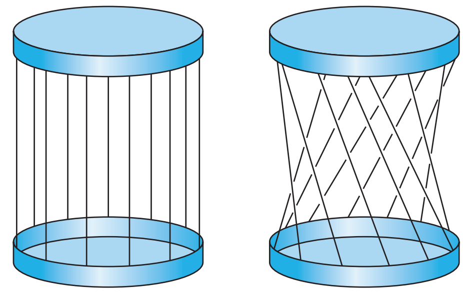
   First, draw a rectangle. Divide the rectangle in half by a vertical line segment, and label the top endpoint V. Next, divide the length and width of each half rectangle into an equal number of parts to form grid lines, as shown in the figure below. Draw lines from V to the endpoints of horizontal grid line 1, and mark the points where these lines cross the vertical grid lines labeled 1. Next, draw lines from V to the endpoints of horizontal grid line 2, and mark the points where these lines cross the vertical grid lines labeled 2. Continue in this way until you have used all the horizontal grid lines. Now use line segments to connect the points you have marked to obtain an approximation to the desired parabola. Apply this procedure to draw a parabola that fits into a 6 ft by 10 ft rectangle on a lawn.

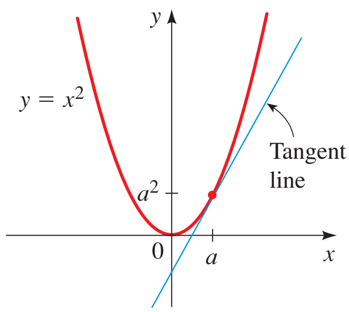


4. **Hyperbolas from Straight Lines** In this problem we construct hyperbolic shapes using straight lines. Punch equally spaced holes into the edges of two large plastic lids. Connect corresponding holes with strings of equal lengths as shown in the figure on the next page. Holding the strings taut, twist one lid against the other. An imaginary surface passing through the strings has hyperbolic cross sections. (An architectural example of this is the Kobe Tower


Martin Mette/Shutterstock.com

in Japan, shown in the photograph.) What happens to the vertices of the hyperbolic cross sections as the lids are twisted more?





5. Tangent Lines to a Parabola In this problem we show that the line tangent to the parabola $y = x^2$ at the point (a, a^2) has the equation $y = 2ax - a^2$.

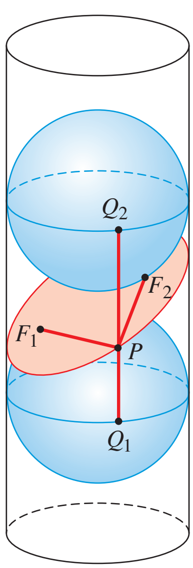
(a) Let m be the slope of the tangent line at (a, a^2). Show that the equation of the tangent line is $y - a^2 = m(x - a)$.

(b) Use the fact that the tangent line intersects the parabola at only one point to show that (a, a^2) is the only solution of the system.

$$\begin{cases} y - a^2 = m(x - a) \\ y = x^2 \end{cases}$$

(c) Eliminate y from the system in part (b) to get a quadratic equation in x. Show that the discriminant of this quadratic is $(m - 2a)^2$. Since the system in part (b) has exactly one solution, the discriminant must equal 0. Find m.

(d) Substitute the value for m you found in part (c) into the equation in part (a), and simpify to get the equation of the tangent line.



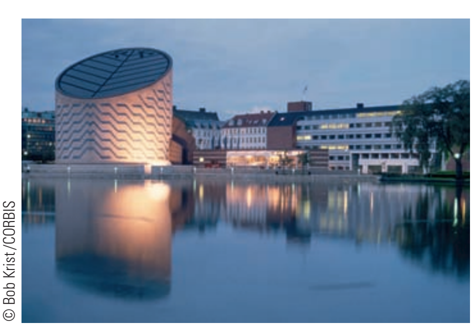
6. A Cut Cylinder In this problem we prove that when a cylinder is cut by a plane, an ellipse is formed. An architectural example of this is the Tycho Brahe Planetarium in Copenhagen (see the photograph). In the figure, a cylinder is cut by a plane, resulting in the red curve. Two spheres with the same radius as the cylinder slide inside the cylinder so that they just touch the plane at F_1 and F_2. Choose an arbitrary point P on the curve, and let Q_1 and Q_2 be the two points on the cylinder where a vertical line through P touches the "equator" of each sphere.

(a) Show that $PF_1 = PQ_1$ and $PF_2 = PQ_2$. [*Hint:* Use the fact that all tangents to a sphere from a given point outside the sphere are of the same length.]

(b) Explain why $PQ_1 + PQ_2$ is the same for all points P on the curve.

(c) Show that $PF_1 + PF_2$ is the same for all points P on the curve.

(d) Conclude that the curve is an ellipse with foci F_1 and F_2.


© Bob Krist/CORBIS

1. Consider the following system of equations.

$$\begin{cases} x^2 + y^2 = 4y \\ x^2 - 2y = 0 \end{cases}$$

(a) Is the system linear or nonlinear? Explain.
(b) Find all solutions of the system.
(c) The graph of each equation is a conic section. Name the type of conic section in each case.
(d) Graph both equations on the same set of axes.
(e) On your graph, shade the region that corresponds to the solution of the system of inequalities.

$$\begin{cases} x^2 + y^2 \le 4y \\ x^2 - 2y \le 0 \end{cases}$$

2. Find the complete solution of each linear system, or show that no solution exists.

(a) $\begin{cases} x + y - z = 2 \\ 2x + 3y - z = 5 \\ 3x + 5y + 2z = 11 \end{cases}$ **(b)** $\begin{cases} y - z = 2 \\ x + 2y - 3z = 3 \\ 3x + 5y - 8z = 7 \end{cases}$

3. Xavier, Yolanda, and Zachary go fishing. Yolanda catches as many fish as Xavier and Zachary put together. Zachary catches 2 more fish than Xavier. The total catch for all three people is 20 fish. How many did each person catch?

4. Let $A = \begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 6 & 5 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Calculate each of the following, or explain why the calculation can't be done.

$$A + B, \quad C - D, \quad AB, \quad CB, \quad BD, \quad \det(B), \quad \det(C), \quad \det(D)$$

(b) Based on the values you calculated for $\det(C)$ and $\det(D)$, which matrix, C or D, has an inverse? Find the inverse of the invertible one.

5. Consider the following system of equations.

$$\begin{cases} 5x - 3y = 5 \\ 6x - 4y = 0 \end{cases}$$

(a) Write a matrix equation of the form $AX = B$ that is equivalent to this system.
(b) Find A^{-1}, the inverse of the coefficient matrix.
(c) Solve the matrix equation by multiplying each side by A^{-1}.
(d) Now solve the system using the Cramer's Rule. Did you get the same solution as in part (b)?

6. Find the partial fraction decomposition of the rational function $r(x) = \dfrac{4x + 8}{x^4 + 4x^2}$.

7. Find an equation for the parabola with vertex at the origin and focus $F(0, 3)$.

8. Find the focus and directrix of each parabola, and sketch its graph.

(a) $x^2 + 6y = 0$ **(b)** $x - 2y^2 + 4y = 2$

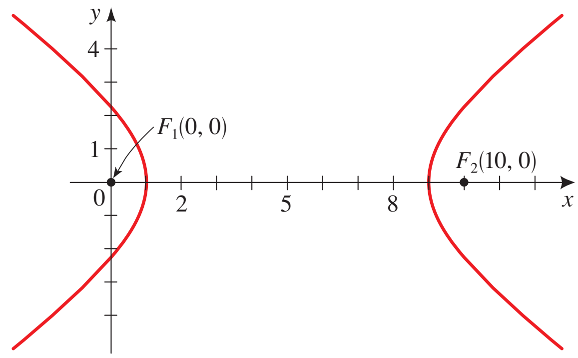
9. Determine whether the equation represents an ellipse or a hyperbola. If it is an ellipse, find the coordinates of its vertices and foci, and sketch its graph. If it is a hyperbola, find the coordinates of its vertices and foci, find the equations of its asymptotes, and sketch its graph.

(a) $\dfrac{x^2}{9} - y^2 = 1$ **(b)** $\dfrac{x^2}{9} + y^2 = 1$ **(c)** $-\dfrac{x^2}{9} + y^2 = 1$

10. Sketch the graph of each conic section, and find the coordinates of its foci. What type of conic section does each equation represent?

(a) $9x^2 + 4y^2 = 24y$ **(b)** $x^2 + 6x - y^2 + 8y = 16$

11. Find an equation for the conic whose graph is shown.



Zia Soleil / Getty Images

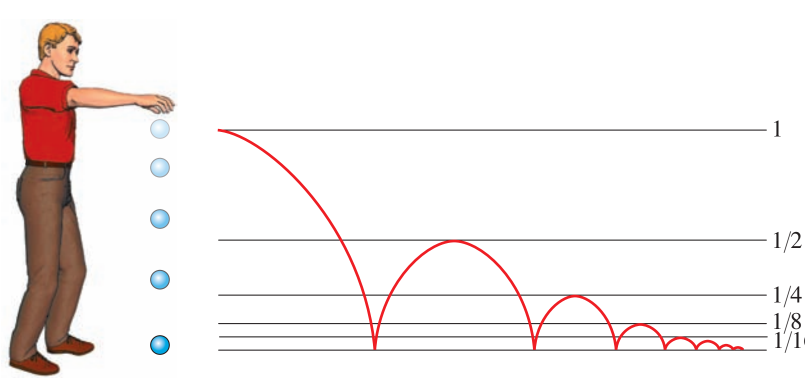
CHAPTER 8

SEQUENCES AND SERIES

Functions on the Natural Numbers Throughout this book we have used functions to model real-world situations. The functions we've used have always had real numbers as inputs. For example, a function that models temperature in terms of time has real numbers (representing time) as inputs. But many real-world situations occur in stages: stage 1, 2, 3, ... To model such situations, we need functions whose inputs are the natural numbers 1, 2, 3, . . . (representing the stages). For example, the peaks of a bouncing ball are represented by the natural numbers 1, 2, 3, . . . (representing peak 1, 2, 3, . . .). A function f that models the height of the ball at each peak has natural numbers 1, 2, 3, . . . as inputs and gives the heights as $f(1)$, $f(2)$, $f(3)$, ... In general a function whose inputs are the natural numbers is called a *sequence*. We can think of a sequence as simply a list of numbers written in a specific order.

The amount in a bank account at the end of each month, mortgage payments, and the amount of an annuity are sequences. The formulas that generate these sequences drive our economy—they allow us to borrow money to buy our dream home closer to graduation than to retirement. In this chapter we study these and other applications of sequences.

In *Focus on Modeling* at the end of the chapter we investigate how sequences are used in modeling real-world situations that occur in stages, where each stage depends on what happened at the preceding stage(s).



8.1 Sequences and Summation Notation

LEARNING OBJECTIVES After completing this section, you will be able to:

Find the terms of a sequence ▶ Find the terms of a recursive sequence ▶ Find the partial sums of a sequence ▶ Use sigma notation

Roughly speaking, a sequence is an infinite list of numbers. The numbers in the sequence are often written as $a_1, a_2, a_3, \ldots$. The dots mean that the list continues forever. A simple example is the sequence

$$\begin{array}{ccccc} 5, & 10, & 15, & 20, & 25, \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ a_1 & a_2 & a_3 & a_4 & a_5 \ldots \end{array}$$

We can describe the pattern of the sequence displayed above by the following *formula*:

$$a_n = 5n$$

You may have already thought of a different way to describe the pattern—namely, "you go from one number to the next by adding 5." This natural way of describing the sequence is expressed by the *recursive formula*:

$$a_n = a_{n-1} + 5$$

starting with $a_1 = 5$. Try substituting $n = 1, 2, 3, \ldots$ in each of these formulas to see how they produce the numbers in the sequence. In this section we see how these different ways are used to describe specific sequences.

▼ Sequences

Any ordered list of numbers can be viewed as a function whose input values are 1, 2, 3, . . . and whose output values are the numbers in the list. So we define a sequence as follows.

DEFINITION OF A SEQUENCE

A **sequence** is a function f whose domain is the set of natural numbers. The **terms of the sequence** are the function values

$$f(1), f(2), f(3), \ldots, f(n), \ldots$$

We usually write a_n instead of the function notation $f(n)$. So the terms of the sequence are written as

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

The number a_1 is called the **first term**, a_2 is called the **second term**, and in general, a_n is called the ***n*th term**.

Here is a simple example of a sequence:

$$2, 4, 6, 8, 10, \ldots$$

We can write a sequence in this way when it's clear what the subsequent terms of the sequence are. This sequence consists of even numbers. To be more accurate, however, we need to specify a procedure for finding *all* the terms of the sequence. This can be done by giving a formula for the nth term a_n of the sequence. In this case,

Another way to write this sequence is to use function notation:

$$a(n) = 2n$$

so $a(1) = 2, a(2) = 4, a(3) = 6, \ldots$

$$a_n = 2n$$

and the sequence can be written as

$$\underset{\text{1st term}}{2}, \quad \underset{\text{2nd term}}{4}, \quad \underset{\text{3rd term}}{6}, \quad \underset{\text{4th term}}{8}, \quad \ldots, \quad \underset{n\text{th term}}{2n}, \quad \ldots$$

Notice how the formula $a_n = 2n$ gives all the terms of the sequence. For instance, substituting 1, 2, 3, and 4 for n gives the first four terms:

$$a_1 = 2 \cdot 1 = 2 \qquad a_2 = 2 \cdot 2 = 4$$

$$a_3 = 2 \cdot 3 = 6 \qquad a_4 = 2 \cdot 4 = 8$$

To find the 103rd term of this sequence, we use $n = 103$ to get

$$a_{103} = 2 \cdot 103 = 206$$

EXAMPLE 1 | Finding the Terms of a Sequence

Find the first five terms and the 100th term of the sequence defined by each formula.

(a) $a_n = 2n - 1$ **(b)** $c_n = n^2 - 1$

(c) $t_n = \dfrac{n}{n+1}$ **(d)** $r_n = \dfrac{(-1)^n}{2^n}$

SOLUTION To find the first five terms, we substitute $n = 1, 2, 3, 4$, and 5 in the formula for the nth term. To find the 100th term, we substitute $n = 100$. This gives the following.

nth term	First five terms	100th term
(a) $2n - 1$	1, 3, 5, 7, 9	199
(b) $n^2 - 1$	0, 3, 8, 15, 24	9999
(c) $\dfrac{n}{n+1}$	$\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}$	$\dfrac{100}{101}$
(d) $\dfrac{(-1)^n}{2^n}$	$-\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{8}, \dfrac{1}{16}, -\dfrac{1}{32}$	$\dfrac{1}{2^{100}}$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **3**, **5**, **7**, AND **9**

In Example 1(d) the presence of $(-1)^n$ in the sequence has the effect of making successive terms alternately negative and positive.

It is often useful to picture a sequence by sketching its graph. Since a sequence is a function whose domain is the natural numbers, we can draw its graph in the Cartesian plane. For instance, the graph of the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{n}, \ldots$$

is shown in Figure 1.

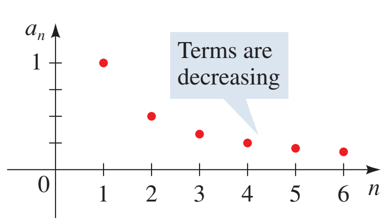


FIGURE 1

Compare the graph of the sequence shown in Figure 1 to the graph of

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots, \frac{(-1)^{n+1}}{n}, \ldots$$

shown in Figure 2. The graph of every sequence consists of isolated points that are *not* connected.

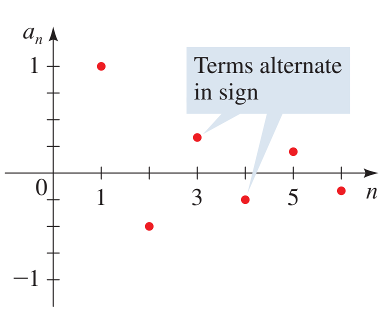
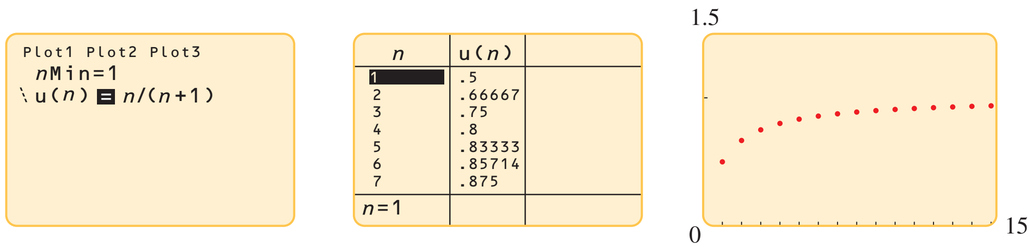


FIGURE 2

See Appendix C, *Using the TI-83/84 Graphing Calculator*, for additional instructions on working with sequences.

Graphing calculators are useful in analyzing sequences. To work with sequences on a TI-83, we put the calculator in `Seq` mode ("sequence" mode) as in Figure 3(a). If we enter the sequence $u(n) = n/(n + 1)$ of Example 1(c), we can display the terms using the `TABLE` command as shown in Figure 3(b). We can also graph the sequence as shown in Figure 3(c).



(a) (b) (c)

FIGURE 3
$u(n) = n/(n + 1)$

Not all sequences can be defined by a formula. For example, there is no known formula for the sequence of prime numbers:*

2, 3, 5, 7, 11, 13, 17, 19, 23, . . .

ERATOSTHENES (circa 276–195 B.C.) was a renowned Greek geographer, mathematician, and astronomer. He accurately calculated the circumference of the earth by an ingenious method. He is most famous, however, for his method for finding primes, now called the *sieve of Eratosthenes*. The method consists of listing the integers, beginning with 2 (the first prime), and then crossing out all the multiples of 2, which are not prime. The next number remaining on the list is 3 (the second prime), so we again cross out all multiples of it. The next remaining number is 5 (the third prime number), and we cross out all multiples of it, and so on. In this way all numbers that are not prime are crossed out, and the remaining numbers are the primes.

Finding patterns is an important part of mathematics. Consider a sequence that begins

$$1, 4, 9, 16, \ldots$$

Can you detect a pattern in these numbers? In other words, can you define a sequence whose first four terms are these numbers? The answer to this question seems easy; these numbers are the squares of the numbers 1, 2, 3, 4. Thus the sequence we are looking for is defined by $a_n = n^2$. However, this is not the *only* sequence whose first four terms are 1, 4, 9, 16. In other words, the answer to our problem is not unique (see Exercise 86). In the next example we are interested in finding an *obvious* sequence whose first few terms agree with the given ones.

EXAMPLE 2 | Finding the nth Term of a Sequence

Find the nth term of a sequence whose first several terms are given.

(a) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$ **(b)** $-2, 4, -8, 16, -32, \ldots$

SOLUTION

(a) We notice that the numerators of these fractions are the odd numbers and the denominators are the even numbers. Even numbers are of the form $2n$, and odd numbers are of the form $2n - 1$ (an odd number differs from an even number by 1). So a sequence that has these numbers for its first four terms is given by

$$a_n = \frac{2n - 1}{2n}$$

(b) These numbers are powers of 2, and they alternate in sign, so a sequence that agrees with these terms is given by

$$a_n = (-1)^n 2^n$$

You should check that these formulas do indeed generate the given terms.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **29** AND **35**

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* A *prime number* is a whole number p whose only divisors are p and 1. (By convention the number 1 is not considered prime.)

Large Prime Numbers

The search for large primes fascinates many people. As of this writing, the largest known prime number is

$$2^{43,112,609} - 1$$

It was discovered by Edson Smith of the Department of Mathematics at UCLA. In decimal notation this number contains 12,978,189 digits. If it were written in full, it would occupy more than three times as many pages as this book contains. Smith was working with a large Internet group known as GIMPS (the Great Internet Mersenne Prime Search). Numbers of the form $2^p - 1$, where p is prime, are called Mersenne numbers and are more easily checked for primality than others. That is why the largest known primes are of this form.

Recursively Defined Sequences

Some sequences do not have simple defining formulas like those of the preceding example. The nth term of a sequence may depend on some or all of the terms preceding it. A sequence defined in this way is called **recursive**. Here are two examples.

EXAMPLE 3 | Finding the Terms of a Recursively Defined Sequence

A sequence is defined recursively by $a_1 = 1$ and

$$a_n = 3(a_{n-1} + 2)$$

(a) Find the first five terms of the sequence.

(b) Use a graphing calculator to find the 20th term of the sequence.

SOLUTION

(a) The defining formula for this sequence is recursive. It allows us to find the nth term a_n if we know the preceding term a_{n-1}. Thus, we can find the second term from the first term, the third term from the second term, the fourth term from the third term, and so on. Since we are given the first term $a_1 = 1$, we can proceed as follows.

$$a_2 = 3(a_1 + 2) = 3(1 + 2) = 9$$

$$a_3 = 3(a_2 + 2) = 3(9 + 2) = 33$$

$$a_4 = 3(a_3 + 2) = 3(33 + 2) = 105$$

$$a_5 = 3(a_4 + 2) = 3(105 + 2) = 321$$

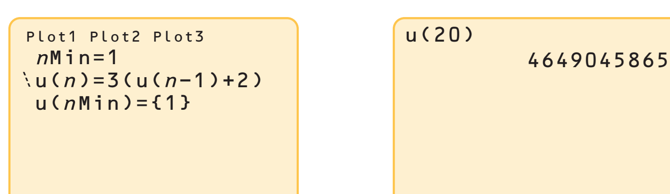
Thus the first five terms of this sequence are

$$1, 9, 33, 105, 321, \ldots$$

(b) Note that to find the 20th term of the recursive sequence, we must first find all 19 preceding terms. This is most easily done by using a graphing calculator. Figure 4(a) shows how to enter this sequence on the TI-83 calculator. From Figure 4(b) we see that the 20th term of the sequence is

$$a_{20} = 4,649,045,865$$

See Appendix C, *Using the TI-83/84 Graphing Calculator,* for additional instructions on working with sequences.



(a) (b)

FIGURE 4 $u(n) = 3(u(n - 1) + 2), u(1) = 1$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **15** AND **25**

EXAMPLE 4 | The Fibonacci Sequence

Find the first 11 terms of the sequence defined recursively by $F_1 = 1$, $F_2 = 1$ and

$$F_n = F_{n-1} + F_{n-2}$$

FIBONACCI (1175–1250) was born in Pisa, Italy, and was educated in North Africa. He traveled widely in the Mediterranean area and learned the various methods then in use for writing numbers. On returning to Pisa in 1202, Fibonacci advocated the use of the Hindu-Arabic decimal system, the one we use today, over the Roman numeral system that was used in Europe in his time. His most famous book, *Liber Abaci,* expounds on the advantages of the Hindu-Arabic numerals. In fact, multiplication and division were so complicated using Roman numerals that a college degree was necessary to master these skills. Interestingly, in 1299 the city of Florence outlawed the use of the decimal system for merchants and businesses, requiring numbers to be written in Roman numerals or words. One can only speculate about the reasons for this law.

SOLUTION To find F_n, we need to find the two preceding terms, F_{n-1} and F_{n-2}. Since we are given F_1 and F_2, we proceed as follows.

$$F_3 = F_2 + F_1 = 1 + 1 = 2$$

$$F_4 = F_3 + F_2 = 2 + 1 = 3$$

$$F_5 = F_4 + F_3 = 3 + 2 = 5$$

It's clear what is happening here. Each term is simply the sum of the two terms that precede it, so we can easily write down as many terms as we please. Here are the first 11 terms. (You can also find these using a graphing calculator.)

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **19**

The sequence in Example 4 is called the **Fibonacci sequence**, named after the 13th century Italian mathematician who used it to solve a problem about the breeding of rabbits (see Exercise 85). The sequence also occurs in numerous other applications in nature. (See Figures 5 and 6.) In fact, so many phenomena behave like the Fibonacci sequence that one mathematical journal, the *Fibonacci Quarterly*, is devoted entirely to its properties.

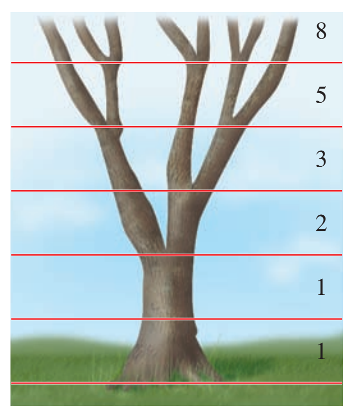


FIGURE 5 The Fibonacci sequence in the branching of a tree

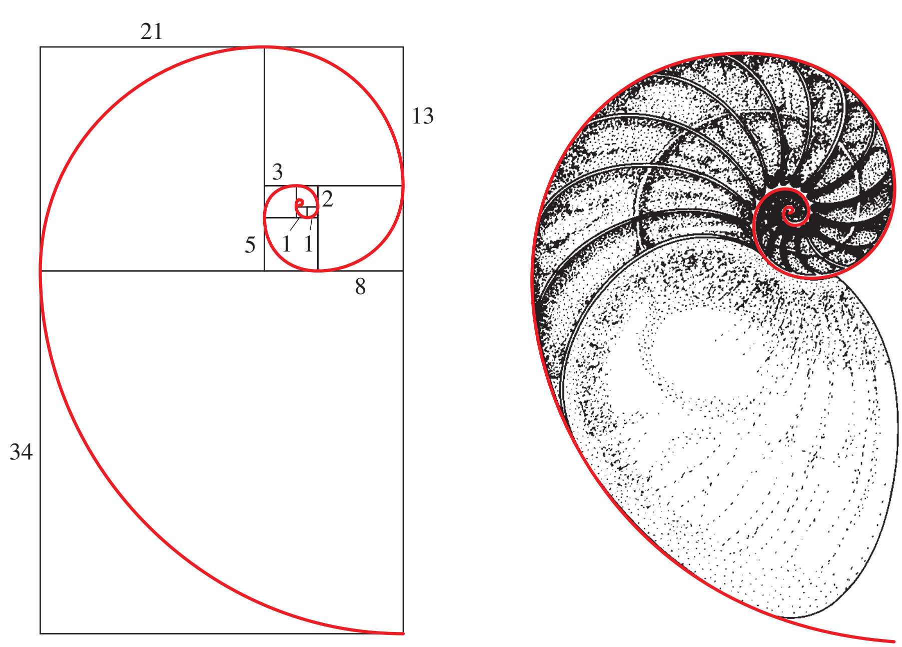


FIGURE 6 Fibonacci spiral Nautilus shell

The Partial Sums of a Sequence

In calculus we are often interested in adding the terms of a sequence. This leads to the following definition.

THE PARTIAL SUMS OF A SEQUENCE

For the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

the **partial sums** are

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
S_4 &= a_1 + a_2 + a_3 + a_4 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\
&\vdots
\end{aligned}$$

S_1 is called the **first partial sum**, S_2 is the **second partial sum**, and so on. S_n is called the ***n*th partial sum**. The sequence $S_1, S_2, S_3, \ldots, S_n, \ldots$ is called the **sequence of partial sums**.

EXAMPLE 5 | Finding the Partial Sums of a Sequence

Find the first four partial sums and the nth partial sum of the sequence given by $a_n = 1/2^n$.

SOLUTION The terms of the sequence are

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

The first four partial sums are

$$\begin{aligned}
S_1 &= \frac{1}{2} &&= \frac{1}{2} \\
S_2 &= \frac{1}{2} + \frac{1}{4} &&= \frac{3}{4} \\
S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &&= \frac{7}{8} \\
S_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &&= \frac{15}{16}
\end{aligned}$$

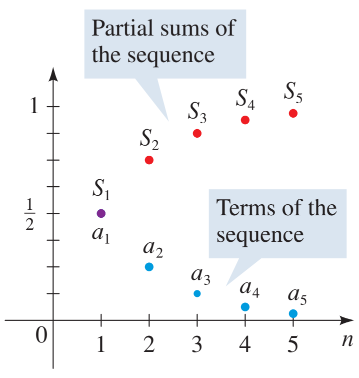


FIGURE 7 Graph of the sequence a_n and the sequence of partial sums S_n

Notice that in the value of each partial sum, the denominator is a power of 2 and the numerator is one less than the denominator. In general, the nth partial sum is

$$S_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

The first five terms of a_n and S_n are graphed in Figure 7.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **43**

EXAMPLE 6 | Finding the Partial Sums of a Sequence

Find the first four partial sums and the nth partial sum of the sequence given by

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

SOLUTION The first four partial sums are

$$S_1 = \left(1 - \frac{1}{2}\right) = 1 - \frac{1}{2}$$

$$S_2 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$S_4 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right) = 1 - \frac{1}{5}$$

Do you detect a pattern here? Of course. The nth partial sum is

$$S_n = 1 - \frac{1}{n+1}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45**

Sigma Notation

Given a sequence

$$a_1, a_2, a_3, a_4, \ldots$$

we can write the sum of the first n terms using **summation notation**, or **sigma notation**. This notation derives its name from the Greek letter Σ (capital sigma, corresponding to our S for "sum"). Sigma notation is used as follows:

This tells us to end with $k = n$

This tells us to add

$$\sum_{k=1}^{n} a_k$$

This tells us to start with $k = 1$

$$\boxed{\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + \cdots + a_n}$$

The left side of this expression is read, "The sum of a_k from $k = 1$ to $k = n$." The letter k is called the **index of summation**, or the **summation variable**, and the idea is to replace k in the expression after the sigma by the integers $1, 2, 3, \ldots, n$, and add the resulting expressions, arriving at the right side of the equation.

EXAMPLE 7 | Sigma Notation

Find each sum.

(a) $\displaystyle\sum_{k=1}^{5} k^2$ **(b)** $\displaystyle\sum_{j=3}^{5} \frac{1}{j}$ **(c)** $\displaystyle\sum_{i=5}^{10} i$ **(d)** $\displaystyle\sum_{i=1}^{6} 2$

SOLUTION

(a) $$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

(b) $$\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$$

(c) $\displaystyle\sum_{i=5}^{10} i = 5 + 6 + 7 + 8 + 9 + 10 = 45$

(d) $\displaystyle\sum_{i=1}^{6} 2 = 2 + 2 + 2 + 2 + 2 + 2 = 12$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **47** AND **49**

```
sum(seq(K²,K,1,5,1))
                  55
sum(seq(1/J,J,3,5,
1))►Frac
               47/60
```

FIGURE 8

We can use a graphing calculator to evaluate sums. For instance, Figure 8 shows how the TI-83 can be used to evaluate the sums in parts (a) and (b) of Example 7.

EXAMPLE 8 | Writing Sums in Sigma Notation

Write each sum using sigma notation.

(a) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$

(b) $\sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{77}$

SOLUTION

(a) We can write

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = \sum_{k=1}^{7} k^3$$

(b) A natural way to write this sum is

$$\sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{77} = \sum_{k=3}^{77} \sqrt{k}$$

However, there is no unique way of writing a sum in sigma notation. We could also write this sum as

$$\sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{77} = \sum_{k=0}^{74} \sqrt{k+3}$$

or

$$\sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{77} = \sum_{k=1}^{75} \sqrt{k+2}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **67** AND **69**

The Golden Ratio

The ancient Greeks considered a line segment to be divided into the **golden ratio** if the ratio of the shorter part to the longer part is the same as the ratio of the longer part to the whole segment.

Thus the segment shown is divided into the golden ratio if

$$\frac{1}{x} = \frac{x}{1+x}$$

This leads to a quadratic equation whose positive solution is

$$x = \frac{1+\sqrt{5}}{2} \approx 1.618$$

This ratio occurs naturally in many places. For instance, psychological experiments show that the most pleasing shape of rectangle is one whose sides are in golden ratio. The ancient Greeks agreed with this and built their temples in this ratio.

The golden ratio is related to the Fibonacci sequence. In fact, it can be shown by using calculus* that the ratio of two successive Fibonacci numbers

$$\frac{F_{n+1}}{F_n}$$

gets closer to the golden ratio the larger the value of n. Try finding this ratio for $n = 10$.

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*See *Principles of Problem Solving* 13 at the book companion website: **www.stewartmath.com**

The following properties of sums are natural consequences of properties of the real numbers.

PROPERTIES OF SUMS

Let $a_1, a_2, a_3, a_4, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$ be sequences. Then for every positive integer n and any real number c, the following properties hold.

1. $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. $\displaystyle\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. $\displaystyle\sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right)$

PROOF To prove Property 1, we write out the left side of the equation to get

$$\sum_{k=1}^{n}(a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n)$$

Because addition is commutative and associative, we can rearrange the terms on the right side to read

$$\sum_{k=1}^{n}(a_k + b_k) = (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n)$$

Rewriting the right side using sigma notation gives Property 1. Property 2 is proved in a similar manner. To prove Property 3, we use the Distributive Property:

$$\begin{aligned}\sum_{k=1}^{n} ca_k &= ca_1 + ca_2 + ca_3 + \cdots + ca_n \\ &= c(a_1 + a_2 + a_3 + \cdots + a_n) = c\left(\sum_{k=1}^{n} a_k\right)\end{aligned}$$

8.1 EXERCISES

CONCEPTS

1. A sequence is a function whose domain is ________.

2. The nth partial sum of a sequence is the sum of the first ________ terms of the sequence. So for the sequence $a_n = n^2$ the fourth partial sum is $S_4 =$ ____ + ____ + ____ + ____ = ____.

SKILLS

3–14 ■ Find the first four terms and the 100th term of the sequence.

3. $a_n = n + 1$

4. $a_n = 2n + 3$

5. $a_n = \dfrac{1}{n + 1}$

6. $a_n = n^2 + 1$

7. $a_n = 5^n$

8. $a_n = \left(\dfrac{-1}{3}\right)^n$

9. $a_n = \dfrac{(-1)^n}{n^2}$

10. $a_n = \dfrac{1}{n^2}$

11. $a_n = 1 + (-1)^n$

12. $a_n = (-1)^{n+1}\dfrac{n}{n + 1}$

13. $a_n = n^n$

14. $a_n = 3$

15–20 ■ Find the first five terms of the given recursively defined sequence.

15. $a_n = 2(a_{n-1} - 2)$ and $a_1 = 3$

16. $a_n = \dfrac{a_{n-1}}{2}$ and $a_1 = -8$

17. $a_n = 2a_{n-1} + 1$ and $a_1 = 1$

18. $a_n = \dfrac{1}{1 + a_{n-1}}$ and $a_1 = 1$

19. $a_n = a_{n-1} + a_{n-2}$ and $a_1 = 1, a_2 = 2$

20. $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ and $a_1 = a_2 = a_3 = 1$

21–26 ■ Use a graphing calculator to do the following. **(a)** Find the first 10 terms of the sequence. **(b)** Graph the first 10 terms of the sequence.

21. $a_n = 4n + 3$

22. $a_n = n^2 + n$

23. $a_n = \dfrac{12}{n}$

24. $a_n = 4 - 2(-1)^n$

25. $a_n = \dfrac{1}{a_{n-1}}$ and $a_1 = 2$

26. $a_n = a_{n-1} - a_{n-2}$ and $a_1 = 1, a_2 = 3$

27–38 ■ Find the nth term of a sequence whose first several terms are given.

27. $2, 4, 6, 8, \ldots$

28. $1, 3, 5, 7, \ldots$

29. $2, 4, 8, 16, \ldots$

30. $-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots$

31. $1, 4, 7, 10, \ldots$

32. $3, 7, 11, 15, \ldots$

33. $5, -25, 125, -625, \ldots$

34. $3, 0.3, 0.03, 0.003, \ldots$

35. $1, \frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \ldots$

36. $\frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$

37. $0, 2, 0, 2, 0, 2, \ldots$

38. $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots$

39–42 ■ Find the first six partial sums $S_1, S_2, S_3, S_4, S_5, S_6$ of the sequence.

39. $1, 3, 5, 7, \ldots$

40. $1^2, 2^2, 3^2, 4^2, \ldots$

41. $\dfrac{1}{3}, \dfrac{1}{3^2}, \dfrac{1}{3^3}, \dfrac{1}{3^4}, \ldots$

42. $-1, 1, -1, 1, \ldots$

43–46 ■ Find the first four partial sums and the nth partial sum of the sequence a_n.

43. $a_n = \dfrac{2}{3^n}$

44. $a_n = \dfrac{1}{n+1} - \dfrac{1}{n+2}$

45. $a_n = \sqrt{n} - \sqrt{n+1}$

46. $a_n = \log\left(\dfrac{n}{n+1}\right)$ [*Hint:* Use a property of logarithms to write the nth term as a difference.]

47–54 ■ Find the sum.

47. $\displaystyle\sum_{k=1}^{4} k$

48. $\displaystyle\sum_{k=1}^{4} k^2$

49. $\displaystyle\sum_{k=1}^{3} \frac{1}{k}$

50. $\displaystyle\sum_{j=1}^{100} (-1)^j$

51. $\displaystyle\sum_{i=1}^{8} [1 + (-1)^i]$

52. $\displaystyle\sum_{i=4}^{12} 10$

53. $\displaystyle\sum_{k=1}^{5} 2^{k-1}$

54. $\displaystyle\sum_{i=1}^{3} i2^i$

55–60 ■ Use a graphing calculator to evaluate the sum.

55. $\displaystyle\sum_{k=1}^{10} k^2$

56. $\displaystyle\sum_{k=1}^{100} (3k + 4)$

57. $\displaystyle\sum_{j=7}^{20} j^2(1 + j)$

58. $\displaystyle\sum_{j=5}^{15} \frac{1}{j^2 + 1}$

59. $\displaystyle\sum_{n=0}^{22} (-1)^n 2n$

60. $\displaystyle\sum_{n=1}^{100} \frac{(-1)^n}{n}$

61–66 ■ Write the sum without using sigma notation.

61. $\displaystyle\sum_{k=1}^{5} \sqrt{k}$

62. $\displaystyle\sum_{i=0}^{4} \frac{2i - 1}{2i + 1}$

63. $\displaystyle\sum_{k=0}^{6} \sqrt{k + 4}$

64. $\displaystyle\sum_{k=6}^{9} k(k + 3)$

65. $\displaystyle\sum_{k=3}^{100} x^k$

66. $\displaystyle\sum_{j=1}^{n} (-1)^{j+1} x^j$

67–74 ■ Write the sum using sigma notation.

67. $1 + 2 + 3 + 4 + \cdots + 100$

68. $2 + 4 + 6 + \cdots + 20$

69. $1^2 + 2^2 + 3^2 + \cdots + 10^2$

70. $\dfrac{1}{2 \ln 2} - \dfrac{1}{3 \ln 3} + \dfrac{1}{4 \ln 4} - \dfrac{1}{5 \ln 5} + \cdots + \dfrac{1}{100 \ln 100}$

71. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{999 \cdot 1000}$

72. $\dfrac{\sqrt{1}}{1^2} + \dfrac{\sqrt{2}}{2^2} + \dfrac{\sqrt{3}}{3^2} + \cdots + \dfrac{\sqrt{n}}{n^2}$

73. $1 + x + x^2 + x^3 + \cdots + x^{100}$

74. $1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots - 100x^{99}$

75. Find a formula for the nth term of the sequence

$$\sqrt{2}, \quad \sqrt{2\sqrt{2}}, \quad \sqrt{2\sqrt{2\sqrt{2}}}, \quad \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$

[*Hint:* Write each term as a power of 2.]

76. Define the sequence

$$G_n = \frac{1}{\sqrt{5}}\left(\frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n}\right)$$

Use the TABLE command on a graphing calculator to find the first 10 terms of this sequence. Compare to the Fibonacci sequence F_n.

APPLICATIONS

77. Compound Interest Julio deposits \$2000 in a savings account that pays 2.4% interest per year compounded monthly. The amount in the account after n months is given by the sequence

$$A_n = 2000\left(1 + \frac{0.024}{12}\right)^n$$

(a) Find the first six terms of the sequence.
(b) Find the amount in the account after 3 years.

78. Compound Interest Helen deposits \$100 at the end of each month into an account that pays 6% interest per year compounded monthly. The amount of interest she has accumulated after n months is given by the sequence

$$I_n = 100\left(\frac{1.005^n - 1}{0.005} - n\right)$$

(a) Find the first six terms of the sequence.
(b) Find the interest she has accumulated after 5 years.

79. Population of a City A city was incorporated in 2004 with a population of 35,000. It is expected that the population will increase at a rate of 2% per year. The population n years after 2004 is given by the sequence

$$P_n = 35{,}000(1.02)^n$$

(a) Find the first five terms of the sequence.
(b) Find the population in 2014.

80. Paying off a Debt Margarita borrows \$10,000 from her uncle and agrees to repay it in monthly installments of \$200. Her uncle charges 0.5% interest per month on the balance.
(a) Show that her balance A_n in the nth month is given recursively by $A_0 = 10{,}000$ and

$$A_n = 1.005A_{n-1} - 200$$

(b) Find her balance after six months.

81. Fish Farming A fish farmer has 5000 catfish in his pond. The number of catfish increases by 8% per month, and the farmer harvests 300 catfish per month.
(a) Show that the catfish population P_n after n months is given recursively by $P_0 = 5000$ and

$$P_n = 1.08P_{n-1} - 300$$

(b) How many fish are in the pond after 12 months?

82. Price of a House The median price of a house in Orange County increases by about 6% per year. In 2002 the median price was \$240,000. Let P_n be the median price n years after 2002.
(a) Find a formula for the sequence P_n.
(b) Find the expected median price in 2010.

83. Salary Increases A newly hired salesman is promised a beginning salary of \$30,000 a year with a \$2000 raise every year. Let S_n be his salary in his nth year of employment.
(a) Find a recursive definition of S_n.
(b) Find his salary in his fifth year of employment.

84. Concentration of a Solution A biologist is trying to find the optimal salt concentration for the growth of a certain species of mollusk. She begins with a brine solution that has 4 g/L of salt and increases the concentration by 10% every day. Let C_0 denote the initial concentration, and let C_n be the concentration after n days.
(a) Find a recursive definition of C_n.
(b) Find the salt concentration after 8 days.

85. Fibonacci's Rabbits Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair that becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month? Show that the answer is F_n, where F_n is the nth term of the Fibonacci sequence.

DISCOVERY ■ DISCUSSION ■ WRITING

86. Different Sequences That Start the Same
(a) Show that the first four terms of the sequence $a_n = n^2$ are

$$1, 4, 9, 16, \ldots$$

(b) Show that the first four terms of the sequence $a_n = n^2 + (n-1)(n-2)(n-3)(n-4)$ are also

$$1, 4, 9, 16, \ldots$$

(c) Find a sequence whose first six terms are the same as those of $a_n = n^2$ but whose succeeding terms differ from this sequence.
(d) Find two different sequences that begin

$$2, 4, 8, 16, \ldots$$

87. A Recursively Defined Sequence Find the first 40 terms of the sequence defined by

$$a_{n+1} = \begin{cases} \dfrac{a_n}{2} & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$$

and $a_1 = 11$. Do the same if $a_1 = 25$. Make a conjecture about this type of sequence. Try several other values for a_1, to test your conjecture.

88. A Different Type of Recursion Find the first 10 terms of the sequence defined by

$$a_n = a_{n-a_{n-1}} + a_{n-a_{n-2}}$$

with

$$a_1 = 1 \qquad \text{and} \qquad a_2 = 1$$

How is this recursive sequence different from the others in this section?

8.2 ARITHMETIC SEQUENCES

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find the terms of an arithmatic sequence ▶ Find the partial sums of an arithmetic sequence

In this section we study a special type of sequence, called an arithmetic sequence.

▼ Arithmetic Sequences

Perhaps the simplest way to generate a sequence is to start with a number a and add to it a fixed constant d, over and over again.

DEFINITION OF AN ARITHMETIC SEQUENCE

An **arithmetic sequence** is a sequence of the form

$$a, a + d, a + 2d, a + 3d, a + 4d, \ldots$$

The number a is the **first term**, and d is the **common difference** of the sequence. The ***n*th term** of an arithmetic sequence is given by

$$a_n = a + (n - 1)d$$

The number d is called the common difference because any two consecutive terms of an arithmetic sequence differ by d.

EXAMPLE 1 | Arithmetic Sequences

(a) If $a = 2$ and $d = 3$, then we have the arithmetic sequence

$$2, 2 + 3, 2 + 6, 2 + 9, \ldots$$

or

$$2, 5, 8, 11, \ldots$$

Any two consecutive terms of this sequence differ by $d = 3$. The nth term is $a_n = 2 + 3(n - 1)$.

(b) Consider the arithmetic sequence

$$9, 4, -1, -6, -11, \ldots$$

Here the common difference is $d = -5$. The terms of an arithmetic sequence decrease if the common difference is negative. The nth term is $a_n = 9 - 5(n - 1)$.

(c) The graph of the arithmetic sequence $a_n = 1 + 2(n - 1)$ is shown in Figure 1. Notice that the points in the graph lie on the straight line $y = 2x - 1$, which has slope $d = 2$.

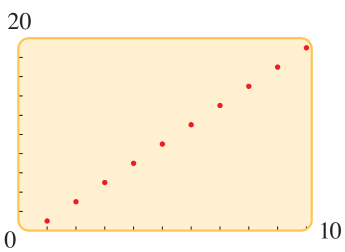


FIGURE 1

See Appendix C, *Using the TI-83/84 Graphing Calculator,* for instructions on how to graph sequences.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **5**, **9**, AND **13**

An arithmetic sequence is determined completely by the first term a and the common difference d. Thus if we know the first two terms of an arithmetic sequence, then we can find a formula for the nth term, as the next example shows.

EXAMPLE 2 | Finding Terms of an Arithmetic Sequence

Find the first six terms and the 300th term of the arithmetic sequence

$$13, 7, \ldots$$

SOLUTION Since the first term is 13, we have $a = 13$. The common difference is $d = 7 - 13 = -6$. Thus the nth term of this sequence is

$$a_n = 13 - 6(n - 1)$$

From this we find the first six terms:

$$13, 7, 1, -5, -11, -17, \ldots$$

The 300th term is $a_{300} = 13 - 6(300 - 1) = -1781$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **29**

The next example shows that an arithmetic sequence is determined completely by *any* two of its terms.

MATHEMATICS IN THE MODERN WORLD

Fair Division of Assets

Dividing an asset fairly among a number of people is of great interest to mathematicians. Problems of this nature include dividing the national budget, disputed land, or assets in divorce cases. In 1994 Brams and Taylor found a mathematical way of dividing things fairly. Their solution has been applied to division problems in political science, legal proceedings, and other areas. To understand the problem, consider the following example. Suppose persons *A* and *B* want to divide a property fairly between them. To divide it *fairly* means that both *A* and *B* must be satisfied with the outcome of the division. Solution: *A* gets to divide the property into two pieces, then *B* gets to choose the piece he or she wants. Since both *A* and *B* had a part in the division process, each should be satisfied. The situation becomes much more complicated if three or more people are involved (and that's where mathematics comes in). Dividing things fairly involves much more than simply cutting things in half; it must take into account the *relative worth* each person attaches to the thing being divided. A story from the Bible illustrates this clearly. Two women appear before King Solomon, each claiming to be the mother of the same newborn baby. To discover which of these two women is the real mother, King Solomon ordered his swordsman to cut the baby in half! The real mother, who attaches far more worth to the baby than anyone else does, immediately gives up her claim to the baby to save the baby's life.

Mathematical solutions to fair-division problems have recently been applied in an international treaty, the Convention on the Law of the Sea. If a country wants to develop a portion of the sea floor, it is required to divide the portion into two parts, one part to be used by itself and the other by a consortium that will preserve it for later use by a less developed country. The consortium gets first pick.

EXAMPLE 3 | Finding Terms of an Arithmetic Sequence

The 11th term of an arithmetic sequence is 52, and the 19th term is 92. Find the 1000th term.

SOLUTION To find the nth term of this sequence, we need to find a and d in the formula

$$a_n = a + (n - 1)d$$

From this formula we get

$$a_{11} = a + (11 - 1)d = a + 10d$$
$$a_{19} = a + (19 - 1)d = a + 18d$$

Since $a_{11} = 52$ and $a_{19} = 92$, we get the following two equations:

$$\begin{cases} 52 = a + 10d \\ 92 = a + 18d \end{cases}$$

Solving this system for a and d, we get $a = 2$ and $d = 5$. (Verify this.) Thus the nth term of this sequence is

$$a_n = 2 + 5(n - 1)$$

The 1000th term is $a_{1000} = 2 + 5(1000 - 1) = 4997$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41** ■

Partial Sums of Arithmetic Sequences

Suppose we want to find the sum of the numbers 1, 2, 3, 4, . . . , 100, that is,

$$\sum_{k=1}^{100} k$$

When the famous mathematician C. F. Gauss (see page 306) was a schoolboy, his teacher posed this problem to the class and expected that it would keep the students busy for a long time. But Gauss answered the question almost immediately. His idea was this: Since we are adding numbers produced according to a fixed pattern, there must also be a pattern (or formula) for finding the sum. He started by writing the numbers from 1 to 100 and then below them wrote the same numbers in reverse order. Writing S for the sum and adding corresponding terms give

$$\begin{array}{rcrcrcrcrcrcrcr} S &=& 1 &+& 2 &+& 3 &+& \cdots &+& 98 &+& 99 &+& 100 \\ S &=& 100 &+& 99 &+& 98 &+& \cdots &+& 3 &+& 2 &+& 1 \\ \hline 2S &=& 101 &+& 101 &+& 101 &+& \cdots &+& 101 &+& 101 &+& 101 \end{array}$$

It follows that $2S = 100(101) = 10{,}100$ and so $S = 5050$.

Of course, the sequence of natural numbers 1, 2, 3, . . . is an arithmetic sequence (with $a = 1$ and $d = 1$), and the method for summing the first 100 terms of this sequence can be used to find a formula for the nth partial sum of any arithmetic sequence. We want to find the sum of the first n terms of the arithmetic sequence whose terms are $a_k = a + (k - 1)d$; that is, we want to find

$$\begin{aligned} S_n &= \sum_{k=1}^{n} [a + (k - 1)d] \\ &= a + (a + d) + (a + 2d) + (a + 3d) + \cdots + [a + (n - 1)d] \end{aligned}$$

Using Gauss's method, we write

$$\begin{array}{rcccccccc} S_n &=& a &+& (a + d) &+ \cdots +& [a + (n - 2)d] &+& [a + (n - 1)d] \\ S_n &=& [a + (n - 1)d] &+& [a + (n - 2)d] &+ \cdots +& (a + d) &+& a \\ \hline 2S_n &=& [2a + (n - 1)d] &+& [2a + (n - 1)d] &+ \cdots +& [2a + (n - 1)d] &+& [2a + (n - 1)d] \end{array}$$

There are n identical terms on the right side of this equation, so

$$2S_n = n[2a + (n-1)d]$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Notice that $a_n = a + (n-1)d$ is the nth term of this sequence. So we can write

$$S_n = \frac{n}{2}[a + a + (n-1)d] = n\left(\frac{a + a_n}{2}\right)$$

This last formula says that the sum of the first n terms of an arithmetic sequence is the average of the first and nth terms multiplied by n, the number of terms in the sum. We now summarize this result.

PARTIAL SUMS OF AN ARITHMETIC SEQUENCE

For the arithmetic sequence $a_n = a + (n-1)d$ the ***n*th partial sum**

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \cdots + [a + (n-1)d]$$

is given by either of the following formulas.

1. $S_n = \frac{n}{2}[2a + (n-1)d]$ **2.** $S_n = n\left(\frac{a + a_n}{2}\right)$

EXAMPLE 4 | Finding a Partial Sum of an Arithmetic Sequence

Find the sum of the first 40 terms of the arithmetic sequence

$$3, 7, 11, 15, \ldots$$

SOLUTION For this arithmetic sequence, $a = 3$ and $d = 4$. Using Formula 1 for the partial sum of an arithmetic sequence, we get

$$S_{40} = \tfrac{40}{2}[2(3) + (40-1)4] = 20(6 + 156) = 3240$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **47**

EXAMPLE 5 | Finding a Partial Sum of an Arithmetic Sequence

Find the sum of the first 50 odd numbers.

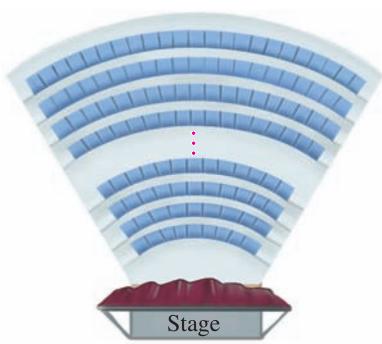
SOLUTION The odd numbers form an arithmetic sequence with $a = 1$ and $d = 2$. The nth term is $a_n = 1 + 2(n-1) = 2n - 1$, so the 50th odd number is $a_{50} = 2(50) - 1 = 99$. Substituting in Formula 2 for the partial sum of an arithmetic sequence, we get

$$S_{50} = 50\left(\frac{a + a_{50}}{2}\right) = 50\left(\frac{1 + 99}{2}\right) = 50 \cdot 50 = 2500$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **53**

EXAMPLE 6 | Finding the Seating Capacity of an Amphitheater

An amphitheater has 50 rows of seats with 30 seats in the first row, 32 in the second, 34 in the third, and so on. Find the total number of seats.



SOLUTION The numbers of seats in the rows form an arithmetic sequence with $a = 30$ and $d = 2$. Since there are 50 rows, the total number of seats is the sum

$$S_{50} = \tfrac{50}{2}[2(30) + 49(2)] \qquad S_n = \frac{n}{2}[2a + (n-1)d]$$
$$= 3950$$

Thus the amphitheater has 3950 seats.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **71**

EXAMPLE 7 | Finding the Number of Terms in a Partial Sum

How many terms of the arithmetic sequence 5, 7, 9, . . . must be added to get 572?

SOLUTION We are asked to find n when $S_n = 572$. Substituting $a = 5$, $d = 2$, and $S_n = 572$ in Formula 1 for the partial sum of an arithmetic sequence, we get

$$572 = \frac{n}{2}[2 \cdot 5 + (n-1)2] \qquad S_n = \frac{n}{2}[2a + (n-1)d]$$
$$572 = 5n + n(n-1) \qquad \text{Distributive Property}$$
$$0 = n^2 + 4n - 572 \qquad \text{Expand}$$
$$0 = (n - 22)(n + 26) \qquad \text{Factor}$$

This gives $n = 22$ or $n = -26$. But since n is the *number* of terms in this partial sum, we must have $n = 22$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **65**

8.2 EXERCISES

CONCEPTS

1. An arithmetic sequence is a sequence in which the ________ between successive terms is constant.

2. The sequence $a_n = a + (n-1)d$ is an arithmetic sequence in which a is the first term and d is the ________ ________. So for the arithmetic sequence $a_n = 2 + 5(n-1)$ the first term is ________, and the common difference is ________.

3. *True or false*? The nth partial sum of an arithmetic sequence is the average of the first and last terms times n.

4. *True or false*? If we know the first and second terms of an arithmetic sequence, then we can find any other term.

SKILLS

5–8 ■ A sequence is given. **(a)** Find the first five terms of the sequence. **(b)** What is the common difference d? **(c)** Graph the terms you found in (a).

5. $a_n = 5 + 2(n-1)$ **6.** $a_n = 3 - 4(n-1)$

7. $a_n = \frac{5}{2} - (n-1)$ **8.** $a_n = \frac{1}{2}(n-1)$

9–12 ■ Find the nth term of the arithmetic sequence with given first term a and common difference d. What is the 10th term?

9. $a = 3, d = 5$ **10.** $a = -6, d = 3$

11. $a = \frac{5}{2}, d = -\frac{1}{2}$ **12.** $a = \sqrt{3}, d = \sqrt{3}$

13–22 ■ Determine whether the sequence is arithmetic. If it is arithmetic, find the common difference.

13. 5, 8, 11, 14, . . . **14.** 3, 6, 9, 13, . . .

15. 56, 31, 6, −19, . . . **16.** 115, 101, 87, 73, . . .

17. 2, 4, 8, 16, . . . **18.** 2, 4, 6, 8, . . .

19. $3, \frac{3}{2}, 0, -\frac{3}{2}, \ldots$ **20.** ln 2, ln 4, ln 8, ln 16, . . .

21. 2.6, 4.3, 6.0, 7.7, . . . **22.** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

23–28 ■ Find the first five terms of the sequence, and determine whether it is arithmetic. If it is arithmetic, find the common difference, and express the nth term of the sequence in the standard form $a_n = a + (n-1)d$.

23. $a_n = 4 + 7n$ **24.** $a_n = 4 + 2^n$

25. $a_n = \dfrac{1}{1 + 2n}$ **26.** $a_n = 1 + \dfrac{n}{2}$

27. $a_n = 6n - 10$ **28.** $a_n = 3 + (-1)^n n$

29–40 ■ Determine the common difference, the fifth term, the nth term, and the 100th term of the arithmetic sequence.

29. $2, 5, 8, 11, \ldots$

30. $1, 5, 9, 13, \ldots$

31. $21, 13, 5, -3, \ldots$

32. $90, 66, 42, 18, \ldots$

33. $4, 9, 14, 19, \ldots$

34. $11, 8, 5, 2, \ldots$

35. $-12, -8, -4, 0, \ldots$

36. $\frac{7}{6}, \frac{5}{3}, \frac{13}{6}, \frac{8}{3}, \ldots$

37. $25, 26.5, 28, 29.5, \ldots$

38. $15, 12.3, 9.6, 6.9, \ldots$

39. $2, 2 + s, 2 + 2s, 2 + 3s, \ldots$

40. $-t, -t + 3, -t + 6, -t + 9, \ldots$

41. The tenth term of an arithmetic sequence is $\frac{55}{2}$, and the second term is $\frac{7}{2}$. Find the first term.

42. The 12th term of an arithmetic sequence is 32, and the fifth term is 18. Find the 20th term.

43. The 100th term of an arithmetic sequence is 98, and the common difference is 2. Find the first three terms.

44. The 20th term of an arithmetic sequence is 101, and the common difference is 3. Find a formula for the nth term.

45. Which term of the arithmetic sequence $1, 4, 7, \ldots$ is 88?

46. The first term of an arithmetic sequence is 1, and the common difference is 4. Is 11,937 a term of this sequence? If so, which term is it?

47–52 ■ Find the partial sum S_n of the arithmetic sequence that satisfies the given conditions.

47. $a = 1, d = 2, n = 10$

48. $a = 3, d = 2, n = 12$

49. $a = 5, d = -4, n = 20$

50. $a = 100, d = -5, n = 8$

51. $a_1 = 55, d = 12, n = 10$

52. $a_2 = 8, a_5 = 9.5, n = 15$

53–60 ■ A partial sum of an arithmetic sequence is given. Find the sum.

53. $1 + 5 + 9 + \cdots + 401$

54. $-3 + \left(-\frac{3}{2}\right) + 0 + \frac{3}{2} + 3 + \cdots + 30$

55. $250 + 233 + 216 + \cdots + 97$

56. $89 + 85 + 81 + \cdots + 13$

57. $0.7 + 2.7 + 4.7 + \cdots + 56.7$

58. $-10 - 9.9 - 9.8 - \cdots - 0.1$

59. $\sum_{k=0}^{10} (3 + 0.25k)$

60. $\sum_{n=0}^{20} (1 - 2n)$

61. Show that a right triangle whose sides are in arithmetic progression is similar to a 3–4–5 triangle.

62. Find the product of the numbers

$$10^{1/10}, 10^{2/10}, 10^{3/10}, 10^{4/10}, \ldots, 10^{19/10}$$

63. A sequence is **harmonic** if the reciprocals of the terms of the sequence form an arithmetic sequence. Determine whether the following sequence is harmonic:

$$1, \tfrac{3}{5}, \tfrac{3}{7}, \tfrac{1}{3}, \ldots$$

64. The **harmonic mean** of two numbers is the reciprocal of the average of the reciprocals of the two numbers. Find the harmonic mean of 3 and 5.

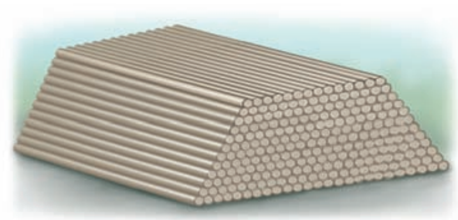
65. An arithmetic sequence has first term $a = 5$ and common difference $d = 2$. How many terms of this sequence must be added to get 2700?

66. An arithmetic sequence has first term $a_1 = 1$ and fourth term $a_4 = 16$. How many terms of this sequence must be added to get 2356?

APPLICATIONS

67. Depreciation The purchase value of an office computer is \$12,500. Its annual depreciation is \$1875. Find the value of the computer after 6 years.

68. Poles in a Pile Telephone poles are being stored in a pile with 25 poles in the first layer, 24 in the second, and so on. If there are 12 layers, how many telephone poles does the pile contain?



69. Salary Increases A man gets a job with a salary of \$30,000 a year. He is promised a \$2300 raise each subsequent year. Find his total earnings for a 10-year period.

70. Drive-In Theater A drive-in theater has spaces for 20 cars in the first parking row, 22 in the second, 24 in the third, and so on. If there are 21 rows in the theater, find the number of cars that can be parked.

71. Theater Seating An architect designs a theater with 15 seats in the first row, 18 in the second, 21 in the third, and so on. If the theater is to have a seating capacity of 870, how many rows must the architect use in his design?

72. Falling Ball When an object is allowed to fall freely near the surface of the earth, the gravitational pull is such that the object falls 16 ft in the first second, 48 ft in the next second, 80 ft in the next second, and so on.

(a) Find the total distance a ball falls in 6 s.

(b) Find a formula for the total distance a ball falls in n seconds.

73. The Twelve Days of Christmas In the well-known song "The Twelve Days of Christmas," a person gives his sweetheart k gifts on the kth day for each of the 12 days of Christmas. The person also repeats each gift identically on each subsequent day. Thus, on the 12th day the sweetheart receives a gift for the first day, 2 gifts for the second, 3 gifts for the third, and so on. Show that the number of gifts received on the 12th day is a partial sum of an arithmetic sequence. Find this sum.

DISCOVERY ■ DISCUSSION ■ WRITING

74. Arithmetic Means The **arithmetic mean** (or average) of two numbers a and b is

$$m = \frac{a + b}{2}$$

Note that m is the same distance from a as from b, so a, m, b

is an arithmetic sequence. In general, if $m_1, m_2, \ldots, m_k$ are equally spaced between a and b so that

$$a, m_1, m_2, \ldots, m_k, b$$

is an arithmetic sequence, then $m_1, m_2, \ldots, m_k$ are called k arithmetic means between a and b.

(a) Insert two arithmetic means between 10 and 18.
(b) Insert three arithmetic means between 10 and 18.
(c) Suppose a doctor needs to increase a patient's dosage of a certain medicine from 100 mg to 300 mg per day in five equal steps. How many arithmetic means must be inserted between 100 and 300 to give the progression of daily doses, and what are these means?

8.3 Geometric Sequences

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find the terms of a geometric sequence ▶ Find the partial sums of a geometric sequence ▶ Find the sum of an infinite geometric sequence

In this section we study geometric sequences. This type of sequence occurs frequently in applications to finance, population growth, and other fields.

▼ Geometric Sequences

Recall that an arithmetic sequence is generated when we repeatedly add a number d to an initial term a. A *geometric* sequence is generated when we start with a number a and repeatedly *multiply* by a fixed nonzero constant r.

DEFINITION OF A GEOMETRIC SEQUENCE

A **geometric sequence** is a sequence of the form

$$a, ar, ar^2, ar^3, ar^4, \ldots$$

The number a is the **first term**, and r is the **common ratio** of the sequence. The ***n*th term** of a geometric sequence is given by

$$a_n = ar^{n-1}$$

The number r is called the common ratio because the ratio of any two consecutive terms of the sequence is r.

EXAMPLE 1 | Geometric Sequences

(a) If $a = 3$ and $r = 2$, then we have the geometric sequence

$$3, \quad 3 \cdot 2, \quad 3 \cdot 2^2, \quad 3 \cdot 2^3, \quad 3 \cdot 2^4, \quad \ldots$$

or

$$3, 6, 12, 24, 48, \ldots$$

Notice that the ratio of any two consecutive terms is $r = 2$. The nth term is $a_n = 3(2)^{n-1}$.

(b) The sequence

$$2, -10, 50, -250, 1250, \ldots$$

is a geometric sequence with $a = 2$ and $r = -5$. When r is negative, the terms of the sequence alternate in sign. The nth term is $a_n = 2(-5)^{n-1}$.

(c) The sequence

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$$

is a geometric sequence with $a = 1$ and $r = \frac{1}{3}$. The nth term is $a_n = 1\left(\frac{1}{3}\right)^{n-1}$.

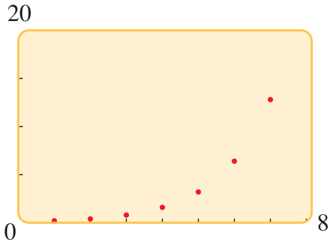


FIGURE 1

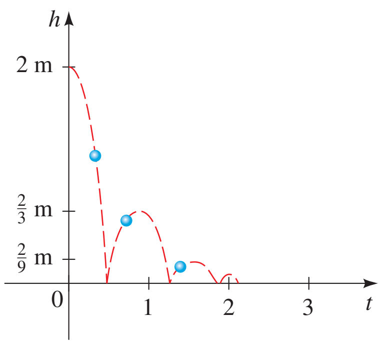


FIGURE 2

(d) The graph of the geometric sequence $a_n = \frac{1}{5} \cdot 2^{n-1}$ is shown in Figure 1. Notice that the points in the graph lie on the graph of the exponential function $y = \frac{1}{5} \cdot 2^{x-1}$.

If $0 < r < 1$, then the terms of the geometric sequence ar^{n-1} decrease, but if $r > 1$, then the terms increase. (What happens if $r = 1$?)

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **5**, **9**, AND **13**

Geometric sequences occur naturally. Here is a simple example. Suppose a ball has elasticity such that when it is dropped, it bounces up one-third of the distance it has fallen. If this ball is dropped from a height of 2 m, then it bounces up to a height of $2\left(\frac{1}{3}\right) = \frac{2}{3}$ m. On its second bounce, it returns to a height of $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9}$ m, and so on (see Figure 2). Thus the height h_n that the ball reaches on its nth bounce is given by the geometric sequence

$$h_n = \tfrac{2}{3}\left(\tfrac{1}{3}\right)^{n-1} = 2\left(\tfrac{1}{3}\right)^n$$

We can find the nth term of a geometric sequence if we know any two terms, as the following examples show.

EXAMPLE 2 | Finding Terms of a Geometric Sequence

Find the eighth term of the geometric sequence 5, 15, 45, ...

SOLUTION To find a formula for the nth term of this sequence, we need to find a and r. Clearly, $a = 5$. To find r, we find the ratio of any two consecutive terms. For instance, $r = \frac{45}{15} = 3$. Thus

$$a_n = 5(3)^{n-1}$$

The eighth term is $a_8 = 5(3)^{8-1} = 5(3)^7 = 10{,}935$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **29**

EXAMPLE 3 | Finding Terms of a Geometric Sequence

The third term of a geometric sequence is $\frac{63}{4}$, and the sixth term is $\frac{1701}{32}$. Find the fifth term.

SOLUTION Since this sequence is geometric, its nth term is given by the formula $a_n = ar^{n-1}$. Thus

$$a_3 = ar^{3-1} = ar^2$$
$$a_6 = ar^{6-1} = ar^5$$

From the values we are given for these two terms, we get the following system of equations:

$$\begin{cases} \frac{63}{4} = ar^2 \\ \frac{1701}{32} = ar^5 \end{cases}$$

We solve this system by dividing:

$$\frac{ar^5}{ar^2} = \frac{\frac{1701}{32}}{\frac{63}{4}}$$
$$r^3 = \tfrac{27}{8} \qquad \text{Simplify}$$
$$r = \tfrac{3}{2} \qquad \text{Take cube root of each side}$$

Substituting for r in the first equation gives

$$\tfrac{63}{4} = a\left(\tfrac{3}{2}\right)^2 \qquad \text{Substitute } r = \tfrac{3}{2} \text{ in } \tfrac{63}{4} = ar^2$$
$$a = 7 \qquad \text{Solve for } a$$

SRINIVASA RAMANUJAN (1887–1920) was born into a poor family in the small town of Kumbakonam in India. Self-taught in mathematics, he worked in virtual isolation from other mathematicians. At the age of 25 he wrote a letter to G. H. Hardy, the leading British mathematician at the time, listing some of his discoveries. Hardy immediately recognized Ramanujan's genius, and for the next six years the two worked together in London until Ramanujan fell ill and returned to his hometown in India, where he died a year later. Ramanujan was a genius with phenomenal ability to see hidden patterns in the properties of numbers. Most of his discoveries were written as complicated infinite series, the importance of which was not recognized until many years after his death. In the last year of his life he wrote 130 pages of mysterious formulas, many of which still defy proof. Hardy tells the story that when he visited Ramanujan in a hospital and arrived in a taxi, he remarked to Ramanujan that the cab's number, 1729, was uninteresting. Ramanujan replied "No, it is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways."

It follows that the nth term of this sequence is

$$a_n = 7\left(\tfrac{3}{2}\right)^{n-1}$$

Thus the fifth term is

$$a_5 = 7\left(\tfrac{3}{2}\right)^{5-1} = 7\left(\tfrac{3}{2}\right)^4 = \tfrac{567}{16}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41** ■

▼ Partial Sums of Geometric Sequences

For the geometric sequence $a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}, \ldots$, the nth partial sum is

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}$$

To find a formula for S_n, we multiply S_n by r and subtract from S_n:

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} \\ rS_n &= \phantom{a + {}} ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n \\ \hline S_n - rS_n &= a - ar^n \end{aligned}$$

So

$$S_n(1 - r) = a(1 - r^n)$$

$$S_n = \frac{a(1 - r^n)}{1 - r} \qquad (r \neq 1)$$

We summarize this result.

PARTIAL SUMS OF A GEOMETRIC SEQUENCE

For the geometric sequence $a_n = ar^{n-1}$, the **nth partial sum**

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} \qquad (r \neq 1)$$

is given by

$$S_n = a\frac{1 - r^n}{1 - r}$$

EXAMPLE 4 | Finding a Partial Sum of a Geometric Sequence

Find the sum of the first five terms of the geometric sequence

$$1, 0.7, 0.49, 0.343, \ldots$$

SOLUTION The required sum is the sum of the first five terms of a geometric sequence with $a = 1$ and $r = 0.7$. Using the formula for S_n with $n = 5$, we get

$$S_5 = 1 \cdot \frac{1 - (0.7)^5}{1 - 0.7} = 2.7731$$

Thus the sum of the first five terms of this sequence is 2.7731.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **49** AND **53** ■

EXAMPLE 5 | Finding a Partial Sum of a Geometric Sequence

Find the sum $\displaystyle\sum_{k=1}^{5} 7\left(-\tfrac{2}{3}\right)^k$.

SOLUTION The given sum is the fifth partial sum of a geometric sequence with first term $a = 7\left(-\frac{2}{3}\right)^1 = -\frac{14}{3}$ and common ratio $r = -\frac{2}{3}$. Thus by the formula for S_n we have

$$S_5 = -\frac{14}{3} \cdot \frac{1 - \left(-\frac{2}{3}\right)^5}{1 - \left(-\frac{2}{3}\right)} = -\frac{14}{3} \cdot \frac{1 + \frac{32}{243}}{\frac{5}{3}} = -\frac{770}{243}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **55**

What Is an Infinite Series?

An expression of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots$$

is called an **infinite series**. The dots mean that we are to continue the addition indefinitely. What meaning can we attach to the sum of infinitely many numbers? It seems at first that it is not possible to add infinitely many numbers and arrive at a finite number. But consider the following problem. You have a cake, and you want to eat it by first eating half the cake, then eating half of what remains, then again eating half of what remains. This process can continue indefinitely because at each stage, some of the cake remains. (See Figure 3.)

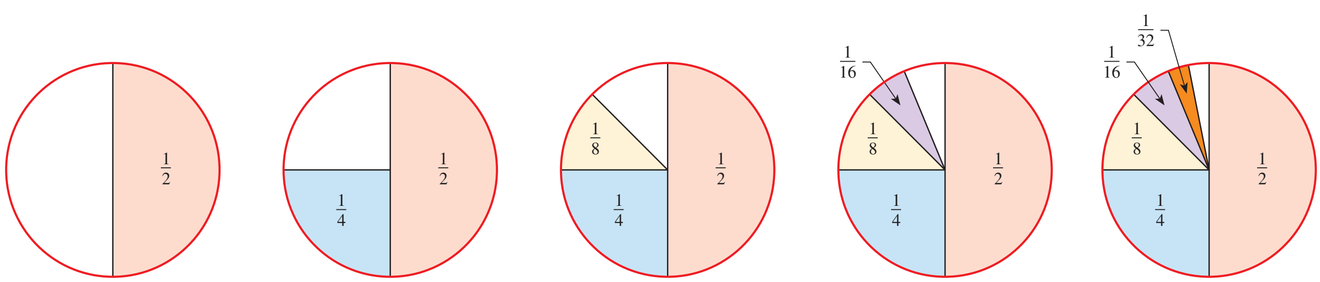


FIGURE 3

Does this mean that it's impossible to eat all of the cake? Of course not. Let's write down what you have eaten from this cake:

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

This is an infinite series, and we note two things about it: First, from Figure 3 it's clear that no matter how many terms of this series we add, the total will never exceed 1. Second, the more terms of this series we add, the closer the sum is to 1 (see Figure 3). This suggests that the number 1 can be written as the sum of infinitely many smaller numbers:

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$

To make this more precise, let's look at the partial sums of this series:

$$\begin{aligned} S_1 &= \frac{1}{2} &&= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{1}{4} &&= \frac{3}{4} \\ S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &&= \frac{7}{8} \\ S_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &&= \frac{15}{16} \end{aligned}$$

and, in general (see Example 5 of Section 8.1),

$$S_n = 1 - \frac{1}{2^n}$$

As n gets larger and larger, we are adding more and more of the terms of this series. Intuitively, as n gets larger, S_n gets closer to the sum of the series. Now notice that as n gets large, $1/2^n$ gets closer and closer to 0. Thus S_n gets close to $1 - 0 = 1$. Using the notation of Section 3.7, we can write

$$S_n \to 1 \quad \text{as} \quad n \to \infty$$

In general, if S_n gets close to a finite number S as n gets large, we say that the infinite series **converges** (or is **convergent**). The number S is called the **sum of the infinite series**. If an infinite series does not converge, we say that the series **diverges** (or is **divergent**).

Infinite Geometric Series

An **infinite geometric series** is a series of the form

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \cdots$$

We can apply the reasoning used earlier to find the sum of an infinite geometric series. The nth partial sum of such a series is given by the formula

$$S_n = a\frac{1 - r^n}{1 - r} \qquad (r \neq 1)$$

It can be shown that if $|r| < 1$, then r^n gets close to 0 as n gets large (you can easily convince yourself of this using a calculator). It follows that S_n gets close to $a/(1 - r)$ as n gets large, or

$$S_n \to \frac{a}{1 - r} \quad \text{as} \quad n \to \infty$$

Thus the sum of this infinite geometric series is $a/(1 - r)$.

Here is another way to arrive at the formula for the sum of an infinite geometric series:

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \cdots \\ &= a + r(a + ar + ar^2 + \cdots) \\ &= a + rS \end{aligned}$$

Solve the equation $S = a + rS$ for S to get

$$\begin{aligned} S - rS &= a \\ (1 - r)S &= a \\ S &= \frac{a}{1 - r} \end{aligned}$$

SUM OF AN INFINITE GEOMETRIC SERIES

If $|r| < 1$, then the infinite geometric series

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$$

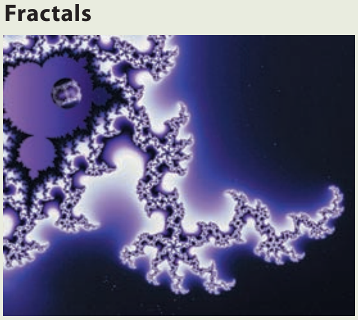
converges and has the sum

$$S = \frac{a}{1 - r}$$

If $|r| \geq 1$, the series diverges.

MATHEMATICS IN THE MODERN WORLD

Fractals



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Many of the things we model in this book have regular predictable shapes. But recent advances in mathematics have made it possible to model such seemingly random or even chaotic shapes as those of a cloud, a flickering flame, a mountain, or a jagged coastline. The basic tools in this type of modeling are the fractals invented by the mathematician Benoit Mandelbrot. A *fractal* is a geometric shape built up from a simple basic shape by scaling and repeating the shape indefinitely according to a given rule. Fractals have infinite detail; this means the closer you look, the more you see. They are also *self-similar*; that is, zooming in on a portion of the fractal yields the same detail as the original shape. Because of their beautiful shapes, fractals are used by movie makers to create fictional landscapes and exotic backgrounds.

Although a fractal is a complex shape, it is produced according to very simple rules. This property of fractals is exploited in a process of storing pictures on a computer called *fractal image compression*. In this process a picture is stored as a simple basic shape and a rule; repeating the shape according to the rule produces the original picture. This is an extremely efficient method of storage; that's how thousands of color pictures can be put on a single compact disc.

EXAMPLE 6 | Infinite Series

Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

(a) $2 + \dfrac{2}{5} + \dfrac{2}{25} + \dfrac{2}{125} + \cdots$ **(b)** $1 + \dfrac{7}{5} + \left(\dfrac{7}{5}\right)^2 + \left(\dfrac{7}{5}\right)^3 + \cdots$

SOLUTION

(a) This is an infinite geometric series with $a = 2$ and $r = \frac{1}{5}$. Since $|r| = \left|\frac{1}{5}\right| < 1$, the series converges. By the formula for the sum of an infinite geometric series we have

$$S = \frac{2}{1 - \frac{1}{5}} = \frac{5}{2}$$

(b) This is an infinite geometric series with $a = 1$ and $r = \frac{7}{5}$. Since $|r| = \left|\frac{7}{5}\right| > 1$, the series diverges.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **57** AND **61**

EXAMPLE 7 | Writing a Repeated Decimal as a Fraction

Find the fraction that represents the rational number $2.3\overline{51}$.

SOLUTION This repeating decimal can be written as a series:

$$\frac{23}{10} + \frac{51}{1000} + \frac{51}{100{,}000} + \frac{51}{10{,}000{,}000} + \frac{51}{1{,}000{,}000{,}000} + \cdots$$

After the first term, the terms of this series form an infinite geometric series with

$$a = \frac{51}{1000} \quad \text{and} \quad r = \frac{1}{100}$$

Thus the sum of this part of the series is

$$S = \frac{\frac{51}{1000}}{1 - \frac{1}{100}} = \frac{\frac{51}{1000}}{\frac{99}{100}} = \frac{51}{1000} \cdot \frac{100}{99} = \frac{51}{990}$$

So
$$2.3\overline{51} = \frac{23}{10} + \frac{51}{990} = \frac{2328}{990} = \frac{388}{165}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **69**

8.3 EXERCISES

CONCEPTS

1. A geometric sequence is a sequence in which the ________ of successive terms is constant.

2. The sequence $a_n = ar^{n-1}$ is a geometric sequence in which a is the first term and r is the ________ ________. So for the geometric sequence $a_n = 2(5)^{n-1}$ the first term is ________, and the common ratio is ________.

3. *True or false*? If we know the first and second terms of a geometric sequence, then we can find any other term.

4. **(a)** The nth partial sum of a geometric sequence $a_n = ar^{n-1}$ is given by $S_n =$ ________.

(b) The series $\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$ is an infinite ________ series. If $|r| < 1$, then this series ________, and its sum is $S =$ ________. If $|r| \geq 1$, the series ________.

SKILLS

5–8 ■ The nth term of a sequence is given. **(a)** Find the first five terms of the sequence. **(b)** What is the common ratio r? **(c)** Graph the terms you found in (a).

5. $a_n = 5(2)^{n-1}$

6. $a_n = 3(-4)^{n-1}$

7. $a_n = \frac{5}{2}\left(-\frac{1}{2}\right)^{n-1}$

8. $a_n = 3^{n-1}$

9–12 ■ Find the nth term of the geometric sequence with given first term a and common ratio r. What is the fourth term?

9. $a = 3, \quad r = 5$ **10.** $a = -6, \quad r = 3$

11. $a = \frac{5}{2}, \quad r = -\frac{1}{2}$ **12.** $a = \sqrt{3}, \quad r = \sqrt{3}$

13–22 ■ Determine whether the sequence is geometric. If it is geometric, find the common ratio.

13. $2, 4, 8, 16, \ldots$ **14.** $2, 6, 18, 36, \ldots$

15. $144, 96, 64, \frac{128}{3}, \ldots$ **16.** $48, 36, 27, \frac{81}{4}, \ldots$

17. $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots$ **18.** $27, -9, 3, -1, \ldots$

19. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ **20.** $e^2, e^4, e^6, e^8, \ldots$

21. $1.0, 1.1, 1.21, 1.331, \ldots$ **22.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$

23–28 ■ Find the first five terms of the sequence, and determine whether it is geometric. If it is geometric, find the common ratio, and express the nth term of the sequence in the standard form $a_n = ar^{n-1}$.

23. $a_n = 2(3)^n$ **24.** $a_n = 4 + 3^n$

25. $a_n = \dfrac{1}{4^n}$ **26.** $a_n = (-1)^n 2^n$

27. $a_n = \ln(5^{n-1})$ **28.** $a_n = n^n$

29–38 ■ Determine the common ratio, the fifth term, and the nth term of the geometric sequence.

29. $2, 6, 18, 54, \ldots$ **30.** $7, \frac{14}{3}, \frac{28}{9}, \frac{56}{27}, \ldots$

31. $0.3, -0.09, 0.027, -0.0081, \ldots$

32. $1, \sqrt{2}, 2, 2\sqrt{2}, \ldots$

33. $144, -12, 1, -\frac{1}{12}, \ldots$ **34.** $-8, -2, -\frac{1}{2}, -\frac{1}{8}, \ldots$

35. $3, 3^{5/3}, 3^{7/3}, 27, \ldots$ **36.** $t, \dfrac{t^2}{2}, \dfrac{t^3}{4}, \dfrac{t^4}{8}, \ldots$

37. $1, s^{2/7}, s^{4/7}, s^{6/7}, \ldots$ **38.** $5, 5^{c+1}, 5^{2c+1}, 5^{3c+1}, \ldots$

39. The first term of a geometric sequence is 8, and the second term is 4. Find the fifth term.

40. The first term of a geometric sequence is 3, and the third term is $\frac{4}{3}$. Find the fifth term.

41. The third term of a geometric sequence is $\frac{100}{9}$, and the sixth term is $\frac{800}{243}$. Find the first and the second term.

42. The fourth term of a geometric sequence is $\frac{135}{8}$, and the seventh term is $\frac{3645}{64}$. Find the first term and the third term.

43. The eighth term of a geometric sequence is 640, and the third term is 20. Find the first term and the nth term.

44. The third term of a geometric sequence is 12, and the sixth term is 768. Find the first term and the nth term.

45. The common ratio in a geometric sequence is $\frac{2}{5}$, and the fourth term is $\frac{5}{2}$. Find the third term.

46. The common ratio in a geometric sequence is $\frac{3}{2}$, and the fifth term is 1. Find the first three terms.

47. Which term of the geometric sequence $2, 6, 18, \ldots$ is 118,098?

48. The second and fifth terms of a geometric sequence are 10 and 1250, respectively. Is 31,250 a term of this sequence? If so, which term is it?

49–52 ■ Find the partial sum S_n of the geometric sequence that satisfies the given conditions.

49. $a = 5, \quad r = 2, \quad n = 6$ **50.** $a = \frac{2}{3}, \quad r = \frac{1}{3}, \quad n = 4$

51. $a_3 = 28, \quad a_6 = 224, \quad n = 6$

52. $a_2 = 0.12, \quad a_5 = 0.00096, \quad n = 4$

53–56 ■ Find the sum.

53. $1 + 3 + 9 + \cdots + 2187$

54. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{512}$

55. $\displaystyle\sum_{k=0}^{10} 3\left(\tfrac{1}{2}\right)^k$ **56.** $\displaystyle\sum_{j=0}^{5} 7\left(\tfrac{3}{2}\right)^j$

57–68 ■ Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

57. $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \cdots$ **58.** $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \cdots$

59. $1 - \dfrac{1}{3} + \dfrac{1}{9} - \dfrac{1}{27} + \cdots$ **60.** $\dfrac{2}{5} + \dfrac{4}{25} + \dfrac{8}{125} + \cdots$

61. $1 + \dfrac{3}{2} + \left(\dfrac{3}{2}\right)^2 + \left(\dfrac{3}{2}\right)^3 + \cdots$

62. $\dfrac{1}{3^6} + \dfrac{1}{3^8} + \dfrac{1}{3^{10}} + \dfrac{1}{3^{12}} + \cdots$

63. $3 - \dfrac{3}{2} + \dfrac{3}{4} - \dfrac{3}{8} + \cdots$

64. $1 - 1 + 1 - 1 + \cdots$

65. $3 - 3(1.1) + 3(1.1)^2 - 3(1.1)^3 + \cdots$

66. $-\dfrac{100}{9} + \dfrac{10}{3} - 1 + \dfrac{3}{10} - \cdots$

67. $\dfrac{1}{\sqrt{2}} + \dfrac{1}{2} + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{4} + \cdots$

68. $1 - \sqrt{2} + 2 - 2\sqrt{2} + 4 - \cdots$

69–74 ■ Express the repeating decimal as a fraction.

69. $0.777\ldots$ **70.** $0.2\overline{53}$ **71.** $0.030303\ldots$

72. $2.11\overline{25}$ **73.** $0.\overline{112}$ **74.** $0.123123123\ldots$

75. If the numbers $a_1, a_2, \ldots, a_n$ form a geometric sequence, then $a_2, a_3, \ldots, a_{n-1}$ are **geometric means** between a_1 and a_n. Insert three geometric means between 5 and 80.

76. Find the sum of the first ten terms of the sequence

$$a + b, a^2 + 2b, a^3 + 3b, a^4 + 4b, \ldots$$

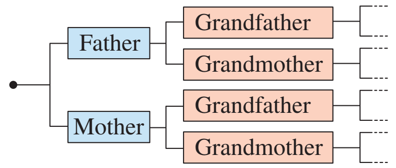
APPLICATIONS

77. Depreciation A construction company purchases a bulldozer for \$160,000. Each year the value of the bulldozer depreciates by 20% of its value in the preceding year. Let V_n be the value of the bulldozer in the nth year. (Let $n = 1$ be the year the bulldozer is purchased.)

(a) Find a formula for V_n.

(b) In what year will the value of the bulldozer be less than \$100,000?

78. Family Tree A person has two parents, four grandparents, eight great-grandparents, and so on. How many ancestors does a person have 15 generations back?

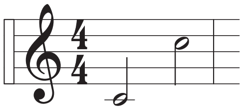


79. Bouncing Ball A ball is dropped from a height of 80 ft. The elasticity of this ball is such that it rebounds three-fourths of the distance it has fallen. How high does the ball rebound on the fifth bounce? Find a formula for how high the ball rebounds on the nth bounce.

80. Bacteria Culture A culture initially has 5000 bacteria, and its size increases by 8% every hour. How many bacteria are present at the end of 5 hours? Find a formula for the number of bacteria present after n hours.

81. Mixing Coolant A truck radiator holds 5 gal and is filled with water. A gallon of water is removed from the radiator and replaced with a gallon of antifreeze; then a gallon of the mixture is removed from the radiator and again replaced by a gallon of antifreeze. This process is repeated indefinitely. How much water remains in the tank after this process is repeated 3 times? 5 times? n times?

82. Musical Frequencies The frequencies of musical notes (measured in cycles per second) form a geometric sequence. Middle C has a frequency of 256, and the C that is an octave higher has a frequency of 512. Find the frequency of C two octaves below middle C.



83. Bouncing Ball A ball is dropped from a height of 9 ft. The elasticity of the ball is such that it always bounces up one-third the distance it has fallen.

(a) Find the total distance the ball has traveled at the instant it hits the ground the fifth time.

(b) Find a formula for the total distance the ball has traveled at the instant it hits the ground the nth time.

84. Geometric Savings Plan A very patient woman wishes to become a billionaire. She decides to follow a simple scheme: She puts aside 1 cent the first day, 2 cents the second day, 4 cents the third day, and so on, doubling the number of cents each day. How much money will she have at the end of 30 days? How many days will it take this woman to realize her wish?

85. St. Ives The following is a well-known children's rhyme:

As I was going to St. Ives,
I met a man with seven wives;
Every wife had seven sacks;
Every sack had seven cats;
Every cat had seven kits;
Kits, cats, sacks, and wives,
How many were going to St. Ives?

Assuming that the entire group is actually going to St. Ives, show that the answer to the question in the rhyme is a partial sum of a geometric sequence, and find the sum.

86. Drug Concentration A certain drug is administered once a day. The concentration of the drug in the patient's bloodstream increases rapidly at first, but each successive dose has less effect than the preceding one. The total amount of the drug (in mg) in the bloodstream after the nth dose is given by

$$\sum_{k=1}^{n} 50\left(\tfrac{1}{2}\right)^{k-1}$$

(a) Find the amount of the drug in the bloodstream after $n = 10$ days.

(b) If the drug is taken on a long-term basis, the amount in the bloodstream is approximated by the infinite series $\sum_{k=1}^{\infty} 50\left(\tfrac{1}{2}\right)^{k-1}$. Find the sum of this series.

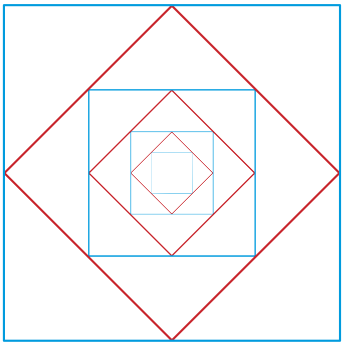
87. Bouncing Ball A certain ball rebounds to half the height from which it is dropped. Use an infinite geometric series to approximate the total distance the ball travels after being dropped from 1 m above the ground until it comes to rest.

88. Bouncing Ball If the ball in Exercise 87 is dropped from a height of 8 ft, then 1 s is required for its first complete bounce—from the instant it first touches the ground until it next touches the ground. Each subsequent complete bounce requires $1/\sqrt{2}$ as long as the preceding complete bounce. Use an infinite geometric series to estimate the time interval from the instant the ball first touches the ground until it stops bouncing.

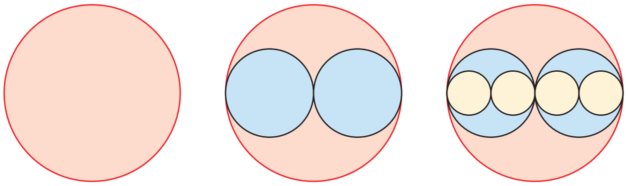
89. Geometry The midpoints of the sides of a square of side 1 are joined to form a new square. This procedure is repeated for each new square. (See the figure.)

(a) Find the sum of the areas of all the squares.

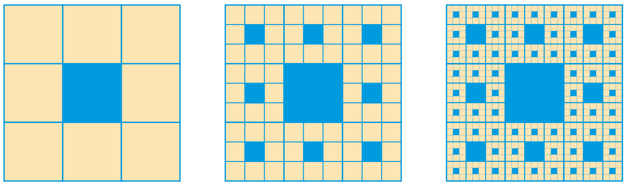
(b) Find the sum of the perimeters of all the squares.



90. Geometry A circular disk of radius R is cut out of paper, as shown in figure (a). Two disks of radius $\frac{1}{2}R$ are cut out of paper and placed on top of the first disk, as in figure (b), and then four disks of radius $\frac{1}{4}R$ are placed on these two disks, as in figure (c). Assuming that this process can be repeated indefinitely, find the total area of all the disks.


(a) (b) (c)

91. Geometry A yellow square of side 1 is divided into nine smaller squares, and the middle square is colored blue as shown in the figure. Each of the smaller yellow squares is in turn divided into nine squares, and each middle square is colored blue. If this process is continued indefinitely, what is the total area that is colored blue?



DISCOVERY ■ DISCUSSION ■ WRITING

92. Arithmetic or Geometric? The first four terms of a sequence are given. Determine whether these terms can be the terms of an arithmetic sequence, a geometric sequence, or neither. Find the next term if the sequence is arithmetic or geometric.

(a) $5, -3, 5, -3, \ldots$ **(b)** $\frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, \ldots$

(c) $\sqrt{3}, 3, 3\sqrt{3}, 9, \ldots$ **(d)** $1, -1, 1, -1, \ldots$

(e) $2, -1, \frac{1}{2}, 2, \ldots$ **(f)** $x - 1, x, x + 1, x + 2, \ldots$

(g) $-3, -\frac{3}{2}, 0, \frac{3}{2}, \ldots$ **(h)** $\sqrt{5}, \sqrt[3]{5}, \sqrt[6]{5}, 1, \ldots$

93. Reciprocals of a Geometric Sequence If $a_1, a_2, a_3, \ldots$ is a geometric sequence with common ratio r, show that the sequence

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$$

is also a geometric sequence, and find the common ratio.

94. Logarithms of a Geometric Sequence If $a_1, a_2, a_3, \ldots$ is a geometric sequence with a common ratio $r > 0$ and $a_1 > 0$, show that the sequence

$$\log a_1, \log a_2, \log a_3, \ldots$$

is an arithmetic sequence, and find the common difference.

95. Exponentials of an Arithmetic Sequence If $a_1, a_2, a_3, \ldots$ is an arithmetic sequence with common difference d, show that the sequence

$$10^{a_1}, 10^{a_2}, 10^{a_3}, \ldots$$

is a geometric sequence, and find the common ratio.

DISCOVERY PROJECT **Finding Patterns**

In this project we investigate the process of finding patterns in sequences by using "difference sequences." You can find the project at the book companion website: **www.stewartmath.com**

8.4 Mathematics of Finance

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find the amount of an annuity ▶ Find the present value of an annuity ▶ Find the amount of the installment payments on a loan

Many financial transactions involve payments that are made at regular intervals. For example, if you deposit \$100 each month in an interest-bearing account, what will the value of your account be at the end of 5 years? If you borrow \$100,000 to buy a house, how much must your monthly payments be in order to pay off the loan in 30 years? Each of these questions involves the sum of a sequence of numbers; we use the results of the preceding section to answer them here.

▼ The Amount of an Annuity

An **annuity** is a sum of money that is paid in regular equal payments. Although the word *annuity* suggests annual (or yearly) payments, they can be made semiannually, quarterly, monthly, or at some other regular interval. Payments are usually made at the end of the payment interval. The **amount of an annuity** is the sum of all the individual payments from the time of the first payment until the last payment is made, together with all the interest. We denote this sum by A_f (the subscript f here is used to denote *final* amount).

EXAMPLE 1 | Calculating the Amount of an Annuity

An investor deposits \$400 every December 15 and June 15 for 10 years in an account that earns interest at the rate of 8% per year, compounded semiannually. How much will be in the account immediately after the last payment?

When using interest rates in calculators, remember to convert percentages to decimals. For example, 8% is 0.08.

SOLUTION We need to find the amount of an annuity consisting of 20 semiannual payments of \$400 each. Since the interest rate is 8% per year, compounded semiannually, the interest rate per time period is $i = 0.08/2 = 0.04$. The first payment is in the account for 19 time periods, the second for 18 time periods, and so on.

The last payment receives no interest. The situation can be illustrated by the time line in Figure 1.

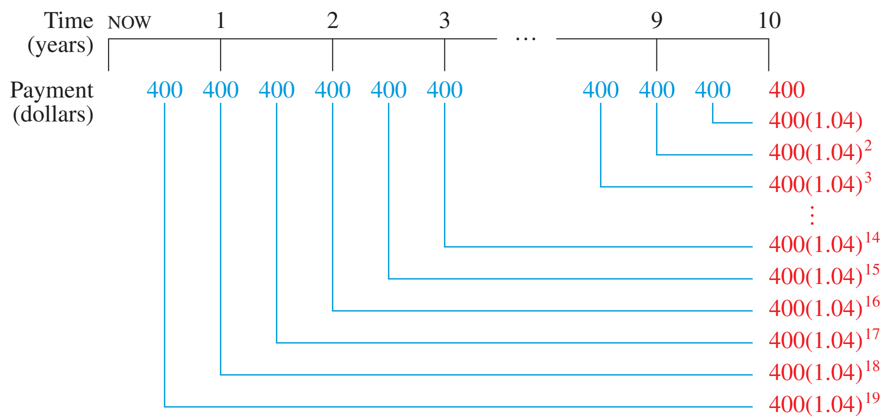


FIGURE 1

The amount A_f of the annuity is the sum of these 20 amounts. Thus

$$A_f = 400 + 400(1.04) + 400(1.04)^2 + \cdots + 400(1.04)^{19}$$

But this is a geometric series with $a = 400$, $r = 1.04$, and $n = 20$, so

$$A_f = 400\,\frac{1 - (1.04)^{20}}{1 - 1.04} \approx 11{,}911.23$$

Thus the amount in the account after the last payment is \$11,911.23.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **3**

In general, the regular annuity payment is called the **periodic rent** and is denoted by R. We also let i denote the interest rate per time period and let n denote the number of payments. *We always assume that the time period in which interest is compounded is equal to the time between payments*. By the same reasoning as in Example 1, we see that the amount A_f of an annuity is

$$A_f = R + R(1 + i) + R(1 + i)^2 + \cdots + R(1 + i)^{n-1}$$

Since this is the nth partial sum of a geometric sequence with $a = R$ and $r = 1 + i$, the formula for the partial sum gives

$$A_f = R\,\frac{1 - (1 + i)^n}{1 - (1 + i)} = R\,\frac{1 - (1 + i)^n}{-i} = R\,\frac{(1 + i)^n - 1}{i}$$

AMOUNT OF AN ANNUITY

The amount A_f of an annuity consisting of n regular equal payments of size R with interest rate i per time period is given by

$$A_f = R\,\frac{(1 + i)^n - 1}{i}$$

MATHEMATICS IN THE MODERN WORLD

Mathematical Economics

The health of the global economy is determined by such interrelated factors as supply, demand, production, consumption, pricing, distribution, and thousands of other factors. These factors are in turn determined by economic decisions (for example, whether or not you buy a certain brand of toothpaste) made by billions of different individuals each day. How will today's creation and distribution of goods affect tomorrow's economy? Such questions are tackled by mathematicians who work on mathematical models of the economy. In the 1940s Wassily Leontief, a pioneer in this area, created a model consisting of thousands of equations that describe how different sectors of the economy, such as the oil industry, transportation, and communication, interact with each other. A different approach to economic models, one dealing with individuals in the economy as opposed to large sectors, was pioneered by John Nash in the 1950s. In his model, which uses *game theory,* the economy is a game where individual players make decisions that often lead to mutual gain. Leontief and Nash were awarded the Nobel Prize in Economics in 1973 and 1994, respectively. Economic theory continues to be a major area of mathematical research.

EXAMPLE 2 | Calculating the Amount of an Annuity

How much money should be invested every month at 12% per year, compounded monthly, in order to have \$4000 in 18 months?

SOLUTION In this problem $i = 0.12/12 = 0.01$, $A_f = 4000$, and $n = 18$. We need to find the amount R of each payment. By the formula for the amount of an annuity,

$$4000 = R\frac{(1 + 0.01)^{18} - 1}{0.01}$$

Solving for R, we get

$$R = \frac{4000(0.01)}{(1 + 0.01)^{18} - 1} \approx 203.928$$

Thus the monthly investment should be \$203.93.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **9**

The Present Value of an Annuity

If you were to receive \$10,000 five years from now, it would be worth much less than if you got \$10,000 right now. This is because of the interest you could accumulate during the next five years if you invested the money now. What smaller amount would you be willing to accept *now* instead of receiving \$10,000 in five years? This is the amount of money that, together with interest, would be worth \$10,000 in five years. The amount that we are looking for here is called the *discounted value* or *present value*. If the interest rate is 8% per year, compounded quarterly, then the interest per time period is $i = 0.08/4 = 0.02$, and there are $4 \times 5 = 20$ time periods. If we let PV denote the present value, then by the formula for compound interest (Section 4.1), we have

$$10{,}000 = PV(1 + i)^n = PV(1 + 0.02)^{20}$$

so

$$PV = 10{,}000(1 + 0.02)^{-20} \approx 6729.713$$

Thus in this situation the present value of \$10,000 is \$6729.71. This reasoning leads to a general formula for present value. If an amount A_f is to be paid in a lump sum n time periods from now and the interest rate per time period is i, then its **present value** A_p is given by

$$A_p = A_f(1 + i)^{-n}$$

Similarly, the **present value of an annuity** is the amount A_p that must be invested now at the interest rate i per time period to provide n payments, each of amount R. Clearly, A_p is the sum of the present values of each individual payment (see Exercise 29). Another way of finding A_p is to note that A_p is the present value of A_f:

$$A_p = A_f(1 + i)^{-n} = R\frac{(1 + i)^n - 1}{i}(1 + i)^{-n} = R\frac{1 - (1 + i)^{-n}}{i}$$

THE PRESENT VALUE OF AN ANNUITY

The **present value** A_p of an annuity consisting of n regular equal payments of size R and interest rate i per time period is given by

$$A_p = R\frac{1 - (1 + i)^{-n}}{i}$$

EXAMPLE 3 | Calculating the Present Value of an Annuity

A person wins $10,000,000 in the California lottery, and the amount is paid in yearly installments of half a million dollars each for 20 years. What is the present value of his winnings? Assume that he can earn 10% interest, compounded annually.

SOLUTION Since the amount won is paid as an annuity, we need to find its present value. Here, $i = 0.1$, $R = \$500{,}000$, and $n = 20$. Thus

$$A_p = 500{,}000 \frac{1 - (1 + 0.1)^{-20}}{0.1} \approx 4{,}256{,}781.859$$

This means that the winner really won only $4,256,781.86 if it were paid immediately.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **11**

Installment Buying

When you buy a house or a car by installment, the payments that you make are an annuity whose present value is the amount of the loan.

EXAMPLE 4 | The Amount of a Loan

A student wishes to buy a car. She can afford to pay $200 per month but has no money for a down payment. If she can make these payments for four years and the interest rate is 12%, what purchase price can she afford?

SOLUTION The payments that the student makes constitute an annuity whose present value is the price of the car (which is also the amount of the loan, in this case). Here, we have $i = 0.12/12 = 0.01$, $R = 200$, and $n = 12 \times 4 = 48$, so

$$A_p = R\frac{1 - (1 + i)^{-n}}{i} = 200\frac{1 - (1 + 0.01)^{-48}}{0.01} \approx 7594.792$$

Thus the student can buy a car priced at $7594.79.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **19**

When a bank makes a loan that is to be repaid with regular equal payments R, then the payments form an annuity whose present value A_p is the amount of the loan. So to find the size of the payments, we solve for R in the formula for the amount of an annuity. This gives the following formula for R.

INSTALLMENT BUYING

If a loan A_p is to be repaid in n regular equal payments with interest rate i per time period, then the size R of each payment is given by

$$R = \frac{iA_p}{1 - (1 + i)^{-n}}$$

EXAMPLE 5 | Calculating Monthly Mortgage Payments

A couple borrows $100,000 at 9% interest as a mortage loan on a house. They expect to make monthly payments for 30 years to repay the loan. What is the size of each payment?

SOLUTION The mortgage payments form an annuity whose present value is $A_p = \$100{,}000$. Also, $i = 0.09/12 = 0.0075$, and $n = 12 \times 30 = 360$. We are looking for the amount R of each payment.

From the formula for installment buying, we get

$$R = \frac{iA_p}{1 - (1 + i)^{-n}} = \frac{(0.0075)(100{,}000)}{1 - (1 + 0.0075)^{-360}} \approx 804.623$$

Thus the monthly payments are \$804.62.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **15**

We now illustrate the use of graphing devices in solving problems related to installment buying.

EXAMPLE 6 | Calculating the Interest Rate from the Size of Monthly Payments

A car dealer sells a new car for \$18,000. He offers the buyer payments of \$405 per month for 5 years. What interest rate is this car dealer charging?

SOLUTION The payments form an annuity with present value $A_p = \$18{,}000$, $R = 405$, and $n = 12 \times 5 = 60$. To find the interest rate, we must solve for i in the equation

$$R = \frac{iA_p}{1 - (1 + i)^{-n}}$$

A little experimentation will convince you that it is not possible to solve this equation for i algebraically. So to find i, we use a graphing device to graph R as a function of the interest rate x, and we then use the graph to find the interest rate corresponding to the value of R we want (\$405 in this case). Since $i = x/12$, we graph the function

$$R(x) = \frac{\frac{x}{12}(18{,}000)}{1 - \left(1 + \frac{x}{12}\right)^{-60}}$$

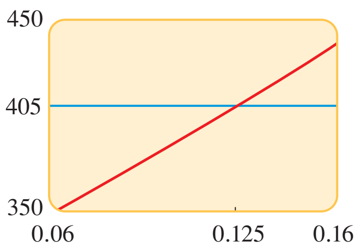


FIGURE 2

in the viewing rectangle $[0.06, 0.16] \times [350, 450]$, as shown in Figure 2. We also graph the horizontal line $R(x) = 405$ in the same viewing rectangle. Then, by moving the cursor to the point of intersection of the two graphs, we find that the corresponding x-value is approximately 0.125. Thus the interest rate is about $12\frac{1}{2}\%$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **25**

8.4 EXERCISES

CONCEPTS

1. An annuity is a sum of money that is paid in regular equal payments. The ________ of an annuity is the sum of all the individual payments together with all the interest.

2. The ________ ________ of an annuity is the amount that must be invested now at interest rate i per time period to provide n payments each of amount R.

APPLICATIONS

3. **Annuity** Find the amount of an annuity that consists of 10 annual payments of \$1000 each into an account that pays 6% interest per year.

4. **Annuity** Find the amount of an annuity that consists of 24 monthly payments of \$500 each into an account that pays 8% interest per year, compounded monthly.

5. **Annuity** Find the amount of an annuity that consists of 20 annual payments of \$5000 each into an account that pays interest of 12% per year.

6. **Annuity** Find the amount of an annuity that consists of 20 semiannual payments of \$500 each into an account that pays 6% interest per year, compounded semiannually.

7. **Annuity** Find the amount of an annuity that consists of 16 quarterly payments of \$300 each into an account that pays 8% interest per year, compounded quarterly.

8. **Annuity** Find the amount of an annuity that consists of 40 annual payments of \$2000 each into an account that pays interest of 5% per year.

9. **Saving** How much money should be invested every quarter at 10% per year, compounded quarterly, to have \$5000 in 2 years?

10. **Saving** How much money should be invested monthly at 6% per year, compounded monthly, to have \$2000 in 8 months?

11. **Annuity** What is the present value of an annuity that consists of 20 semiannual payments of \$1000 at an interest rate of 9% per year, compounded semiannually?

12. **Annuity** What is the present value of an annuity that consists of 30 monthly payments of \$300 at an interest rate of 8% per year, compounded monthly.

13. **Funding an Annuity** How much money must be invested now at 9% per year, compounded semiannually, to fund an annuity of 20 payments of \$200 each, paid every 6 months, the first payment being 6 months from now?

14. **Funding an Annuity** A 55-year-old man deposits \$50,000 to fund an annuity with an insurance company. The money will be invested at 8% per year, compounded semiannually. He is to draw semiannual payments until he reaches age 65. What is the amount of each payment?

15. **Financing a Car** A woman wants to borrow \$12,000 to buy a car. She wants to repay the loan by monthly installments for 4 years. If the interest rate on this loan is $10\frac{1}{2}\%$ per year, compounded monthly, what is the amount of each payment?

16. **Mortgage** What is the monthly payment on a 30-year mortgage of \$80,000 at 9% interest? What is the monthly payment on this same mortgage if it is to be repaid over a 15-year period?

17. **Mortgage** What is the monthly payment on a 30-year mortgage of \$100,000 at 8% interest per year, compounded monthly? What is the total amount paid on this loan over the 30-year period?

18. **Mortgage** What is the monthly payment on a 15-year mortgage of \$200,000 at 6% interest? What is the total amount paid on this loan over the 15-year period?

19. **Mortgage** Dr. Gupta is considering a 30-year mortgage at 6% interest. She can make payments of \$3500 a month. What size loan can she afford?

20. **Mortgage** A couple can afford to make a monthly mortgage payment of \$650. If the mortgage rate is 9% and the couple intends to secure a 30-year mortgage, how much can they borrow?

21. **Financing a Car** Jane agrees to buy a car for a down payment of \$2000 and payments of \$220 per month for 3 years. If the interest rate is 8% per year, compounded monthly, what is the actual purchase price of her car?

22. **Financing a Ring** Mike buys a ring for his fiancee by paying \$30 a month for one year. If the interest rate is 10% per year, compounded monthly, what is the price of the ring?

23. **Mortgage** A couple secures a 30-year loan of \$100,000 at $9\frac{3}{4}\%$ per year, compounded monthly, to buy a house.
   (a) What is the amount of their monthly payment?
   (b) What total amount will they pay over the 30-year period?
   (c) If, instead of taking the loan, the couple deposits the monthly payments in an account that pays $9\frac{3}{4}\%$ interest per year, compounded monthly, how much will be in the account at the end of the 30-year period?

24. **Mortgage** A couple needs a mortgage of \$300,000. Their mortgage broker presents them with two options: a 30-year mortgage at $6\frac{1}{2}\%$ interest or a 15-year mortgage at $5\frac{3}{4}\%$ interest.
   (a) Find the monthly payment on the 30-year mortgage and on the 15-year mortgage. Which mortgage has the larger monthly payment?
   (b) Find the total amount to be paid over the life of each loan. Which mortgage has the lower total payment over its lifetime?

25. **Interest Rate** John buys a stereo system for \$640. He agrees to pay \$32 a month for 2 years. Assuming that interest is compounded monthly, what interest rate is he paying?

26. **Interest Rate** Janet's payments on her \$12,500 car are \$420 a month for 3 years. Assuming that interest is compounded monthly, what interest rate is she paying on the car loan?

27. **Interest Rate** An item at a department store is priced at \$189.99 and can be bought by making 20 payments of \$10.50. Find the interest rate, assuming that interest is compounded monthly.

28. **Interest Rate** A man purchases a \$2000 diamond ring for a down payment of \$200 and monthly installments of \$88 for 2 years. Assuming that interest is compounded monthly, what interest rate is he paying?

DISCOVERY ▪ DISCUSSION ▪ WRITING

29. **Present Value of an Annuity**
   (a) Draw a time line as in Example 1 to show that the present value of an annuity is the sum of the present values of each payment, that is,

$$A_p = \frac{R}{1+i} + \frac{R}{(1+i)^2} + \frac{R}{(1+i)^3} + \cdots + \frac{R}{(1+i)^n}$$

   (b) Use part (a) to derive the formula for A_p given in the text.

30. **An Annuity That Lasts Forever** An **annuity in perpetuity** is one that continues forever. Such annuities are useful in setting up scholarship funds to ensure that the award continues.

(a) Draw a time line (as in Example 1) to show that to set up an annuity in perpetuity of amount R per time period, the amount that must be invested now is

$$A_p = \frac{R}{1+i} + \frac{R}{(1+i)^2} + \frac{R}{(1+i)^3} + \cdots + \frac{R}{(1+i)^n} + \cdots$$

where i is the interest rate per time period.

(b) Find the sum of the infinite series in part (a) to show that

$$A_p = \frac{R}{i}$$

(c) How much money must be invested now at 10% per year, compounded annually, to provide an annuity in perpetuity of \$5000 per year? The first payment is due in one year.

(d) How much money must be invested now at 8% per year, compounded quarterly, to provide an annuity in perpetuity of \$3000 per year? The first payment is due in one year.

31. Amortizing a Mortgage When they bought their house, John and Mary took out a \$90,000 mortgage at 9% interest, repayable monthly over 30 years. Their payment is \$724.17 per month (check this, using the formula in the text). The bank gave them an **amortization schedule**, which is a table showing how much of each payment is interest, how much goes toward the principal, and the remaining principal after each payment. The table below shows the first few entries in the amortization schedule.

Payment number	Total payment	Interest payment	Principal payment	Remaining principal
1	724.17	675.00	49.17	89,950.83
2	724.17	674.63	49.54	89,901.29
3	724.17	674.26	49.91	89,851.38
4	724.17	673.89	50.28	89,801.10

After 10 years they have made 120 payments and are wondering how much they still owe, but they have lost the amortization schedule.

(a) How much do John and Mary still owe on their mortgage? [*Hint:* The remaining balance is the present value of the 240 remaining payments.]

(b) How much of their next payment is interest, and how much goes toward the principal? [*Hint:* Since $9\% \div 12 = 0.75\%$, they must pay 0.75% of the remaining principal in interest each month.]

8.5 Mathematical Induction

LEARNING OBJECTIVES After completing this section, you will be able to:

Prove a statement using the Principle of Mathematical Induction

There are two aspects to mathematics—discovery and proof—and they are of equal importance. We must discover something before we can attempt to prove it, and we cannot be certain of its truth until it has been proved. In this section we examine the relationship between these two key components of mathematics more closely.

Conjecture and Proof

Let's try a simple experiment. We add more and more of the odd numbers as follows:

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$
$$1 + 3 + 5 + 7 + 9 = 25$$

What do you notice about the numbers on the right side of these equations? They are, in fact, all perfect squares. These equations say the following:

The sum of the first 1 odd number is 1^2.

The sum of the first 2 odd numbers is 2^2.

The sum of the first 3 odd numbers is 3^2.

The sum of the first 4 odd numbers is 4^2.

The sum of the first 5 odd numbers is 5^2.

Consider the polynomial

$$p(n) = n^2 - n + 41$$

Here are some values of $p(n)$:

$$p(1) = 41 \quad p(2) = 43$$
$$p(3) = 47 \quad p(4) = 53$$
$$p(5) = 61 \quad p(6) = 71$$
$$p(7) = 83 \quad p(8) = 97$$

All the values so far are prime numbers. In fact, if you keep going, you will find that $p(n)$ is prime for all natural numbers up to $n = 40$. It might seem reasonable at this point to conjecture that $p(n)$ is prime for *every* natural number n. But that conjecture would be too hasty, because it is easily seen that $p(41)$ is *not* prime. This illustrates that we cannot be certain of the truth of a statement no matter how many special cases we check. We need a convincing argument—a *proof*—to determine the truth of a statement.

This leads naturally to the following question: Is it true that for every natural number n, the sum of the first n odd numbers is n^2? Could this remarkable property be true? We could try a few more numbers and find that the pattern persists for the first 6, 7, 8, 9, and 10 odd numbers. At this point we feel fairly confident that this is always true, so we make a *conjecture*:

The sum of the first n odd numbers is n^2.

Since we know that the nth odd number is $2n - 1$, we can write this statement more precisely as

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

It is important to realize that this is still a conjecture. We cannot conclude by checking a finite number of cases that a property is true for all numbers (there are infinitely many). To see this more clearly, suppose someone tells us that he has added up the first trillion odd numbers and found that they do *not* add up to 1 trillion squared. What would you tell this person? It would be silly to say that you're sure it's true because you have already checked the first five cases. You could, however, take out paper and pencil and start checking it yourself, but this task would probably take the rest of your life. The tragedy would be that after completing this task, you would still not be sure of the truth of the conjecture! Do you see why?

Herein lies the power of mathematical proof. A **proof** is a clear argument that demonstrates the truth of a statement beyond doubt.

▼ Mathematical Induction

Let's consider a special kind of proof called **mathematical induction**. Here is how it works: Suppose we have a statement that says something about all natural numbers n. For example, for any natural number n, let $P(n)$ be the following statement:

$P(n)$: The sum of the first n odd numbers is n^2

Since this statement is about all natural numbers, it contains infinitely many statements; we will call them $P(1)$, $P(2)$, ...

$P(1)$: The sum of the first 1 odd number is 1^2.

$P(2)$: The sum of the first 2 odd numbers is 2^2.

$P(3)$: The sum of the first 3 odd numbers is 3^2.

$\vdots \qquad\qquad \vdots$

How can we prove all of these statements at once? Mathematical induction is a clever way of doing just that.

The crux of the idea is this: Suppose we can prove that whenever one of these statements is true, then the one following it in the list is also true. In other words,

For every k, if $P(k)$ is true, then $P(k + 1)$ is true.

This is called the **induction step** because it leads us from the truth of one statement to the truth of the next. Now suppose that we can also prove that

$P(1)$ is true.

The induction step now leads us through the following chain of statements:

$P(1)$ is true, so $P(2)$ is true.
$P(2)$ is true, so $P(3)$ is true.
$P(3)$ is true, so $P(4)$ is true.
$\vdots \qquad\qquad \vdots$

So we see that if both the induction step and $P(1)$ are proved, then statement $P(n)$ is proved for all n. Here is a summary of this important method of proof.

PRINCIPLE OF MATHEMATICAL INDUCTION

For each natural number n, let $P(n)$ be a statement depending on n. Suppose that the following two conditions are satisfied.

1. $P(1)$ is true.

2. For every natural number k, if $P(k)$ is true then $P(k + 1)$ is true.

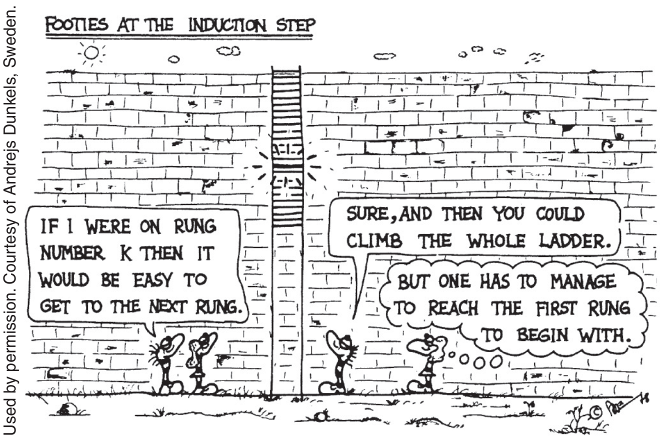
Then $P(n)$ is true for all natural numbers n.

To apply this principle, there are two steps:

Step 1 Prove that $P(1)$ is true.

Step 2 Assume that $P(k)$ is true, and use this assumption to prove that $P(k + 1)$ is true.

Notice that in Step 2 we do not prove that $P(k)$ is true. We only show that *if* $P(k)$ is true, *then* $P(k + 1)$ is also true. The assumption that $P(k)$ is true is called the **induction hypothesis**.



We now use mathematical induction to prove that the conjecture that we made at the beginning of this section is true.

EXAMPLE 1 | A Proof by Mathematical Induction

Prove that for all natural numbers n,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

SOLUTION Let $P(n)$ denote the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Step 1 We need to show that $P(1)$ is true. But $P(1)$ is simply the statement that $1 = 1^2$, which is of course true.

Step 2 We assume that $P(k)$ is true. Thus our induction hypothesis is

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

We want to use this to show that $P(k + 1)$ is true, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2$$

[Note that we get $P(k + 1)$ by substituting $k + 1$ for each n in the statement $P(n)$.] We start with the left side and use the induction hypothesis to obtain the right side of the equation:

$$\begin{aligned}
&1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] \\
&\quad = [1 + 3 + 5 + \cdots + (2k - 1)] + [2(k + 1) - 1] && \text{Group the first } k \text{ terms} \\
&\quad = k^2 + [2(k + 1) - 1] && \text{Induction hypothesis} \\
&\quad = k^2 + [2k + 2 - 1] && \text{Distributive Property} \\
&\quad = k^2 + 2k + 1 && \text{Simplify} \\
&\quad = (k + 1)^2 && \text{Factor}
\end{aligned}$$

This equals k^2 by the induction hypothesis

Thus $P(k + 1)$ follows from $P(k)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers n.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **3**

EXAMPLE 2 | A Proof by Mathematical Induction

Prove that for every natural number n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$$

SOLUTION Let $P(n)$ be the statement $1 + 2 + 3 + \cdots + n = n(n + 1)/2$. We want to show that $P(n)$ is true for all natural numbers n.

Step 1 We need to show that $P(1)$ is true. But $P(1)$ says that

$$1 = \frac{1(1 + 1)}{2}$$

and this statement is clearly true.

Step 2 Assume that $P(k)$ is true. Thus our induction hypothesis is

$$1 + 2 + 3 + \cdots + k = \frac{k(k + 1)}{2}$$

We want to use this to show that $P(k + 1)$ is true, that is,

$$1 + 2 + 3 + \cdots + k + (k + 1) = \frac{(k + 1)[(k + 1) + 1]}{2}$$

So we start with the left side and use the induction hypothesis to obtain the right side:

$$\begin{aligned}
&1 + 2 + 3 + \cdots + k + (k + 1) \\
&\quad = [1 + 2 + 3 + \cdots + k] + (k + 1) && \text{Group the first } k \text{ terms} \\
&\quad = \frac{k(k + 1)}{2} + (k + 1) && \text{Induction hypothesis} \\
&\quad = (k + 1)\left(\frac{k}{2} + 1\right) && \text{Factor } k + 1 \\
&\quad = (k + 1)\left(\frac{k + 2}{2}\right) && \text{Common denominator} \\
&\quad = \frac{(k + 1)[(k + 1) + 1]}{2} && \text{Write } k + 2 \text{ as } k + 1 + 1
\end{aligned}$$

This equals $\frac{k(k + 1)}{2}$ by the induction hypothesis

Thus $P(k + 1)$ follows from $P(k)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers n.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5**

The following box gives formulas for the sums of powers of the first n natural numbers. These formulas are important in calculus. Formula 1 is proved in Example 2. The other formulas are also proved by using mathematical induction (see Exercises 6 and 9).

SUMS OF POWERS

0. $\displaystyle\sum_{k=1}^{n} 1 = n$

1. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

2. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

It might happen that a statement $P(n)$ is false for the first few natural numbers but true from some number on. For example, we might want to prove that $P(n)$ is true for $n \geq 5$. Notice that if we prove that $P(5)$ is true, then this fact, together with the induction step, would imply the truth of $P(5), P(6), P(7), \ldots$. The next example illustrates this point.

EXAMPLE 3 | Proving an Inequality by Mathematical Induction

Prove that $4n < 2^n$ for all $n \geq 5$.

SOLUTION Let $P(n)$ denote the statement $4n < 2^n$.

Step 1 $P(5)$ is the statement that $4 \cdot 5 < 2^5$, or $20 < 32$, which is true.

Step 2 Assume that $P(k)$ is true. Thus our induction hypothesis is

$$4k < 2^k$$

We want to use this to show that $P(k+1)$ is true, that is,

$$4(k+1) < 2^{k+1}$$

We get $P(k+1)$ by replacing n by $k+1$ in the statement $P(n)$.

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BLAISE PASCAL (1623–1662) is considered one of the most versatile minds in modern history. He was a writer and philosopher as well as a gifted mathematician and physicist. Among his contributions that appear in this book are Pascal's triangle and the Principle of Mathematical Induction.

Pascal's father, himself a mathematician, believed that his son should not study mathematics until he was 15 or 16. But at age 12, Blaise insisted on learning geometry and proved most of its elementary theorems himself. At 19 he invented the first mechanical adding machine. In 1647, after writing a major treatise on the conic sections, he abruptly abandoned mathematics because he felt that his intense studies were contributing to his ill health. He devoted himself instead to frivolous recreations such as gambling, but this only served to pique his interest in probability. In 1654 he miraculously survived a carriage accident in which his horses ran off a bridge. Taking this to be a sign from God, Pascal entered a monastery, where he pursued theology and philosophy, writing his famous *Pensées.* He also continued his mathematical research. He valued faith and intuition more than reason as the source of truth, declaring that "the heart has its own reasons, which reason cannot know."

So we start with the left-hand side of the inequality and use the induction hypothesis to show that it is less than the right-hand side. For $k \geq 5$ we have

$$\begin{aligned} 4(k+1) &= 4k + 4 && \text{Distributive Property} \\ &< 2^k + 4 && \text{Induction hypothesis} \\ &< 2^k + 4k && \text{Because } 4 < 4k \\ &< 2^k + 2^k && \text{Induction hypothesis} \\ &= 2 \cdot 2^k \\ &= 2^{k+1} && \text{Property of exponents} \end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers $n \geq 5$.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **21**

8.5 EXERCISES

CONCEPTS

1. Mathematical induction is a method of proving that a statement $P(n)$ is true for all ________ numbers n. In Step 1 we prove that ________ is true.
2. Which of the following is true about Step 2 in a proof by mathematical induction?
   (i) We prove "$P(k+1)$ is true."
   (ii) We prove "If $P(k)$ is true, then $P(k+1)$ is true."

SKILLS

3–14 ■ Use mathematical induction to prove that the formula is true for all natural numbers n.

3. $2 + 4 + 6 + \cdots + 2n = n(n+1)$
4. $1 + 4 + 7 + \cdots + (3n-2) = \dfrac{n(3n-1)}{2}$
5. $5 + 8 + 11 + \cdots + (3n+2) = \dfrac{n(3n+7)}{2}$
6. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
7. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$
8. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$
9. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$
10. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$
11. $2^3 + 4^3 + 6^3 + \cdots + (2n)^3 = 2n^2(n+1)^2$
12. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{(n+1)}$
13. $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n$
    $= 2[1 + (n-1)2^n]$
14. $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$
15. Show that $n^2 + n$ is divisible by 2 for all natural numbers n.
16. Show that $5^n - 1$ is divisible by 4 for all natural numbers n.
17. Show that $n^2 - n + 41$ is odd for all natural numbers n.
18. Show that $n^3 - n + 3$ is divisible by 3 for all natural numbers n.
19. Show that $8^n - 3^n$ is divisible by 5 for all natural numbers n.
20. Show that $3^{2n} - 1$ is divisible by 8 for all natural numbers n.
21. Prove that $n < 2^n$ for all natural numbers n.
22. Prove that $(n+1)^2 < 2n^2$ for all natural numbers $n \geq 3$.
23. Prove that if $x > -1$, then $(1+x)^n \geq 1 + nx$ for all natural numbers n.
24. Show $100n \leq n^2$ for all $n \geq 100$.
25. Let $a_{n+1} = 3a_n$ and $a_1 = 5$. Show that $a_n = 5 \cdot 3^{n-1}$ for all natural numbers n.

26. A sequence is defined recursively by $a_{n+1} = 3a_n - 8$ and $a_1 = 4$. Find an explicit formula for a_n, and then use mathematical induction to prove that the formula you found is true.

27. Show that $x - y$ is a factor of $x^n - y^n$ for all natural numbers n. [*Hint:* $x^{k+1} - y^{k+1} = x^k(x - y) + (x^k - y^k)y$.]

28. Show that $x + y$ is a factor of $x^{2n-1} + y^{2n-1}$ for all natural numbers n.

29–33 ■ F_n denotes the nth term of the Fibonacci sequence discussed in Section 8.1. Use mathematical induction to prove the statement.

29. F_{3n} is even for all natural numbers n.

30. $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$

31. $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_nF_{n+1}$

32. $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$

33. For all $n \geq 2$,

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

34. Let a_n be the nth term of the sequence defined recursively by

$$a_{n+1} = \frac{1}{1 + a_n}$$

and let $a_1 = 1$. Find a formula for a_n in terms of the Fibonacci numbers F_n. Prove that the formula you found is valid for all natural numbers n.

35. Let F_n be the nth term of the Fibonacci sequence. Find and prove an inequality relating n and F_n for natural numbers n.

36. Find and prove an inequality relating $100n$ and n^3.

DISCOVERY ■ DISCUSSION ■ WRITING

37. True or False? Determine whether each statement is true or false. If you think the statement is true, prove it. If you think it is false, give an example in which it fails.

(a) $p(n) = n^2 - n + 11$ is prime for all n.

(b) $n^2 > n$ for all $n \geq 2$.

(c) $2^{2n+1} + 1$ is divisible by 3 for all $n \geq 1$.

(d) $n^3 \geq (n + 1)^2$ for all $n \geq 2$.

(e) $n^3 - n$ is divisible by 3 for all $n \geq 2$.

(f) $n^3 - 6n^2 + 11n$ is divisible by 6 for all $n \geq 1$.

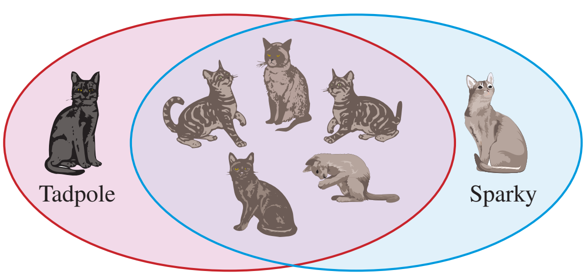
38. All Cats Are Black? What is wrong with the following "proof" by mathematical induction that all cats are black? Let $P(n)$ denote the statement "In any group of n cats, if one cat is black, then they are all black."

Step 1 The statement is clearly true for $n = 1$.

Step 2 Suppose that $P(k)$ is true. We show that $P(k + 1)$ is true.

Suppose we have a group of $k + 1$ cats, one of whom is black; call this cat "Tadpole." Remove some other cat (call it "Sparky") from the group. We are left with k cats, one of whom (Tadpole) is black, so by the induction hypothesis, all k of these are black. Now put Sparky back in the group and take out Tadpole. We again have a group of k cats, all of whom—except possibly Sparky—are black. Then by the induction hypothesis, Sparky must be black too. So all $k + 1$ cats in the original group are black.

Thus by induction $P(n)$ is true for all n. Since everyone has seen at least one black cat, it follows that all cats are black.



8.6 THE BINOMIAL THEOREM

LEARNING OBJECTIVES After completing this section, you will be able to:

Expand powers of binomials using Pascal's triangle ▶ Find binomial coefficients ▶ Expand powers of binomials using the Binomial Theorem ▶ Find a particular term in a binomial expansion

An expression of the form $a + b$ is called a **binomial**. Although in principle it's easy to raise $a + b$ to any power, raising it to a very high power would be tedious. In this section we find a formula that gives the expansion of $(a + b)^n$ for any natural number n and then prove it using mathematical induction.

Expanding $(a+b)^n$

To find a pattern in the expansion of $(a+b)^n$, we first look at some special cases.

$$\begin{aligned}
(a+b)^1 &= a+b\\
(a+b)^2 &= a^2+2ab+b^2\\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\\
&\vdots
\end{aligned}$$

The following simple patterns emerge for the expansion of $(a+b)^n$.

1. There are $n+1$ terms, the first being a^n and the last being b^n.
2. The exponents of a decrease by 1 from term to term, while the exponents of b increase by 1.
3. The sum of the exponents of a and b in each term is n.

For instance, notice how the exponents of a and b behave in the expansion of $(a+b)^5$.

The exponents of *a* decrease:

$$(a+b)^5 = a^{5} + 5a^{4}b^1 + 10a^{3}b^2 + 10a^{2}b^3 + 5a^{1}b^4 + b^5$$

The exponents of *b* increase:

$$(a+b)^5 = a^5 + 5a^4b^{1} + 10a^3b^{2} + 10a^2b^{3} + 5a^1b^{4} + b^{5}$$

With these observations we can write the form of the expansion of $(a+b)^n$ for any natural number n. For example, writing a question mark for the missing coefficients, we have

$$(a+b)^8 = a^8 + \boxed{?}a^7b + \boxed{?}a^6b^2 + \boxed{?}a^5b^3 + \boxed{?}a^4b^4 + \boxed{?}a^3b^5 + \boxed{?}a^2b^6 + \boxed{?}ab^7 + b^8$$

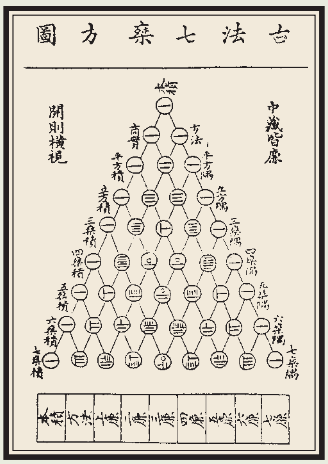
To complete the expansion, we need to determine these coefficients. To find a pattern, let's write the coefficients in the expansion of $(a+b)^n$ for the first few values of n in a triangular array as shown in the following array, which is called **Pascal's triangle**.

$$\begin{array}{lccccccccccc}
(a+b)^0 & & & & & & 1 & & & & & \\
(a+b)^1 & & & & & 1 & & 1 & & & & \\
(a+b)^2 & & & & 1 & & 2 & & 1 & & & \\
(a+b)^3 & & & 1 & & 3 & & 3 & & 1 & & \\
(a+b)^4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(a+b)^5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The row corresponding to $(a+b)^0$ is called the zeroth row and is included to show the symmetry of the array. The key observation about Pascal's triangle is the following property.

> **KEY PROPERTY OF PASCAL'S TRIANGLE**
>
> Every entry (other than a 1) is the sum of the two entries diagonally above it.

What we now call **Pascal's triangle** appears in this Chinese document by Chu Shikie, dated 1303. The title reads "The Old Method Chart of the Seven Multiplying Squares." The triangle was rediscovered by Pascal (see page 604).



University of York, Department of Mathematics

From this property it is easy to find any row of Pascal's triangle from the row above it. For instance, we find the sixth and seventh rows, starting with the fifth row:

$$\begin{array}{lccccccccccccccc}
(a+b)^5 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
(a+b)^6 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
(a+b)^7 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

To see why this property holds, let's consider the following expansions:

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

We arrive at the expansion of $(a+b)^6$ by multiplying $(a+b)^5$ by $(a+b)$. Notice, for instance, that the circled term in the expansion of $(a+b)^6$ is obtained via this multiplication from the two circled terms above it. We get this term when the two terms above it are multiplied by b and a, respectively. Thus its coefficient is the sum of the coefficients of these two terms. We will use this observation at the end of this section when we prove the Binomial Theorem.

Having found these patterns, we can now easily obtain the expansion of any binomial, at least to relatively small powers.

EXAMPLE 1 | Expanding a Binomial Using Pascal's Triangle

Find the expansion of $(a+b)^7$ using Pascal's triangle.

SOLUTION The first term in the expansion is a^7, and the last term is b^7. Using the fact that the exponent of a decreases by 1 from term to term and that of b increases by 1 from term to term, we have

$$(a+b)^7 = a^7 + \boxed{?}\,a^6b + \boxed{?}\,a^5b^2 + \boxed{?}\,a^4b^3 + \boxed{?}\,a^3b^4 + \boxed{?}\,a^2b^5 + \boxed{?}\,ab^6 + b^7$$

The appropriate coefficients appear in the seventh row of Pascal's triangle. Thus

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5** ■

EXAMPLE 2 | Expanding a Binomial Using Pascal's Triangle

Use Pascal's triangle to expand $(2-3x)^5$.

SOLUTION We find the expansion of $(a+b)^5$ and then substitute 2 for a and $-3x$ for b. Using Pascal's triangle for the coefficients, we get

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Substituting $a = 2$ and $b = -3x$ gives

$$\begin{aligned}(2-3x)^5 &= (2)^5 + 5(2)^4(-3x) + 10(2)^3(-3x)^2 + 10(2)^2(-3x)^3 + 5(2)(-3x)^4 + (-3x)^5 \\ &= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **13** ■

▼ The Binomial Coefficients

Although Pascal's triangle is useful in finding the binomial expansion for reasonably small values of n, it isn't practical for finding $(a+b)^n$ for large values of n. The reason is that the method we use for finding the successive rows of Pascal's triangle is recursive. Thus, to find the 100th row of this triangle, we must first find the preceding 99 rows.

We need to examine the pattern in the coefficients more carefully to develop a formula that allows us to calculate directly any coefficient in the binomial expansion. Such a formula exists, and the rest of this section is devoted to finding and proving it. However, to state this formula, we need some notation.

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$

$10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 3{,}628{,}800$

The product of the first n natural numbers is denoted by $\boldsymbol{n!}$ and is called $\boldsymbol{n}$ **factorial**.

$$\boxed{n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n}$$

We also define 0! as follows:

$$\boxed{0! = 1}$$

This definition of 0! makes many formulas involving factorials shorter and easier to write.

THE BINOMIAL COEFFICIENT

Let n and r be nonnegative integers with $r \le n$. The **binomial coefficient** is denoted by $\binom{n}{r}$ and is defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

EXAMPLE 3 | Calculating Binomial Coefficients

(a) $$\binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{9!}{4!\,5!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{(1 \cdot 2 \cdot 3 \cdot 4)(\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5})}$$
$$= \frac{6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} = 126$$

(b) $$\binom{100}{3} = \frac{100!}{3!(100-3)!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 97} \cdot 98 \cdot 99 \cdot 100}{(1 \cdot 2 \cdot 3)(\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 97})}$$
$$= \frac{98 \cdot 99 \cdot 100}{1 \cdot 2 \cdot 3} = 161{,}700$$

(c) $$\binom{100}{97} = \frac{100!}{97!(100-97)!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 97} \cdot 98 \cdot 99 \cdot 100}{(\cancel{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 97})(1 \cdot 2 \cdot 3)}$$
$$= \frac{98 \cdot 99 \cdot 100}{1 \cdot 2 \cdot 3} = 161{,}700$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **17** AND **19**

Although the binomial coefficient $\binom{n}{r}$ is defined in terms of a fraction, all the results of Example 3 are natural numbers. In fact, $\binom{n}{r}$ is always a natural number (see Exercise 54). Notice that the binomial coefficients in parts (b) and (c) of Example 3 are equal. This is a special case of the following relation, which you are asked to prove in Exercise 52.

$$\boxed{\binom{n}{r} = \binom{n}{n-r}}$$

$\binom{5}{2} = \frac{5!}{2!(5-2)!} = 10$

To see the connection between the binomial coefficients and the binomial expansion of $(a + b)^n$, let's calculate the following binomial coefficients:

$$\binom{5}{0} = 1 \qquad \binom{5}{1} = 5 \qquad \binom{5}{2} = 10 \qquad \binom{5}{3} = 10 \qquad \binom{5}{4} = 5 \qquad \binom{5}{5} = 1$$

These are precisely the entries in the fifth row of Pascal's triangle. In fact, we can write Pascal's triangle as follows.

$$\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
 & \cdot & & \cdot & & \cdot & \cdot & \cdot & & \cdot & & \cdot & \\
\binom{n}{0} & & \binom{n}{1} & & \binom{n}{2} & & \cdots & & & \binom{n}{n-1} & & \binom{n}{n} &
\end{array}$$

To demonstrate that this pattern holds, we need to show that any entry in this version of Pascal's triangle is the sum of the two entries diagonally above it. In other words, we must show that each entry satisfies the key property of Pascal's triangle. We now state this property in terms of the binomial coefficients.

KEY PROPERTY OF THE BINOMIAL COEFFICIENTS

For any nonnegative integers r and k with $r \le k$,

$$\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r}$$

Notice that the two terms on the left-hand side of this equation are adjacent entries in the kth row of Pascal's triangle and the term on the right-hand side is the entry diagonally below them, in the $(k + 1)$st row. Thus this equation is a restatement of the key property of Pascal's triangle in terms of the binomial coefficients. A proof of this formula is outlined in Exercise 53.

▼ The Binomial Theorem

We are now ready to state the Binomial Theorem.

THE BINOMIAL THEOREM

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

We prove this theorem at the end of this section. First, let's look at some of its applications.

EXAMPLE 4 | Expanding a Binomial Using the Binomial Theorem

Use the Binomial Theorem to expand $(x + y)^4$.

SOLUTION By the Binomial Theorem,

$$(x + y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

Verify that

$$\binom{4}{0} = 1 \qquad \binom{4}{1} = 4 \qquad \binom{4}{2} = 6 \qquad \binom{4}{3} = 4 \qquad \binom{4}{4} = 1$$

It follows that

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **25**

EXAMPLE 5 | Expanding a Binomial Using the Binomial Theorem

Use the Binomial Theorem to expand $(\sqrt{x} - 1)^8$.

SOLUTION We first find the expansion of $(a + b)^8$ and then substitute $\sqrt{x}$ for a and -1 for b. Using the Binomial Theorem, we have

$$(a + b)^8 = \binom{8}{0}a^8 + \binom{8}{1}a^7b + \binom{8}{2}a^6b^2 + \binom{8}{3}a^5b^3 + \binom{8}{4}a^4b^4 + \binom{8}{5}a^3b^5 + \binom{8}{6}a^2b^6 + \binom{8}{7}ab^7 + \binom{8}{8}b^8$$

Verify that

$$\binom{8}{0} = 1 \qquad \binom{8}{1} = 8 \qquad \binom{8}{2} = 28 \qquad \binom{8}{3} = 56 \qquad \binom{8}{4} = 70$$

$$\binom{8}{5} = 56 \qquad \binom{8}{6} = 28 \qquad \binom{8}{7} = 8 \qquad \binom{8}{8} = 1$$

So

$$(a + b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$$

Performing the substitutions $a = x^{1/2}$ and $b = -1$ gives

$$(\sqrt{x} - 1)^8 = (x^{1/2})^8 + 8(x^{1/2})^7(-1) + 28(x^{1/2})^6(-1)^2 + 56(x^{1/2})^5(-1)^3 + 70(x^{1/2})^4(-1)^4 + 56(x^{1/2})^3(-1)^5 + 28(x^{1/2})^2(-1)^6 + 8(x^{1/2})(-1)^7 + (-1)^8$$

This simplifies to

$$(\sqrt{x} - 1)^8 = x^4 - 8x^{7/2} + 28x^3 - 56x^{5/2} + 70x^2 - 56x^{3/2} + 28x - 8x^{1/2} + 1$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **27**

The Binomial Theorem can be used to find a particular term of a binomial expansion without having to find the entire expansion.

GENERAL TERM OF THE BINOMIAL EXPANSION

The term that contains a^r in the expansion of $(a + b)^n$ is

$$\binom{n}{n-r}a^r b^{n-r}$$

EXAMPLE 6 | Finding a Particular Term in a Binomial Expansion

Find the term that contains x^5 in the expansion of $(2x + y)^{20}$.

SOLUTION The term that contains x^5 is given by the formula for the general term with $a = 2x$, $b = y$, $n = 20$, and $r = 5$. So this term is

$$\binom{20}{15}a^5b^{15} = \frac{20!}{15!(20-15)!}(2x)^5y^{15} = \frac{20!}{15!\,5!}32x^5y^{15} = 496{,}128x^5y^{15}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **39**

EXAMPLE 7 | Finding a Particular Term in a Binomial Expansion

Find the coefficient of x^8 in the expansion of $\left(x^2 + \dfrac{1}{x}\right)^{10}$.

SOLUTION Both x^2 and $1/x$ are powers of x, so the power of x in each term of the expansion is determined by both terms of the binomial. To find the required coefficient, we first find the general term in the expansion. By the formula we have $a = x^2$, $b = 1/x$, and $n = 10$, so the general term is

$$\binom{10}{10-r}(x^2)^r\left(\frac{1}{x}\right)^{10-r} = \binom{10}{10-r}x^{2r}(x^{-1})^{10-r} = \binom{10}{10-r}x^{3r-10}$$

Thus the term that contains x^8 is the term in which

$$3r - 10 = 8$$

$$r = 6$$

So the required coefficient is

$$\binom{10}{10-6} = \binom{10}{4} = 210$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41**

▼ Proof of the Binomial Theorem

We now give a proof of the Binomial Theorem using mathematical induction.

PROOF Let $P(n)$ denote the statement

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Step 1 We show that $P(1)$ is true. But $P(1)$ is just the statement

$$(a+b)^1 = \binom{1}{0}a^1 + \binom{1}{1}b^1 = 1a + 1b = a + b$$

which is certainly true.

Step 2 We assume that $P(k)$ is true. Thus our induction hypothesis is

$$(a+b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k$$

We use this to show that $P(k+1)$ is true.

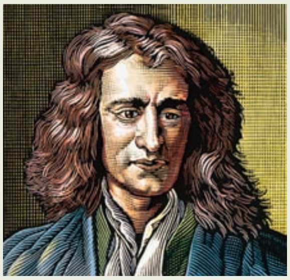
$$\begin{aligned}
(a+b)^{k+1} &= (a+b)[(a+b)^k] \\
&= (a+b)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right] && \text{Induction hypothesis} \\
&= a\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right] \\
&\quad + b\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right] && \text{Distributive Property} \\
&= \binom{k}{0}a^{k+1} + \binom{k}{1}a^kb + \binom{k}{2}a^{k-1}b^2 + \cdots + \binom{k}{k-1}a^2b^{k-1} + \binom{k}{k}ab^k \\
&\quad + \binom{k}{0}a^kb + \binom{k}{1}a^{k-1}b^2 + \binom{k}{2}a^{k-2}b^3 + \cdots + \binom{k}{k-1}ab^k + \binom{k}{k}b^{k+1} && \text{Distributive Property} \\
&= \binom{k}{0}a^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]a^kb + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}b^2 \\
&\quad + \cdots + \left[\binom{k}{k-1} + \binom{k}{k}\right]ab^k + \binom{k}{k}b^{k+1} && \text{Group like terms}
\end{aligned}$$

Using the key property of the binomial coefficients, we can write each of the expressions in square brackets as a single binomial coefficient. Also, writing the first and last coefficients as $\binom{k+1}{0}$ and $\binom{k+1}{k+1}$ (these are equal to 1 by Exercise 50) gives

$$(a+b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^kb + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k}ab^k + \binom{k+1}{k+1}b^{k+1}$$

But this last equation is precisely $P(k+1)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that the theorem is true for all natural numbers n. ■



B. Sanerson/Photo Researchers

SIR ISAAC NEWTON (1642–1727) is universally regarded as one of the giants of physics and mathematics. He is well known for discovering the laws of motion and gravity and for inventing calculus, but he also proved a generalization of the Binomial Theorem, discovered the laws of optics, and developed methods for solving polynomial equations to any desired accuracy. He was born on Christmas Day, a few months after the death of his father. After an unhappy childhood, he entered Cambridge University, where he learned mathematics by studying the writings of Euclid and Descartes.

During the plague years of 1665 and 1666, when the university was closed, Newton thought and wrote about ideas that, once published, instantly revolutionized the sciences. Imbued with a pathological fear of criticism, he published these writings only after many years of encouragement from Edmund Halley (who discovered the now-famous comet) and other colleagues.

Newton's works brought him enormous fame and prestige. Even poets were moved to praise; Alexander Pope wrote:

> Nature and Nature's Laws
> lay hid in Night.
> God said, "Let Newton be"
> and all was Light.

Newton was far more modest about his accomplishments. He said, "I seem to have been only like a boy playing on the seashore . . . while the great ocean of truth lay all undiscovered before me." Newton was knighted by Queen Anne in 1705 and was buried with great honor in Westminster Abbey.

8.6 EXERCISES

CONCEPTS

1. An algebraic expression of the form $a + b$, which consists of a sum of two terms, is called a ________.

2. We can find the coefficients in the expansion of $(a + b)^n$ from the nth row of ________ triangle. So

$$(a + b)^4 = \square a^4 + \square a^3b + \square a^2b^2 + \square ab^3 + \square b^4$$

3. The binomial coefficients can be calculated directly by using the formula $\binom{n}{k} =$ ________. So $\binom{4}{3} =$ ________.

4. To expand $(a + b)^n$, we can use the ________ Theorem. Using this theorem, we find

$$(a + b)^4 = \binom{\square}{\square}a^4 + \binom{\square}{\square}a^3b + \binom{\square}{\square}a^2b^2 + \binom{\square}{\square}ab^3 + \binom{\square}{\square}b^4$$

SKILLS

5–16 ■ Use Pascal's triangle to expand the expression.

5. $(x + y)^6$ **6.** $(2x + 1)^4$ **7.** $\left(x + \dfrac{1}{x}\right)^4$

8. $(x - y)^5$ **9.** $(x - 1)^5$ **10.** $(\sqrt{a} + \sqrt{b})^6$

11. $(x^2y - 1)^5$ **12.** $(1 + \sqrt{2})^6$ **13.** $(2x - 3y)^3$

14. $(1 + x^3)^3$ **15.** $\left(\dfrac{1}{x} - \sqrt{x}\right)^5$ **16.** $\left(2 + \dfrac{x}{2}\right)^5$

17–24 ■ Evaluate the expression.

17. $\binom{6}{4}$ **18.** $\binom{8}{3}$ **19.** $\binom{100}{98}$

20. $\binom{10}{5}$ **21.** $\binom{3}{1}\binom{4}{2}$ **22.** $\binom{5}{2}\binom{5}{3}$

23. $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$

24. $\binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5}$

25–28 ■ Use the Binomial Theorem to expand the expression.

25. $(x + 2y)^4$ **26.** $(1 - x)^5$

27. $\left(1 + \dfrac{1}{x}\right)^6$ **28.** $(2A + B^2)^4$

29. Find the first three terms in the expansion of $(x + 2y)^{20}$.

30. Find the first four terms in the expansion of $(x^{1/2} + 1)^{30}$.

31. Find the last two terms in the expansion of $(a^{2/3} + a^{1/3})^{25}$.

32. Find the first three terms in the expansion of

$$\left(x + \frac{1}{x}\right)^{40}$$

33. Find the middle term in the expansion of $(x^2 + 1)^{18}$.

34. Find the fifth term in the expansion of $(ab - 1)^{20}$.

35. Find the 24th term in the expansion of $(a + b)^{25}$.

36. Find the 28th term in the expansion of $(A - B)^{30}$.

37. Find the 100th term in the expansion of $(1 + y)^{100}$.

38. Find the second term in the expansion of

$$\left(x^2 - \frac{1}{x}\right)^{25}$$

39. Find the term containing x^4 in the expansion of $(x + 2y)^{10}$.

40. Find the term containing y^3 in the expansion of $(\sqrt{2} + y)^{12}$.

41. Find the term containing b^8 in the expansion of $(a + b^2)^{12}$.

42. Find the term that does not contain x in the expansion of

$$\left(8x + \frac{1}{2x}\right)^8$$

43–46 ■ Factor using the Binomial Theorem.

43. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

44. $(x - 1)^5 + 5(x - 1)^4 + 10(x - 1)^3 + 10(x - 1)^2 + 5(x - 1) + 1$

45. $8a^3 + 12a^2b + 6ab^2 + b^3$

46. $x^8 + 4x^6y + 6x^4y^2 + 4x^2y^3 + y^4$

47–52 ■ Simplify using the Binomial Theorem.

47. $\dfrac{(x + h)^3 - x^3}{h}$ **48.** $\dfrac{(x + h)^4 - x^4}{h}$

49. Show that $(1.01)^{100} > 2$. [*Hint:* Note that $(1.01)^{100} = (1 + 0.01)^{100}$, and use the Binomial Theorem to show that the sum of the first two terms of the expansion is greater than 2.]

50. Show that $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$.

51. Show that $\binom{n}{1} = \binom{n}{n-1} = n$.

52. Show that $\binom{n}{r} = \binom{n}{n-r}$ for $0 \le r \le n$.

53. In this exercise we prove the identity

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

(a) Write the left-hand side of this equation as the sum of two fractions.

(b) Show that a common denominator of the expression that you found in part (a) is $r!(n - r + 1)!$.

(c) Add the two fractions using the common denominator in part (b), simplify the numerator, and note that the resulting expression is equal to the right-hand side of the equation.

54. Prove that $\binom{n}{r}$ is an integer for all n and for $0 \le r \le n$. [*Suggestion*: Use induction to show that the statement is true for all n, and use Exercise 53 for the induction step.]

APPLICATIONS

55. Difference in Volumes of Cubes The volume of a cube of side x inches is given by $V(x) = x^3$, so the volume of a cube of side $x + 2$ inches is given by $V(x + 2) = (x + 2)^3$. Use the Binomial Theorem to show that the difference in volume between the larger and smaller cubes is $6x^2 + 12x + 8$ cubic inches.

56. Probability of Hitting a Target The probability that an archer hits the target is $p = 0.9$, so the probability that he misses the target is $q = 0.1$. It is known that in this situation the probability that the archer hits the target exactly r times in n attempts is given by the term containing p^r in the binomial expansion of $(p + q)^n$. Find the probability that the archer hits the target exactly three times in five attempts.

DISCOVERY ■ DISCUSSION ■ WRITING

57. Powers of Factorials Which is larger, $(100!)^{101}$ or $(101!)^{100}$? [*Hint:* Try factoring the expressions. Do they have any common factors?]

58. Sums of Binomial Coefficients Add each of the first five rows of Pascal's triangle, as indicated. Do you see a pattern?

$$1 + 1 = \boxed{?}$$
$$1 + 2 + 1 = \boxed{?}$$
$$1 + 3 + 3 + 1 = \boxed{?}$$
$$1 + 4 + 6 + 4 + 1 = \boxed{?}$$
$$1 + 5 + 10 + 10 + 5 + 1 = \boxed{?}$$

On the basis of the pattern you have found, find the sum of the nth row:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

Prove your result by expanding $(1 + 1)^n$ using the Binomial Theorem.

59. Alternating Sums of Binomial Coefficients Find the sum

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}$$

by finding a pattern as in Exercise 58. Prove your result by expanding $(1 - 1)^n$ using the Binomial Theorem.

CHAPTER 8 | REVIEW

■ PROPERTIES AND FORMULAS

Sequences (p. 570)

A **sequence** is a function whose domain is the set of natural numbers. Instead of writing $a(n)$ for the value of the sequence at n, we generally write a_n, and we refer to this value as the ***n*th term** of the sequence. Sequences are often described in list form:

$$a_1, a_2, a_3, \ldots$$

Partial Sums of a Sequence (pp. 575–576)

For the sequence $a_1, a_2, a_3, \ldots$ the ***n*th partial sum** S_n is the sum of the first n terms of the sequence:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

The nth partial sum of a sequence can also be expressed by using **sigma notation**:

$$S_n = \sum_{k=1}^{n} a_k$$

Arithmetic Sequences (p. 581)

An **arithmetic sequence** is a sequence whose terms are obtained by adding the same fixed constant d to each term to get the next term. Thus an arithmetic sequence has the form

$$a, a + d, a + 2d, a + 3d, \ldots$$

The number a is the **first term** of the sequence, and the number d is the **common difference**. The nth term of the sequence is

$$a_n = a + (n - 1)d$$

Partial Sums of an Arithmetic Sequence (p. 583)

For the arithmetic sequence $a_n = a + (n - 1)d$ the nth partial sum $S_n = \sum_{k=1}^{n} [a + (k - 1)d]$ is given by either of the following equivalent formulas:

1. $S_n = \dfrac{n}{2}[2a + (n - 1)d]$
2. $S_n = n\left(\dfrac{a + a_n}{2}\right)$

Geometric Sequences (p. 586)

A **geometric sequence** is a sequence whose terms are obtained by multiplying each term by the same fixed constant r to get the next term. Thus a geometric sequence has the form

$$a, ar, ar^2, ar^3, \ldots$$

The number a is the **first term** of the sequence, and the number r is the **common ratio**. The nth term of the sequence is

$$a_n = ar^{n-1}$$

Partial Sums of a Geometric Sequence (p. 588)

For the geometric sequence $a_n = ar^{n-1}$ the nth partial sum $S_n = \sum_{k=1}^{n} ar^{k-1}$ (where $r \neq 1$) is given by

$$S_n = a\frac{1 - r^n}{1 - r}$$

Infinite Geometric Series (p. 590)

An **infinite geometric series** is a series of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$$

An infinite series for which $|r| < 1$ has the sum

$$S = \frac{a}{1 - r}$$

Amount of an Annuity (p. 595)

The amount A_f of an **annuity** consisting of n regular equal payments of size R with interest rate i per time period is given by

$$A_f = R\frac{(1 + i)^n - 1}{i}$$

Present Value of an Annuity (p. 596)

The **present value** A_p of an annuity consisting of n regular equal payments of size R with interest rate i per time period is given by

$$A_p = R\frac{1 - (1 + i)^{-n}}{i}$$

Present Value of a Future Amount (p. 596)

If an amount A_f is to be paid in one lump sum, n time periods from now, and the interest rate per time period is i, then its **present value** A_p is given by

$$A_p = A_f(1 + i)^{-n}$$

Installment Buying (p. 597)

If a loan A_p is to be repaid in n regular equal payments with interest rate i per time period, then the size R of each payment is given by

$$R = \frac{iA_p}{1 - (1 + i)^{-n}}$$

Principle of Mathematical Induction (p. 602)

For each natural number n, let $P(n)$ be a statement that depends on n. Suppose that each of the following conditions is satisfied.

1. $P(1)$ is true.
2. For every natural number k, if $P(k)$ is true, then $P(k + 1)$ is true.

Then $P(n)$ is true for all natural numbers n.

Sums of Powers (p. 604)

0. $\displaystyle\sum_{k=1}^{n} 1 = n$

1. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}$

2. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}$

3. $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n + 1)^2}{4}$

Binomial Coefficients (pp. 609–610)

If n and r are positive integers with $n \geq r$, then the **binomial coefficient** $\binom{n}{r}$ is defined by

$$\binom{n}{r} = \frac{n!}{r!(n - r)!}$$

Binomial coefficients satisfy the following properties:

$$\binom{n}{r} = \binom{n}{n - r}$$

$$\binom{k}{r - 1} + \binom{k}{r} = \binom{k + 1}{r}$$

The Binomial Theorem (p. 610)

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

■ LEARNING OBJECTIVES SUMMARY

Section	After completing this chapter, you should be able to . . .	Review Exercises
8.1	▪ Find the terms of a sequence	1–6, 11–14
	▪ Find the terms of a recursive sequence	7–10
	▪ Find the partial sums of a sequence	11–14, 37–40
	▪ Use sigma notation	37–48
8.2	▪ Find the terms of an arithmetic sequence	11, 14, 15–17, 25–26, 30
	▪ Find the partial sums of an arithmetic sequence	50–52
8.3	▪ Find the terms of a geometric sequence	12–13, 18–19, 21–24, 27–29, 31
	▪ Find the partial sums of a geometric sequence	49, 53–54, 60–61
	▪ Find the sum of an infinite geometric sequence	55–60, 62–63
8.4	▪ Find the amount of an annuity	64
	▪ Find the present value of an annuity	65
	▪ Find the amount of the installment payments on a loan	66
8.5	▪ Prove a statement using the Principle of Mathematical Induction	67–72

8.6	■ Expand powers of binomials using Pascal's triangle	77–78
	■ Find binomial coefficients	73–76
	■ Expand powers of binomials using the Binomial Theorem	79–80
	■ Find a particular term in a binomial expansion	81–83

■ EXERCISES

1–6 ■ Find the first four terms as well as the tenth term of the sequence with the given nth term.

1. $a_n = \dfrac{n^2}{n+1}$ **2.** $a_n = (-1)^n \dfrac{2^n}{n}$

3. $a_n = \dfrac{(-1)^n + 1}{n^3}$ **4.** $a_n = \dfrac{n(n+1)}{2}$

5. $a_n = \dfrac{(2n)!}{2^n n!}$ **6.** $a_n = \dbinom{n+1}{2}$

7–10 ■ A sequence is defined recursively. Find the first seven terms of the sequence.

7. $a_n = a_{n-1} + 2n - 1, \quad a_1 = 1$

8. $a_n = \dfrac{a_{n-1}}{n}, \quad a_1 = 1$

9. $a_n = a_{n-1} + 2a_{n-2}, \quad a_1 = 1, a_2 = 3$

10. $a_n = \sqrt{3a_{n-1}}, \quad a_1 = \sqrt{3}$

11–14 ■ The nth term of a sequence is given. **(a)** Find the first five terms of the sequence. **(b)** Graph the terms you found in part (a). **(c)** Find the fifth partial sum of the sequence. **(d)** Determine whether the series is arithmetic or geometric. Find the common difference or the common ratio.

11. $a_n = 2n + 5$ **12.** $a_n = \dfrac{5}{2^n}$

13. $a_n = \dfrac{3^n}{2^{n+1}}$ **14.** $a_n = 4 - \dfrac{n}{2}$

15–22 ■ The first four terms of a sequence are given. Determine whether they can be the terms of an arithmetic sequence, a geometric sequence, or neither. If the sequence is arithmetic or geometric, find the fifth term.

15. $5, 5.5, 6, 6.5, \ldots$

16. $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, \ldots$

17. $t-3, t-2, t-1, t, \ldots$ **18.** $\sqrt{2}, 2, 2\sqrt{2}, 4, \ldots$

19. $t^3, t^2, t, 1, \ldots$ **20.** $1, -\frac{3}{2}, 2, -\frac{5}{2}, \ldots$

21. $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \ldots$ **22.** $a, 1, \dfrac{1}{a}, \dfrac{1}{a^2}, \ldots$

23. Show that $3, 6i, -12, -24i, \ldots$ is a geometric sequence, and find the common ratio. (Here $i = \sqrt{-1}$.)

24. Find the nth term of the geometric sequence $2, 2+2i, 4i, -4+4i, -8, \ldots$ (Here $i = \sqrt{-1}$.)

25. The sixth term of an arithmetic sequence is 17, and the fourth term is 11. Find the second term.

26. The 20th term of an arithmetic sequence is 96, and the common difference is 5. Find the nth term.

27. The third term of a geometric sequence is 9, and the common ratio is $\frac{3}{2}$. Find the fifth term.

28. The second term of a geometric sequence is 10, and the fifth term is $\frac{1250}{27}$. Find the nth term.

29. A teacher makes \$32,000 in his first year at Lakeside School and gets a 5% raise each year.
(a) Find a formula for his salary A_n in his nth year at this school.
(b) List his salaries for his first 8 years at this school.

30. A colleague of the teacher in Exercise 29, hired at the same time, makes \$35,000 in her first year, and gets a \$1200 raise each year.
(a) What is her salary A_n in her nth year at this school?
(b) Find her salary in her eighth year at this school, and compare it to the salary of the teacher in Exercise 29 in his eighth year.

31. A certain type of bacteria divides every 5 s. If three of these bacteria are put into a petri dish, how many bacteria are in the dish at the end of 1 min?

32. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are arithmetic sequences, show that $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$ is also an arithmetic sequence.

33. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are geometric sequences, show that $a_1b_1, a_2b_2, a_3b_3, \ldots$ is also a geometric sequence.

34. (a) If $a_1, a_2, a_3, \ldots$ is an arithmetic sequence, is the sequence $a_1 + 2, a_2 + 2, a_3 + 2, \ldots$ arithmetic?
(b) If $a_1, a_2, a_3, \ldots$ is a geometric sequence, is the sequence $5a_1, 5a_2, 5a_3, \ldots$ geometric?

35. Find the values of x for which the sequence $6, x, 12, \ldots$ is
(a) arithmetic **(b)** geometric

36. Find the values of x and y for which the sequence $2, x, y, 17, \ldots$ is
(a) arithmetic **(b)** geometric

37–40 ■ Find the sum.

37. $\displaystyle\sum_{k=3}^{6} (k+1)^2$ **38.** $\displaystyle\sum_{i=1}^{4} \frac{2i}{2i-1}$

39. $\displaystyle\sum_{k=1}^{6} (k+1)2^{k-1}$ **40.** $\displaystyle\sum_{m=1}^{5} 3^{m-2}$

41–44 ■ Write the sum without using sigma notation. Do not evaluate.

41. $\sum_{k=1}^{10}(k-1)^2$

42. $\sum_{j=2}^{100}\frac{1}{j-1}$

43. $\sum_{k=1}^{50}\frac{3^k}{2^{k+1}}$

44. $\sum_{n=1}^{10}n^2 2^n$

45–48 ■ Write the sum using sigma notation. Do not evaluate.

45. $3+6+9+12+\cdots+99$

46. $1^2+2^2+3^2+\cdots+100^2$

47. $1\cdot 2^3+2\cdot 2^4+3\cdot 2^5+4\cdot 2^6+\cdots+100\cdot 2^{102}$

48. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{999\cdot 1000}$

49–54 ■ Determine whether the expression is a partial sum of an arithmetic or geometric sequence. Then find the sum.

49. $1+0.9+(0.9)^2+\cdots+(0.9)^5$

50. $3+3.7+4.4+\cdots+10$

51. $\sqrt{5}+2\sqrt{5}+3\sqrt{5}+\cdots+100\sqrt{5}$

52. $\frac{1}{3}+\frac{2}{3}+1+\frac{4}{3}+\cdots+33$

53. $\sum_{n=0}^{6}3(-4)^n$

54. $\sum_{k=0}^{8}7(5)^{k/2}$

55–60 ■ Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

55. $1-\frac{2}{5}+\frac{4}{25}-\frac{8}{125}+\cdots$

56. $0.1+0.01+0.001+0.0001+\cdots$

57. $5-5(1.01)+5(1.01)^2-5(1.01)^3+\cdots$

58. $1+\frac{1}{3^{1/2}}+\frac{1}{3}+\frac{1}{3^{3/2}}+\cdots$

59. $-1+\frac{9}{8}-\left(\frac{9}{8}\right)^2+\left(\frac{9}{8}\right)^3-\cdots$

60. $a+ab^2+ab^4+ab^6+\cdots,\quad |b|<1$

61. The first term of an arithmetic sequence is $a=7$, and the common difference is $d=3$. How many terms of this sequence must be added to obtain 325?

62. The sum of the first three terms of a geometric series is 52, and the common ratio is $r=3$. Find the first term.

63. A person has two parents, four grandparents, eight great-grandparents, and so on. What is the total number of a person's ancestors in 15 generations?

64. Find the amount of an annuity consisting of 16 annual payments of \$1000 each into an account that pays 8% interest per year, compounded annually.

65. How much money should be invested every quarter at 12% per year, compounded quarterly, in order to have \$10,000 in one year?

66. What are the monthly payments on a mortgage of \$60,000 at 9% interest if the loan is to be repaid in
(a) 30 years? **(b)** 15 years?

67–69 ■ Use mathematical induction to prove that the formula is true for all natural numbers n.

67. $1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2}$

68. $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

69. $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)=n+1$

70. Show that 7^n-1 is divisible by 6 for all natural numbers n.

71. Let $a_{n+1}=3a_n+4$ and $a_1=4$. Show that $a_n=2\cdot 3^n-2$ for all natural numbers n.

72. Prove that the Fibonacci number F_{4n} is divisible by 3 for all natural numbers n.

73–76 ■ Evaluate the expression.

73. $\binom{5}{2}\binom{5}{3}$

74. $\binom{10}{2}+\binom{10}{6}$

75. $\sum_{k=0}^{5}\binom{5}{k}$

76. $\sum_{k=0}^{8}\binom{8}{k}\binom{8}{8-k}$

77–80 ■ Expand the expression.

77. $(A-B)^3$

78. $(x+2)^5$

79. $(1-x^2)^6$

80. $(2x+y)^4$

81. Find the 20th term in the expansion of $(a+b)^{22}$.

82. Find the first three terms in the expansion of $(b^{-2/3}+b^{1/3})^{20}$.

83. Find the term containing A^6 in the expansion of $(A+3B)^{10}$.

CHAPTER 8 TEST

1. Find the first six terms and the sixth partial sum of the sequence whose nth term is $a_n = 2n^2 - n$.

2. A sequence is defined recursively by $a_{n+1} = 3a_n - n$, $a_1 = 2$. Find the first six terms of the sequence.

3. An arithmetic sequence begins 2, 5, 8, 11, 14, ...
   (a) Find the common difference d for this sequence.
   (b) Find a formula for the nth term a_n of the sequence.
   (c) Find the 35th term of the sequence.

4. A geometric sequence begins 12, 3, $3/4$, $3/16$, $3/64$, ...
   (a) Find the common ratio r for this sequence.
   (b) Find a formula for the nth term a_n of the sequence.
   (c) Find the tenth term of the sequence.

5. The first term of a geometric sequence is 25, and the fourth term is $\frac{1}{5}$.
   (a) Find the common ratio r and the fifth term.
   (b) Find the partial sum of the first eight terms.

6. The first term of an arithmetic sequence is 10, and the tenth term is 2.
   (a) Find the common difference and the 100th term of the sequence.
   (b) Find the partial sum of the first ten terms.

7. Let $a_1, a_2, a_3, \ldots$ be a geometric sequence with initial term a and common ratio r. Show that $a_1^2, a_2^2, a_3^2, \ldots$ is also a geometric sequence by finding its common ratio.

8. Write the expression without using sigma notation, and then find the sum.
   (a) $\displaystyle\sum_{n=1}^{5} (1 - n^2)$ **(b)** $\displaystyle\sum_{n=3}^{6} (-1)^n 2^{n-2}$

9. Find the sum.
   (a) $\displaystyle\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \frac{2^3}{3^4} + \cdots + \frac{2^9}{3^{10}}$
   (b) $\displaystyle 1 + \frac{1}{2^{1/2}} + \frac{1}{2} + \frac{1}{2^{3/2}} + \cdots$

10. Use mathematical induction to prove that for all natural numbers n,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

11. Expand $(2x + y^2)^5$.

12. Find the term containing x^3 in the binomial expansion of $(3x - 2)^{10}$.

13. A puppy weighs 0.85 lb at birth, and each week he gains 24% in weight. Let a_n be his weight in pounds at the end of his nth week of life.
    (a) Find a formula for a_n.
    (b) How much does the puppy weigh when he is six weeks old?
    (c) Is the sequence $a_1, a_2, a_3, \ldots$ arithmetic, geometric, or neither?

Modeling with Recursive Sequences

Many real-world processes occur in stages. Population growth can be viewed in stages—each new generation represents a new stage in population growth. Compound interest is paid in stages—each interest payment creates a new account balance. Many things that change continuously are more easily measured in discrete stages. For example, we can measure the temperature of a continuously cooling object in one-hour intervals. In this *Focus* we learn how recursive sequences are used to model such situations. In some cases we can get an explicit formula for a sequence from the recursion relation that defines it by finding a pattern in the terms of the sequence.

▼ Recursive Sequences as Models

Suppose you deposit some money in an account that pays 6% interest compounded monthly. The bank has a definite rule for paying interest: At the end of each month the bank adds to your account $\frac{1}{2}\%$ (or 0.005) of the amount in your account at that time. Let's express this rule as follows:

$$\boxed{\text{amount at the end of this month}} = \boxed{\text{amount at the end of last month}} + 0.005 \times \boxed{\text{amount at the end of last month}}$$

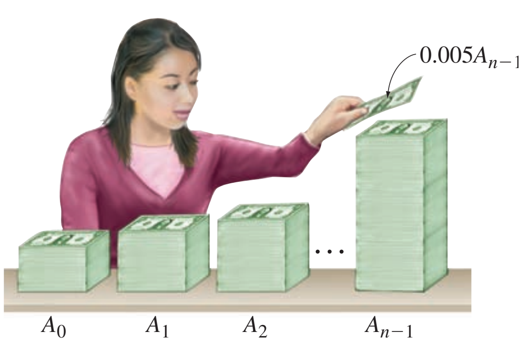
Using the Distributive Property, we can write this as

$$\boxed{\text{amount at the end of this month}} = 1.005 \times \boxed{\text{amount at the end of last month}}$$

To model this statement using algebra, let A_0 be the amount of the original deposit, let A_1 be the amount at the end of the first month, let A_2 be the amount at the end of the second month, and so on. So A_n is the amount at the end of the nth month. Thus

$$A_n = 1.005A_{n-1}$$

We recognize this as a recursively defined sequence—it gives us the amount at each stage in terms of the amount at the preceding stage.



We can use mathematical induction to prove that the formula we found for A_n is valid for all natural numbers n.

To find a formula for A_n, let's find the first few terms of the sequence and look for a pattern:

$$\begin{aligned} A_1 &= 1.005A_0 \\ A_2 &= 1.005A_1 = (1.005)^2A_0 \\ A_3 &= 1.005A_2 = (1.005)^3A_0 \\ A_4 &= 1.005A_3 = (1.005)^4A_0 \end{aligned}$$

We see that in general, $A_n = (1.005)^nA_0$.

EXAMPLE 1 | Population Growth

A certain animal population grows by 2% each year. The initial population is 5000.

(a) Find a recursive sequence that models the population P_n at the end of the nth year.

(b) Find the first five terms of the sequence P_n.

(c) Find a formula for P_n.

SOLUTION

(a) We can model the population using the following rule:

$$\boxed{\text{population at the end of this year}} = 1.02 \times \boxed{\text{population at the end of last year}}$$

Algebraically, we can write this as the recursion relation

$$P_n = 1.02P_{n-1}$$

(b) Since the initial population is 5000, we have

$$\begin{aligned} P_0 &= 5000 \\ P_1 &= 1.02P_0 = (1.02)5000 \\ P_2 &= 1.02P_1 = (1.02)^2 5000 \\ P_3 &= 1.02P_2 = (1.02)^3 5000 \\ P_4 &= 1.02P_3 = (1.02)^4 5000 \end{aligned}$$

(c) We see from the pattern exhibited in part (b) that $P_n = (1.02)^n 5000$. (Note that P_n is a geometric sequence, with common ratio $r = 1.02$.) ■

EXAMPLE 2 | Daily Drug Dose

A patient is to take a 50-mg pill of a certain drug every morning. It is known that the body eliminates 40% of the drug every 24 hours.

(a) Find a recursive sequence that models the amount A_n of the drug in the patient's body after each pill is taken.

(b) Find the first four terms of the sequence A_n.

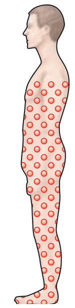
(c) Find a formula for A_n.

(d) How much of the drug remains in the patient's body after 5 days? How much will accumulate in his system after prolonged use?

SOLUTION

(a) Each morning, 60% of the drug remains in his system, plus he takes an additional 50 mg (his daily dose).

$$\boxed{\text{amount of drug this morning}} = 0.6 \times \boxed{\text{amount of drug yesterday morning}} + 50 \text{ mg}$$



We can express this as a recursion relation

$$A_n = 0.6A_{n-1} + 50$$

(b) Since the initial dose is 50 mg, we have

$$\begin{aligned}
A_0 &= 50 \\
A_1 &= 0.6A_0 + 50 = 0.6(50) + 50 \\
A_2 &= 0.6A_1 + 50 = 0.6[0.6(50) + 50] + 50 \\
&= 0.6^2(50) + 0.6(50) + 50 \\
&= 50(0.6^2 + 0.6 + 1) \\
A_3 &= 0.6A_2 + 50 = 0.6[0.6^2(50) + 0.6(50) + 50] + 50 \\
&= 0.6^3(50) + 0.6^2(50) + 0.6(50) + 50 \\
&= 50(0.6^3 + 0.6^2 + 0.6 + 1)
\end{aligned}$$

(c) From the pattern in part (b) we see that

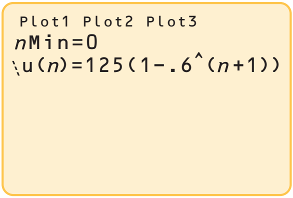
$$\begin{aligned}
A_n &= 50(1 + 0.6 + 0.6^2 + \cdots + 0.6^n) \\
&= 50\left(\frac{1 - 0.6^{n+1}}{1 - 0.6}\right) && \text{Partial sum of a geometric sequence (page 588)} \\
&= 125(1 - 0.6^{n+1}) && \text{Simplify}
\end{aligned}$$

(d) To find the amount remaining after 5 days, we substitute $n = 5$ and get $A_5 = 125(1 - 0.6^{5+1}) \approx 119$ mg.

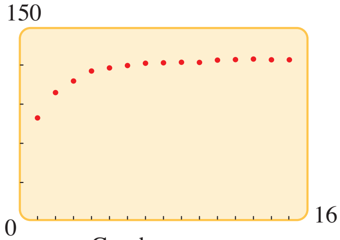
To find the amount remaining after prolonged use, we let n become large. As n gets large, 0.6^n approaches 0. That is, $0.6^n \to 0$ as $n \to \infty$ (see Section 4.1). So as $n \to \infty$,

$$A_n = 125(1 - 0.6^{n+1}) \to 125(1 - 0) = 125$$

Thus after prolonged use the amount of drug in the patient's system approaches 125 mg (see Figure 1, where we have used a graphing calculator to graph the sequence).



Enter sequence



Graph sequence

FIGURE 1

PROBLEMS

1. Retirement Accounts Many college professors keep retirement savings with TIAA, the largest annuity program in the world. Interest on these accounts is compounded and credited *daily*. Professor Brown has \$275,000 on deposit with TIAA at the start of 2011 and receives 3.65% interest per year on his account.

(a) Find a recursive sequence that models the amount A_n in his account at the end of the nth day of 2011.

(b) Find the first eight terms of the sequence A_n, rounded to the nearest cent.

(c) Find a formula for A_n.

2. Fitness Program Sheila decides to embark on a swimming program as the best way to maintain cardiovascular health. She begins by swimming 5 min on the first day, then adds $1\frac{1}{2}$ min every day after that.

(a) Find a recursive formula for the number of minutes T_n that she swims on the nth day of her program.

(b) Find the first 6 terms of the sequence T_n.

(c) Find a formula for T_n. What kind of sequence is this?

(d) On what day does Sheila attain her goal of swimming at least 65 min a day?

(e) What is the total amount of time she will have swum after 30 days?

3. Monthly Savings Program Alice opens a savings account that pays 3% interest per year, compounded monthly. She begins by depositing \$100 at the start of the first month and adds \$100 at the end of each month, when the interest is credited.

(a) Find a recursive formula for the amount A_n in her account at the end of the nth month. (Include the interest credited for that month and her monthly deposit.)

(b) Find the first five terms of the sequence A_n.

(c) Use the pattern you observed in (b) to find a formula for A_n. [*Hint:* To find the pattern most easily, it's best *not* to simplify the terms *too* much.]

(d) How much has she saved after 5 years?

4. Stocking a Fish Pond A pond is stocked with 4000 trout, and through reproduction the population increases by 20% per year. Find a recursive sequence that models the trout population P_n at the end of the nth year under each of the following circumstances. Find the trout population at the end of the fifth year in each case.

(a) The trout population changes only because of reproduction.

(b) Each year 600 trout are harvested.

(c) Each year 250 additional trout are introduced into the pond.

(d) Each year 10% of the trout are harvested, and 300 additional trout are introduced into the pond.

5. Pollution A chemical plant discharges 2400 tons of pollutants every year into an adjacent lake. Through natural runoff, 70% of the pollutants contained in the lake at the beginning of the year are expelled by the end of the year.

(a) Explain why the following sequence models the amount A_n of the pollutant in the lake at the end of the nth year that the plant is operating.

$$A_n = 0.30A_{n-1} + 2400$$

(b) Find the first five terms of the sequence A_n.

(c) Find a formula for A_n.

(d) How much of the pollutant remains in the lake after 6 years? How much will remain after the plant has been operating a long time?

(e) Verify your answer to part (d) by graphing A_n with a graphing calculator for $n = 1$ to $n = 20$.

6. Annual Savings Program Ursula opens a one-year CD that yields 5% interest per year. She begins with a deposit of \$5000. At the end of each year when the CD matures, she reinvests at the same 5% interest rate, also adding 10% to the value of the CD from her other savings. (So for example, after the first year her CD has earned 5% of \$5000 in interest, for a value of \$5250 at maturity. She then adds 10%, or \$525, bringing the total value of her renewed CD to \$5775.)

(a) Find a recursive formula for the amount U_n in Ursula's CD when she reinvests at the end of the nth year.

(b) Find the first five terms of the sequence U_n. Does this appear to be a geometric sequence?

(c) Use the pattern you observed in (b) to find a formula for U_n.

(d) How much has she saved after 10 years?

7. Annual Savings Program Victoria opens a one-year CD with a 5% annual interest yield at the same time as her friend Ursula in Problem 6. She also starts with an initial deposit of \$5000. However, Victoria decides to add \$500 to her CD when she reinvests at the end of the first year, \$1000 at the end of the second, \$1500 at the end of the third, and so on.

(a) Explain why the recursive formula displayed below gives the amount V_n in Victoria's CD when she reinvests at the end of the nth year.

$$V_n = 1.05V_{n-1} + 500n$$

(b) Using the `Seq` ("sequence") mode on your graphing calculator, enter the sequences U_n and V_n as shown in the figure. Then use the `TABLE` command to compare the two sequences. For the first few years, Victoria seems to be accumulating more savings than Ursula. Scroll down in the table to verify that Ursula eventually pulls ahead of Victoria in the savings race. In what year does this occur?

```
Plot1 Plot2 Plot3
\u(n) = 1.05 u(n - 1)
 +0.1 u(n - 1)
 u(nMin) = {5000}
\v(n) = 1.05 v(n - 1)
 +500 n
 v(nMin) = {5000}
```

Entering the sequences

n	u(n)	v(n)
0	5000	5000
1	5750	5750
2	6612.5	7037.5
3	7604.4	8889.4
4	8745	11334
5	10057	14401
6	11565	18121
n=0		

Table of values of the sequences

8. Newton's Law of Cooling A tureen of soup at a temperature of 170°F is placed on a table in a dining room in which the thermostat is set at 70°F. The soup cools according to the following rule, a special case of Newton's Law of Cooling: Each minute, the temperature of the soup declines by 3% of the difference between the soup temperature and the room temperature.

(a) Find a recursive sequence that models the soup temperature T_n at the nth minute.

(b) Enter the sequence T_n in your graphing calculator, and use the `TABLE` command to find the temperature at 10-min increments from $n = 0$ to $n = 60$. (See Problem 7(b).)

(c) Graph the sequence T_n. What temperature will the soup be after a long time?

9. Logistic Population Growth Simple exponential models for population growth do not take into account the fact that when the population increases, survival becomes harder for each individual because of greater competition for food and other resources. We can get a more accurate model by assuming that the birth rate is proportional to the size of the population, but the death rate is proportional to the square of the population. Using this idea, researchers find that the number of raccoons R_n on a certain island is modeled by the following recursive sequence:

Population at end of year — Number of births

$$\underbrace{R_n}_{\text{Population at end of year}} = \underbrace{R_{n-1}}_{\text{Population at beginning of year}} + \underbrace{0.08R_{n-1}}_{\text{Number of births}} - \underbrace{0.0004(R_{n-1})^2}_{\text{Number of deaths}}, \qquad R_0 = 100$$

Here, n represents the number of years since observations began, R_0 is the initial population, 0.08 is the annual birth rate, and 0.0004 is a constant related to the death rate.

(a) Use the `TABLE` command on a graphing calculator to find the raccoon population for each year from $n = 1$ to $n = 7$.

(b) Graph the sequence R_n. What happens to the raccoon population as n becomes large?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CHAPTER 9

PROBABILITY AND STATISTICS

Modeling Patterns in Randomness In the preceding chapters we modeled real-world situations using precise rules, such as equations or functions. But many of our everyday activities are not governed by precise rules; rather, they involve randomness and uncertainty. It is remarkable that we can also use algebra to describe patterns in random events. *Probability,* the main topic of this chapter, gives us a way to quantify the "chance" or "probability" that a particular event occurs.

The importance of probability in the modern world cannot be overestimated. It is used in industry, manufacturing, government, medical research, political polling, and many other areas of human endeavor. For example, insurance companies carefully estimate the probability of lightning striking a house. Knowing this probability allows the company to make certain that the premiums they collect exceed the payouts they would expect to make for lightning damage.

In *Focus on Modeling* at the end of the chapter we use a calculator (or computer) to simulate random events and estimate probabilities.

9.1 Counting

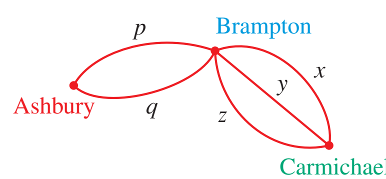
LEARNING OBJECTIVES **After completing this section, you will be able to:**

Use the Fundamental Counting Principle ▶ Count permutations ▶ Count distinguishable permutations ▶ Count combinations ▶ Solve counting problems involving both permutations and combinations

Counting the number of apples in a bag or the number of students in an algebra class is easy. But counting all the different ways in which these students can stand in a row is more difficult. It is this latter kind of counting that we'll study in this section.

▼ The Fundamental Counting Principle

Suppose that three towns—Ashbury, Brampton, and Carmichael—are located in such a way that two roads connect Ashbury to Brampton and three roads connect Brampton to Carmichael.



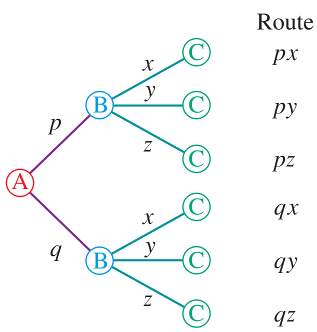


FIGURE 1 Tree diagram

How many different routes can one take to travel from Ashbury to Carmichael via Brampton? The key to answering this question is to consider the problem in stages. At the first stage—from Ashbury to Brampton—there are two choices. For each of these choices there are three choices at the second stage—from Brampton to Carmichael. Thus the number of different routes is $2 \times 3 = 6$. These routes are conveniently enumerated by a *tree diagram* as in Figure 1. The method that we used to solve this problem leads to the following principle.

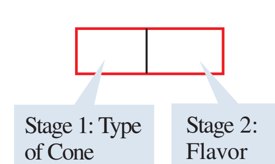
> **THE FUNDAMENTAL COUNTING PRINCIPLE**
>
> Suppose that two events occur in order. If the first event can occur in m ways and the second can occur in n ways (after the first has occurred), then the two events can occur *in order* in $m \times n$ ways.

There is an immediate consequence of this principle for any number of events: If $E_1, E_2, \ldots, E_k$ are events that occur in order and if E_1 can occur in n_1 ways, E_2 in n_2 ways, and so on, then the events can occur in order in $n_1 \times n_2 \times \cdots \times n_k$ ways.

EXAMPLE 1 | Using the Fundamental Counting Principle

An ice-cream store offers three types of cones and 31 flavors. How many different single-scoop ice-cream cones is it possible to buy at this store?

SOLUTION There are two stages for selecting an ice-cream cone. At the first stage we choose a type of cone, and at the second stage we choose a flavor. We can think of the different stages as boxes:



The first box can be filled in three ways, and the second can be filled in 31 ways:

3	31
Stage 1	Stage 2

By the Fundamental Counting Principle there are $3 \times 31 = 93$ ways of choosing a single-scoop ice-cream cone at this store.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17** ■

EXAMPLE 2 | Using the Fundamental Counting Principle

In a certain state, automobile license plates display three letters followed by three digits. How many such plates are possible if repetition of the letters

(a) is allowed? **(b)** is not allowed?

SOLUTION

(a) There are six selection stages, one for each letter or digit on the license plate. As in the preceding example, we sketch a box for each stage:

26	26	26	10	10	10

Letters Digits

At the first stage we choose a letter (from 26 possible choices); at the second stage we choose another letter (again from 26 choices); at the third stage we choose another letter (26 choices); at the fourth stage we choose a digit (from 10 possible choices); at the fifth stage we choose a digit (again from 10 choices); and at the sixth stage, we choose another digit (10 choices). By the Fundamental Counting Principle the number of possible license plates is

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17{,}576{,}000$$

(b) If repetition of letters is not allowed, then we arrange the choices as follows:

26	25	24	10	10	10

Letters Digits

At the first stage we have 26 letters to choose from, but once the first letter has been chosen, there are only 25 letters to choose from at the second stage. Once the first two letters have been chosen, 24 letters are left to choose from for the third stage. The digits are chosen as before. Thus the number of possible license plates in this case is

$$26 \times 25 \times 24 \times 10 \times 10 \times 10 = 15{,}600{,}000$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **29** ■

Let S be a set with n elements. A subset of S can be chosen by making one of two choices for each element: We can choose the element to be *in* or *out* of A. Since S has n elements and we have two choices for each element, by the Fundamental Counting Principle the total number of different subsets is $2 \times 2 \times \cdots \times 2$, where there are n factors. This gives the following formula.

THE NUMBER OF SUBSETS OF A SET

A set with n elements has 2^n different subsets.

EXAMPLE 3 | Finding the Number of Subsets

A pizza parlor offers a basic cheese pizza and a choice of 16 toppings. How many different kinds of pizza can be ordered at this pizza parlor?

SOLUTION We need the number of possible subsets of the 16 toppings (including the empty set, which corresponds to a plain cheese pizza). Thus

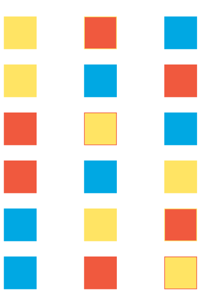
$$2^{16} = 65{,}536$$

different pizzas can be ordered.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **37**

Counting Permutations

Permutations of three colored squares



A **permutation** of a set of distinct objects is an ordering of these objects. For example, some permutations of the letters *ABCD* are

$$ABDC \qquad BACD \qquad DCBA \qquad DABC$$

How many such permutations are possible? There are four choices for the first position, three for the second (after the first has been chosen), two for the third (after the first two have been chosen), and only one choice for the fourth letter (the letter that has not yet been chosen). By the Fundamental Counting Principle the number of possible permutations is

$$4 \times 3 \times 2 \times 1 = 4! = 24$$

The same reasoning with 4 replaced by n leads to the following.

$$\boxed{\text{The number of permutations of } n \text{ objects is } n!}$$

How many permutations consisting of two letters can be made from these same four letters? Some of these permutations are *AB, AC, BD, DB,* . . . . There are 4 choices of the first letter and 3 for the second letter. By the Fundamental Counting Principle there are $4 \times 3 = 12$ such permutations. In general, if a set has n elements, then the number of ways of ordering r elements from the set is denoted by $\boldsymbol{P(n, r)}$ and is called **the number of permutations of *n* objects taken *r* at a time**.

PERMUTATIONS OF *n* OBJECTS TAKEN *r* AT A TIME

The number of permutations of n objects taken r at a time is

$$P(n, r) = \frac{n!}{(n - r)!}$$

PROOF There are n objects and r positions to place them in. Thus there are n choices for the first position, $n-1$ choices for the second, $n-2$ choices for the third, and so on. The last position can be filled in $n-r+1$ ways. By the Fundamental Counting Principle we conclude that

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

We can express this formula using factorial notation by multiplying numerator and denominator by $(n-r)\cdots 3\cdot 2\cdot 1$:

$$P(n, r) = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)\cdots 3\cdot 2\cdot 1}{(n-r)\cdots 3\cdot 2\cdot 1} = \frac{n!}{(n-r)!}$$

EXAMPLE 4 | Finding the Number of Permutations

There are six runners in a race that is completed with no tie.

(a) In how many different ways can the race be completed?

(b) In how many different ways can first, second, and third place be decided?

SOLUTION

(a) The number of ways to complete the race is the number of permutations of the six runners: $6! = 720$.

(b) The number of ways in which the first three positions can be decided is

$$P(6, 3) = \frac{6!}{(6-3)!} = \frac{6 \times 5 \times 4 \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2 \times 1}} = 120$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **41**

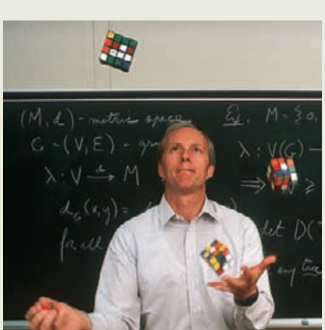
EXAMPLE 5 | Finding the Number of Permutations

A club has nine members. In how many ways can a president, a vice president, and a secretary be chosen from the members of this club?

SOLUTION We need the number of ways of selecting three members, in order, for the positions of president, vice president, and secretary from the nine club members. This number is

$$P(9, 3) = \frac{9!}{(9-3)!} = \frac{9 \times 8 \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}} = 504$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **43**



Ken Regan/Camera 5

RONALD GRAHAM, born in Taft, California, in 1935, is considered the world's leading mathematician in the field of combinatorics, the branch of mathematics that deals with counting. For many years Graham headed the Mathematical Studies Center at Bell Laboratories in Murray Hill, New Jersey, where he solved key problems for the telephone industry. During the *Apollo* program, NASA needed to evaluate mission schedules so that the three astronauts aboard the spacecraft could find the time to perform all the necessary tasks. The number of ways to allot these tasks was astronomical—too vast for even a computer to sort out. Graham, using his knowledge of combinatorics, was able to reassure NASA that there were easy ways of solving their problem that were not too far from the theoretically best possible solution. Besides being a prolific mathematician, Graham is an accomplished juggler (he has been on stage with the Cirque du Soleil and is a past president of the International Jugglers Association). Several of his research papers address the mathematical aspects of juggling. He is also fluent in Mandarin Chinese and Japanese and once spoke with former President Jiang of China in his native language.

EXAMPLE 6 | Finding the Number of Permutations

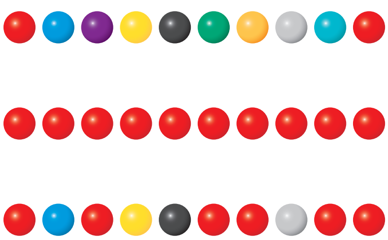
From 20 raffle tickets in a hat, 4 tickets are to be selected in order. The holder of the first ticket wins a car, the second a motorcycle, the third a bicycle, and the fourth a skateboard. In how many different ways can these prizes be awarded?

SOLUTION The order in which the tickets are chosen determines who wins each prize. So we need to find the number of ways of selecting 4 objects, in order, from 20 objects (the tickets). This number is

$$P(20, 4) = \frac{20!}{(20-4)!} = \frac{20 \times 19 \times 18 \times 17 \times \cancel{16 \times 15 \times 14 \times \cdots \times 3 \times 2 \times 1}}{\cancel{16 \times 15 \times 14 \times \cdots \times 3 \times 2 \times 1}} = 116{,}280$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **45** ■

▼ Distinguishable Permutations



If we have a collection of ten balls, each a different color, then the number of permutations of these balls is $P(10, 10) = 10!$. If all ten balls are red, then we have just one distinguishable permutation because all the ways of ordering these balls look exactly the same. In general, in considering a set of objects, some of which are of the same kind, then two permutations are **distinguishable** if one cannot be obtained from the other by interchanging the positions of elements of the same kind. For example, if we have ten balls, of which six are red and the other four are each a different color, then how many distinguishable permutations are possible? The key point here is that balls of the same color are not distinguishable. So each rearrangement of the red balls, keeping all the other balls fixed, gives essentially the same permutation. Since there are 6! rearrangements of the red balls for each fixed position of the other balls, the total number of distinguishable permutations is $10!/6!$. The same type of reasoning gives the following general rule:

DISTINGUISHABLE PERMUTATIONS

If a set of n objects consists of k different kinds of objects with n_1 objects of the first kind, n_2 objects of the second kind, n_3 objects of the third kind, and so on, where $n_1 + n_2 + \cdots + n_k = n$, then the number of distinguishable permutations of these objects is

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}$$

EXAMPLE 7 | Finding the Number of Distinguishable Permutations

Find the number of different ways of placing 15 balls in a row given that 4 are red, 3 are yellow, 6 are black, and 2 are blue.

SOLUTION We want to find the number of distinguishable permutations of these balls. By the formula this number is

$$\frac{15!}{4!\,3!\,6!\,2!} = 6{,}306{,}300$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **55** ■

Suppose we have 15 wooden balls in a row and four colors of paint: red, yellow, black, and blue. In how many different ways can the 15 balls be painted in such a way that we have 4 red, 3 yellow, 6 black, and 2 blue balls? A little thought will show that this number is exactly the same as that calculated in Example 3. This way of looking at the problem is somewhat different, however. Here we think of the number of ways to **partition** the balls into four groups, each containing 4, 3, 6, and 2 balls to be painted red, yellow, black, and blue, respectively. The next example shows how this reasoning is used.

EXAMPLE 8 | Finding the Number of Partitions

Fourteen construction workers are to be assigned to three different tasks. Seven workers are needed for mixing cement, five for laying bricks, and two for carrying the bricks to the brick layers. In how many different ways can the workers be assigned to these tasks?

SOLUTION We need to partition the workers into three groups containing 7, 5, and 2 workers, respectively. This number is

$$\frac{14!}{7!\,5!\,2!} = 72{,}072$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **59**

Counting Combinations

When counting permutations, we are interested in the number of ways of ordering the elements of a set. In many counting problems, however, order is not important. For example, a poker hand is the same hand regardless of how it is ordered. A poker player who is interested in the number of possible hands wants to know the number of ways of drawing five cards from 52 cards, without regard to the order in which the cards are dealt. We now develop a formula for counting in situations in which order doesn't matter.

A **combination** of r elements of a set is any subset of r elements from the set (without regard to order). If the set has n elements, then the number of combinations of r elements is denoted by $\boldsymbol{C(n, r)}$ and is called the **number of combinations of $\boldsymbol{n}$ elements taken $\boldsymbol{r}$ at a time.** For example, consider a set with the four elements A, B, C, and D. The combinations of these four elements taken three at a time are listed below. Compare this with the permutations of these elements listed in the margin.

ABC	*ABD*	*ACD*	*BCD*
ACB	*ADB*	*ADC*	*BDC*
BAC	*BAD*	*CAD*	*CBD*
BCA	*BDA*	*CDA*	*CDB*
CAB	*DAB*	*DAC*	*DBC*
CBA	*DBA*	*DCA*	*DCB*

ABC *ABD* *ACD* *BCD*

We notice that the number of combinations is a lot fewer than the number of permutations. In fact, each combination of three elements generates 3! permutations. So $C(4, 3) = P(4, 3)/3! = 4$. In general, each combination of r objects gives rise to $r!$ permutations of these objects, so we get the following formula.

COMBINATIONS OF $\boldsymbol{n}$ OBJECTS TAKEN $\boldsymbol{r}$ AT A TIME

The number of combinations of n objects taken r at a time is

$$C(n, r) = \frac{n!}{r!\,(n - r)!}$$

The key difference between permutations and combinations is order. If we are interested in ordered arrangements, then we are counting permutations, but if we are concerned with subsets without regard to order, then we are counting combinations. Compare Examples 9 and 10 below (where order doesn't matter) to Examples 5 and 6 (where order does matter).

EXAMPLE 9 | Finding the Number of Combinations

A club has nine members. In how many ways can a committee of three be chosen from the members of this club?

SOLUTION We need the number of ways of choosing three of the nine members. Order is not important here, because the committee is the same no matter how its members are ordered. So we want the number of combinations of nine objects (the club members) taken three at a time. This number is

$$C(9, 3) = \frac{9!}{3!(9-3)!} = \frac{9 \times 8 \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{(3 \times 2 \times 1) \times (\cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1})} = 84$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **61**

EXAMPLE 10 | Finding the Number of Combinations

From 20 raffle tickets in a hat, four tickets are to be chosen at random. The holders of the winning tickets get free trips to the Bahamas. In how many ways can the four winners be chosen?

SOLUTION We need to find the number of ways of choosing four winners from 20 entries. The order in which the tickets are chosen doesn't matter, because the same prize is awarded to each of the four winners. So we want the number of combinations of 20 objects (the tickets) taken four at a time. This number is

$$C(20, 4) = \frac{20!}{4!(20-4)!} = \frac{20 \times 19 \times 18 \times 17 \times \cancel{16 \times 15 \times 14 \times \cdots \times 3 \times 2 \times 1}}{(4 \times 3 \times 2 \times 1) \times (\cancel{16 \times 15 \times 14 \times \cdots \times 3 \times 2 \times 1})} = 4845$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **63**

▼ Problem Solving with Permutations and Combinations

The crucial step in solving counting problems is deciding whether to use permutations, combinations, or the Fundamental Counting Principle. In some cases the solution of a problem may require using more than one of these principles. Here are some general guidelines to help us decide how to apply these principles.

GUIDELINES FOR SOLVING COUNTING PROBLEMS

1. **Fundamental Counting Principle.** When consecutive choices are being made, use the Fundamental Counting Principle.
2. **Does Order Matter?** When we want to find the number of ways of picking r objects from n objects, we need to ask ourselves, "Does the order in which we pick the objects matter?"

   If the order matters, we use permutations.

   If the order doesn't matter, we use combinations.

EXAMPLE 11 | Using Combinations

A group of 25 campers consists of 15 women and 10 men. In how many ways can a scouting party of 6 be chosen if it must consist of 3 women and 2 men?

SOLUTION Three women can be chosen from the 15 women in $C(15, 3)$ ways, and two men can be chosen from the 10 men in $C(10, 2)$ ways. It follows by the Fundamental Counting Principle that the number of ways of choosing the scouting party is

$$C(15, 3) \times C(10, 2) = 455 \times 45 = 20{,}475$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **75**

EXAMPLE 12 | Using Permutations and Combinations

A committee of seven—consisting of a chairman, a vice chairman, a secretary, and four other members—is to be chosen from a class of 20 students. In how many ways can the committee be chosen?

SOLUTION In choosing the three officers, order is important. So the number of ways of choosing them is

$$P(20, 3) = 6840$$

Next, we need to choose four other students from the 17 remaining. Since order doesn't matter in choosing these four members, the number of ways of doing this is

$$C(17, 4) = 2380$$

By the Fundamental Counting Principle the number of ways of choosing this committee is

$$P(20, 3) \times C(17, 4) = 6840 \times 2380 = 16{,}279{,}200$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **77**

EXAMPLE 13 | Using Permutations and Combinations

Twelve employees at a company picnic are to stand in a row for a group photograph. In how many ways can this be done if

(a) Jane and John insist on standing next to each other?

(b) Jane and John refuse to stand next to each other?

SOLUTION Since the order in which the people stand is important, we use permutations. But we can't use permutations directly.

(a) Since Jane and John insist on standing together, let's think of them as one object. So we have 11 objects to arrange in a row, and there are $P(11, 11)$ ways of doing this. For each of these arrangements there are two ways of having Jane and John stand together: Jane-John or John-Jane. By the Fundamental Counting Principle the total number of arrangements is

$$2 \times P(11, 11) = 2 \times 11! = 79{,}833{,}600$$

(b) There are $P(12, 12)$ ways of arranging the 12 people. Of these, $2 \times P(11, 11)$ have Jane and John standing together (by part (a)). All the rest have Jane and John standing apart. So the number of arrangements with Jane and John standing apart is

$$P(12, 12) - 2 \times P(11, 11) = 12! - 2 \times 11! = 399{,}168{,}000$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **85**

9.1 EXERCISES

CONCEPTS

1. The Fundamental Counting Principle says that if one event can occur in m ways and a second event can occur in n ways, then the two events can occur in order in _____ × _____ ways. So if you have two choices for shoes and three choices for hats, then the number of different shoe-hat combinations you can wear is _____ × _____ = _____.

2. The number of ways of arranging r objects from n objects in order is called the number of _____________ of n objects taken r at a time and is given by the formula $P(n, r) =$ ________.

3. The number of ways of choosing r objects from n objects is called the number of __________ of n objects taken r at a time and is given by the formula $C(n, r) =$ __________.

4. *True or false*?
- **(a)** In counting combinations, order matters.
- **(b)** In counting permutations, order matters.
- **(c)** For a set of n distinct objects, the number of different combinations of these objects is more than the number of different permutations.
- **(d)** If we have a set with five distinct objects, then the number of different ways of choosing two members of this set is the same as the number of ways of choosing three members.

SKILLS

5–16 ■ Evaluate the expression.

5. $P(8, 3)$ **6.** $P(9, 2)$
7. $P(11, 4)$ **8.** $P(10, 5)$
9. $P(100, 1)$ **10.** $P(99, 3)$
11. $C(8, 3)$ **12.** $C(9, 2)$
13. $C(11, 4)$ **14.** $C(10, 5)$
15. $C(100, 1)$ **16.** $C(99, 3)$

APPLICATIONS

Exercises 17–36 involve the Fundamental Counting Principle.

17. Ice-Cream Cones A vendor sells ice cream from a cart on the boardwalk. He offers vanilla, chocolate, strawberry, and pistachio ice cream, served in either a waffle, sugar, or plain cone. How many different single-scoop ice-cream cones can you buy from this vendor?

18. Three-Letter Words How many three-letter "words" (strings of letters) can be formed by using the 26 letters of the alphabet if repetition of letters
- **(a)** is allowed?
- **(b)** is not allowed?

19. Horse Race Eight horses compete in a race. (Assume that the race does not end in a tie.)
- **(a)** How many different orders are possible for completing the race?
- **(b)** In how many different ways can first, second, and third places be decided?

20. Multiple-Choice Test A multiple-choice test has five questions with four choices for each question. In how many different ways can the test be completed?

21. Phone Numbers Telephone numbers consist of seven digits; the first digit cannot be 0 or 1. How many telephone numbers are possible?

22. Running a Race In how many different ways can a race with five runners be completed? (Assume that there is no tie.)

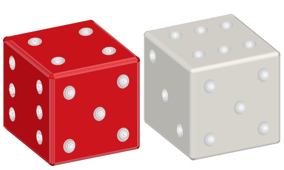
23. Restaurant Meals A restaurant offers the items listed in the table. How many different meals consisting of a main course, a drink, and a dessert can be selected at this restaurant?

Main courses	Drinks	Desserts
Chicken	Iced tea	Ice cream
Beef	Apple juice	Layer cake
Lasagna	Cola	Blueberry pie
Quiche	Ginger ale	
	Coffee	

24. Multiple Routes Towns A, B, C, and D are located in such a way that there are four roads from A to B, five roads from B to C, and six roads from C to D. How many routes are there from town A to town D via towns B and C?

25. Flipping a Coin A coin is flipped five times, and the resulting sequence of heads and tails is recorded. How many such sequences are possible?

26. Rolling a Pair of Dice A red die and a white die are rolled, and the numbers that show are recorded. How many different outcomes are possible? (The singular form of the word *dice* is *die.*)



27. Rolling Three Dice A red die, a blue die, and a white die are rolled, and the numbers that show are recorded. How many different outcomes are possible?

28. Choosing Outfits A girl has five skirts, eight blouses, and 12 pairs of shoes. How many different skirt-blouse-shoe outfits can she wear? (Assume that each item matches all the others, so she is willing to wear any combination.)

29. License Plates Standard automobile license plates in California display a nonzero digit, followed by three letters, followed by three digits. How many different standard plates are possible in this system?

30. ID Numbers A company's employee ID number system consists of one letter followed by three digits. How many different ID numbers are possible with this system?

31. **Combination Lock** A combination lock has 60 different positions. To open the lock, the dial is turned to a certain number in the clockwise direction, then to a number in the counterclockwise direction, and finally to a third number in the clockwise direction. If successive numbers in the combination cannot be the same, how many different combinations are possible?

32. **License Plates** A state has registered 8 million automobiles. To simplify the license plate system, a state employee suggests that each plate display only two letters followed by three digits. Will this system create enough different license plates for all the vehicles that are registered?

33. **Class Executive** In how many ways can a president, vice president, and secretary be chosen from a class of 30 students?

34. **Committee Officers** A senate subcommittee consists of ten Democrats and seven Republicans. In how many ways can a chairman, vice chairman, and secretary be chosen if the chairman must be a Democrat and the vice chairman must be a Republican?

35. **Social Security Numbers** Social Security numbers consist of nine digits, with the first digit between 0 and 6, inclusive. How many Social Security numbers are possible?

36. **Holiday Photos** A couple have seven children: three girls and four boys. In how many ways can the children be arranged for a holiday photo if the girls sit in a row in the front and the boys stand in a row behind the girls?

Exercises 37–40 involve counting subsets.

37. **Subsets** A set has eight elements.
   (a) How many subsets containing five elements does this set have?
   (b) How many subsets does this set have?

38. **Travel Brochures** A travel agency has limited numbers of eight different free brochures about Australia. The agent tells you to take any that you like but no more than one of any kind. In how many different ways can you choose brochures (including not choosing any)?

39. **Hamburgers** A hamburger chain gives their customers a choice of ten different hamburger toppings. In how many different ways can a customer order a hamburger?

40. **To Shop or Not to Shop** Each of 20 shoppers in a shopping mall chooses to enter or not to enter the Dressfastic clothing store. How many different outcomes of their decisions are possible?

Exercises 41–52 involve counting permutations.

41. **Seating Arrangements** Ten people are at a party.
   (a) In how many different ways can they be seated in a row of ten chairs?
   (b) In how many different ways can six of these people be selected and then seated in a row of six chairs?

42. **Three-Letter Words** How many three-letter "words" can be made from the letters FGHIJK? (Letters may not be repeated.)

43. **Class Officers** In how many different ways can a president, vice president, and secretary be chosen from a class of 15 students?

44. **Three-Digit Numbers** How many different three-digit whole numbers can be formed by using the digits 1, 3, 5, and 7 if no repetition of digits is allowed?

45. **Contest Prizes** In how many different ways can first, second, and third prizes be awarded in a game with eight contestants?

46. **Piano Recital** A pianist plans to play eight pieces at a recital. In how many ways can she arrange these pieces in the program?

47. **Running a Race** In how many different ways can a race with nine runners be completed, assuming that there is no tie?

48. **Signal Flags** A ship carries five signal flags of different colors. How many different signals can be sent by hoisting exactly three of the five flags on the ship's flagpole in different orders?

49. **Contest Prizes** In how many ways can first, second, and third prizes be awarded in a contest with 1000 contestants?

50. **Class Officers** In how many ways can a president, vice president, secretary, and treasurer be chosen from a class of 30 students?

51. **Seating Arrangements** In how many ways can five students be seated in a row of five chairs if Jack insists on sitting in the first chair?

52. **Seating Arrangements** In how many ways can the students in Exercise 51 be seated if Jack insists on sitting in the middle chair?

Exercises 53–60 involve distinguishable permutations.

53. **Arrangements** In how many ways can two blue marbles and four red marbles be arranged in a row?

54. **Arrangements** In how many different ways can five red balls, two white balls, and seven blue balls be arranged in a row?

55. Arranging Coins In how many different ways can four pennies, three nickels, two dimes, and three quarters be arranged in a row?

56. Arranging Letters In how many different ways can the letters of the word *ELEEMOSYNARY* be arranged?

57. Distributions A man bought three vanilla ice-cream cones, two chocolate cones, four strawberry cones, and five butterscotch cones for his 14 chidren. In how many ways can he distribute the cones among his children?

58. Room Assignments When seven students take a trip, they find a hotel with three rooms available: a room for one person, a room for two people, and a room for three people. In how many different ways can the students be assigned to these rooms? (One student has to sleep in the car.)

59. Work Assignments Eight workers are cleaning a large house. Five are needed to clean windows, two to clean the carpets, and one to clean the rest of the house. In how many different ways can these tasks be assigned to the eight workers?

60. Transporting Students A group of 30 students is taking a field trip to a science museum. Three vans are available for transporting the students. The first van has room for 8 students, and the other two vans each have room for 11 students. In how many different ways can the students be assigned to the vans?

Exercises 61–74 involve counting combinations.

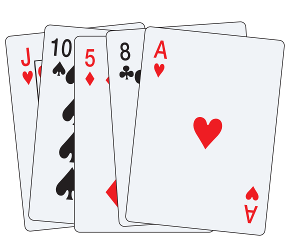
61. Committee In how many ways can a committee of three members be chosen from a club of 25 members?

62. Choosing Books In how many ways can three books be chosen from a group of six different books?

63. Raffle In a raffle with 12 entries, in how many ways can three winners be selected?

64. Choosing a Group In how many ways can six people be chosen from a group of ten?

65. Draw Poker Hands How many different five-card hands can be dealt from a deck of 52 cards?



66. Stud Poker Hands How many different seven-card hands can be picked from a deck of 52 cards?

67. Choosing Exam Questions A student must answer seven of the ten questions on an exam. In how many ways can she choose the seven questions?

68. Three-Topping Pizzas A pizza parlor offers a choice of 16 different toppings. How many three-topping pizzas are possible?

69. Violin Recital A violinist has practiced 12 pieces. In how many ways can he choose eight of these pieces for a recital?

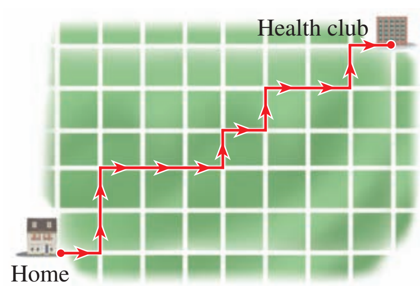
70. Choosing Clothing If a woman has eight skirts, in how many ways can she choose five of these to take on a weekend trip?

71. Choosing Clothing If a man has ten pairs of pants, in how many ways can he choose three of these to take on a business trip?

72. Field Trip From a class with 30 students, seven are to be chosen to go on a field trip. Find the number of different ways that the seven students can be chosen under the given condition.
(a) Jack must go on the field trip.
(b) Jack is not allowed to go on the field trip.
(c) There are no restrictions on who can go on the field trip.

73. Lottery In the 6/49 lottery game, a player picks six numbers from 1 to 49. How many different choices does the player have?

74. Jogging Routes A jogger jogs every morning to his health club, which is eight blocks east and five blocks north of his home. He always takes a route that is as short as possible, but he likes to vary it (see the figure). How many different routes can he take? [*Hint:* The route shown can be thought of as *ENNEEENENEENE,* where *E* is East and *N* is North.]



Solve Exercises 75–90 by using the appropriate counting principle(s).

75. Choosing a Committee A class has 20 students, of whom 12 are females and 8 are males. In how many ways can a committee of five students be picked from this class under each condition?
(a) No restriction is placed on the number of males or females on the committee.
(b) No males are to be included on the committee.
(c) The committee must have three females and two males.

76. Doubles Tennis From a group of ten male and ten female tennis players, two men and two women are to face each other in a men-versus-women doubles match. In how many different ways can this match be arranged?

77. Choosing a Committee A committee of six is to be chosen from a class of 20 students. The committee is to consist of a chair, a secretary and four other members. In how many different ways can the committee be picked?

78. Choosing a Group Sixteen boys and nine girls go on a camping trip. In how many ways can a group of six be selected to gather firewood, given the following conditions?
(a) The group consists of two girls and four boys.
(b) The group contains at least two girls.

79. Dance Committee A school dance committee is to consist of two freshmen, three sophomores, four juniors, and five seniors. If six freshmen, eight sophomores, twelve juniors, and ten seniors are eligible to be on the committee, in how many ways can the committee be chosen?

80. Casting a Play A group of 22 aspiring thespians contains 10 men and 12 women. For the next play, the director wants to choose a leading man, a leading lady, a supporting male role, a supporting female role, and eight extras—three women and five men. In how many ways can the cast be chosen?

81. Hockey Lineup A hockey team has 20 players, of whom 12 play forward, six play defense, and two are goalies. In how many ways can the coach pick a starting lineup consisting of three forwards, two defense players, and one goalie?

82. Choosing a Pizza A pizza parlor offers four sizes of pizza (small, medium, large, and colossus), two types of crust (thick and thin), and 14 different toppings. How many different pizzas can be made with these choices?

83. Choosing a Committee In how many ways can a committee of four be chosen from a group of ten if Barry and Harry refuse to serve together on the same committee?

84. Parking Committee A five-person committee consisting of students and teachers is being formed to study the issue of student parking privileges. Of those who have expressed an interest in serving on the committee, 12 are teachers and 14 are students. In how many ways can the committee be formed if at least one student and one teacher must be included?

85. Arranging Books In how many ways can five different mathematics books be placed on a shelf if the two algebra books are to be placed next to each other?

86. Arranging a Class Picture In how many ways can ten students be arranged in a row for a class picture if John and Jane want to stand next to each other and Mike and Molly also insist on standing next to each other?

87. Seating Arrangements In how many ways can four men and four women be seated in a row of eight seats for each of the following arrangements?

(a) The first seat is to be occupied by a man.
(b) The first and last seats are to be occupied by women.

88. Seating Arrangements In how many ways can four men and four women be seated in a row of eight seats for each of the following arrangements?

(a) The women are to be seated together.
(b) The men and women are to be seated alternately by gender.

89. Selecting Prizewinners From a group of 30 contestants, six are to be chosen as semifinalists, then two of those are chosen as finalists, and then the top prize is awarded to one of the finalists. In how many ways can these choices be made in sequence?

90. Choosing a Delegation Three delegates are to be chosen from a group of four lawyers, a priest, and three professors. In how many ways can the delegation be chosen if it must include at least one professor?

DISCOVERY ▪ DISCUSSION ▪ WRITING

91. Pairs of Initials Explain why in any group of 677 people, at least two people must have the same pair of initials.

92. Complementary Combinations Without performing any calculations, explain in words why the number of ways of choosing two objects from ten objects is the same as the number of ways of choosing eight objects from ten objects. In general, explain why

$$C(n, r) = C(n, n - r)$$

93. An Identity Involving Combinations Kevin has ten different marbles, and he wants to give three of them to Luke and two to Mark. In how many ways can he choose to do this? There are two ways of analyzing this problem: He could first pick three for Luke and then two for Mark, or he could first pick two for Mark and then three for Luke. Explain how these two viewpoints show that

$$C(10, 3) \cdot C(7, 2) = C(10, 2) \cdot C(8, 3)$$

In general, explain why

$$C(n, r) \cdot C(n - r, k) = C(n, k) \cdot C(n - k, r)$$

94. Why Is $\binom{n}{r}$ the Same as $C(n, r)$? This exercise explains why the binomial coefficients $\binom{n}{r}$ that appear in the expansion of $(x + y)^n$ are the same as $C(n, r)$, the number of ways of choosing r objects from n objects. First, note that expanding a binomial using only the Distributive Property gives

$$\begin{aligned}
(x + y)^2 &= (x + y)(x + y) \\
&= (x + y)x + (x + y)y \\
&= xx + xy + yx + yy \\
(x + y)^3 &= (x + y)(xx + xy + yx + yy) \\
&= xxx + xxy + xyx + xyy + yxx \\
&\quad + yxy + yyx + yyy
\end{aligned}$$

(a) Expand $(x + y)^5$ using only the Distributive Property.
(b) Write all the terms that represent x^2y^3. These are all the terms that contain two x's and three y's.
(c) Note that the two x's appear in all possible positions. Conclude that the number of terms that represent x^2y^3 is $C(5, 2)$.
(d) In general, explain why $\binom{n}{r}$ in the Binomial Theorem is the same as $C(n, r)$.

9.2 Probability

LEARNING OBJECTIVES **After completing this section, you will be able to:**

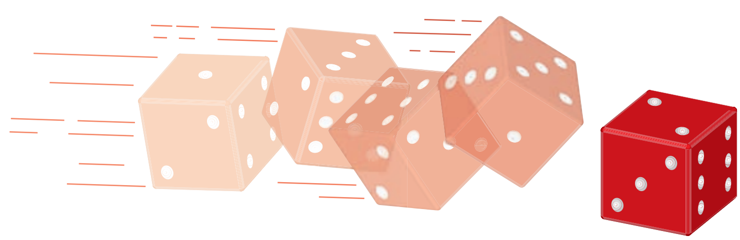
Find the probability of an event by counting ▶ Find the probability of the complement of an event ▶ Find the probability of the union of events ▶ Find conditional probabilities ▶ Find the probability of the intersection of events

In this section we study probability, which is the mathematical study of "chance."

▼ What Is Probability?

The mathematical **theory of probability** was first discussed in 1654 in a series of letters between Pascal (see page 604) and Fermat (see page 107). Their correspondence was prompted by a question raised by the experienced gambler the Chevalier de Méré. The Chevalier was interested in the equitable distribution of the stakes of an interrupted gambling game (see Problem 3, page 666).

Suppose we roll a die, and we're hoping to get a "two." Of course, it's impossible to predict what number will show up. But here's the key idea: If we roll the die many many times, a "two" will show up about one-sixth of the time. If you try this experiment you'll see that it actually works! We say that the *probability* (or chance) of getting a "two" is $\frac{1}{6}$.



To discuss probability, let's begin by defining some terms. An **experiment** is a process, such as tossing a coin, that gives definite results, called the **outcomes** of the experiment. The **sample space** of an experiment is the set of all possible outcomes. If we let H stand for heads and T for tails, then the sample space of the coin-tossing experiment is $S = \{H, T\}$. The table gives some experiments and their sample spaces.

Experiment	Sample space
Tossing a coin	$\{H, T\}$
Rolling a die	$\{1, 2, 3, 4, 5, 6\}$
Tossing a coin twice and observing the sequence of heads and tails	$\{HH, HT, TH, TT\}$
Picking a card from a deck and observing the suit	$\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$
Administering a drug to three patients and observing whether they recover (R) or not (N)	$\{RRR, RRN, RNR, RNN, NRR, NRN, NNR, NNN\}$

We will be concerned only with experiments for which all the outcomes are **equally likely**. For example, when we toss a perfectly balanced coin, heads and tails are equally likely outcomes in the sense that if this experiment is repeated many times, we expect that about as many heads as tails will show up.

In any given experiment we are often concerned with a particular set of outcomes. We might be interested in a die showing an even number or in picking an ace from a deck of cards. Any particular set of outcomes is a subset of the sample space. This leads to the following definition.

DEFINITION OF AN EVENT

If S is the sample space of an experiment, then an **event** E is any subset of the sample space.

EXAMPLE 1 | Events in a Sample Space

An experiment consists of tossing a coin three times and recording the results in order. List the outcomes in the sample space, then list the outcome in each event.

(a) The event E of getting "exactly two heads."

(b) The event F of getting "at least two heads."

(c) The event G of getting "no heads."

SOLUTION We write H for heads and T for tails. So the outcome HTH means that the three tosses resulted in Heads, Tails, Heads, in that order. The sample space is

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

(a) The event E is the subset of the sample space S that consists of all outcomes with exactly two heads. Thus

$$E = \{HHT, HTH, THH\}$$

(b) The event F is the subset of the sample space S that consists of all outcomes with at least two heads. Thus

$$F = \{HHH, HHT, HTH, THH\}$$

(c) The event G is the subset of the sample space S that consists of all outcomes with no heads. Thus

$$G = \{TTT\}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **5**

We are now ready to define the notion of probability. Intuitively, we know that rolling a die may result in any of six equally likely outcomes, so the chance of any particular outcome occurring is $\frac{1}{6}$. What is the chance of showing an even number? Of the six equally likely outcomes possible, three are even numbers. So it is reasonable to say that the chance of showing an even number is $\frac{3}{6} = \frac{1}{2}$. This reasoning is the intuitive basis for the following definition of probability.

DEFINITION OF PROBABILITY

Let S be the sample space of an experiment in which all outcomes are equally likely, and let E be an event. Then the **probability** of E, written $P(E)$, is

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of elements in } E}{\text{number of elements in } S}$$

Notice that $0 \le n(E) \le n(S)$, so the probability $P(E)$ of an event is a number between 0 and 1, that is,

$$0 \le P(E) \le 1$$

PERSI DIACONIS (b. 1945) is currently professor of statistics and mathematics at Stanford University in California. He was born in New York City into a musical family and studied violin until the age of 14. At that time he left home to become a magician. He was a magician (apprentice and master) for ten years. Magic is still his passion, and if there were a professorship for magic, he would certainly qualify for such a post! His interest in card tricks led him to a study of probability and statistics. He is now one of the leading statisticians in the world. With his unusual background he approaches mathematics with an undeniable flair. He says, "Statistics is the physics of numbers. Numbers seem to arise in the world in an orderly fashion. When we examine the world, the same regularities seem to appear again and again." Among his many original contributions to mathematics is a probabilistic study of the perfect card shuffle.

The closer the probability of an event is to 1, the more likely the event is to happen; the closer to 0, the less likely. If $P(E) = 1$, then E is called a **certain event**; if $P(E) = 0$, then E is called an **impossible event**.

EXAMPLE 2 | Finding the Probability of an Event

A coin is tossed three times, and the results are recorded in order. Find the probability of the following.

(a) The event E of getting "exactly two heads."

(b) The event F of getting "at least two heads."

(c) The event G of getting "no heads."

SOLUTION By the results of Example 1 the sample space S of this experiment contains 8 outcomes.

(a) The event E of getting "exactly two heads" contains 3 outcomes, so by the definition of probability,

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$$

(b) The event F of getting "at least two heads" has 4 outcomes, so

$$P(F) = \frac{n(F)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

(c) The event G of getting "no heads" has one outcome, so

$$P(G) = \frac{n(G)}{n(S)} = \frac{1}{8}$$

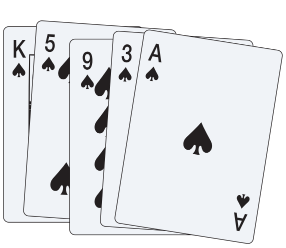
PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **7**

Calculating Probability by Counting

To find the probability of an event, we do not need to list all the elements in the sample space and the event. We need only the *number* of elements in these sets. The counting techniques that we learned in the preceding sections will be very useful here.

EXAMPLE 3 | Finding the Probability of an Event

A five-card poker hand is drawn from a standard deck of 52 cards. What is the probability that all five cards are spades?



SOLUTION The experiment here consists of choosing five cards from the deck, and the sample space S consists of all possible five-card hands. Thus the number of elements in the sample space is

$$n(S) = C(52, 5) = \frac{52!}{5!(52-5)!} = 2{,}598{,}960$$

The event E that we are interested in consists of choosing five spades. Since the deck contains only 13 spades, the number of ways of choosing five spades is

$$n(E) = C(13, 5) = \frac{13!}{5!(13-5)!} = 1287$$

Thus the probability of drawing five spades is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1287}{2{,}598{,}960} \approx 0.0005$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **15**

What does the answer to Example 3 tell us? Because $0.0005 = \frac{1}{2000}$, we conclude that if you play poker many, many times, on average you will be dealt a hand consisting of only spades about once in every 2000 hands.

EXAMPLE 4 | Finding the Probability of an Event

A bag contains 20 tennis balls, of which four are defective. If two balls are selected at random from the bag, what is the probability that both are defective?

SOLUTION The experiment consists of choosing two balls from 20, so the number of elements in the sample space S is $C(20, 2)$. Since there are four defective balls, the number of ways of picking two defective balls is $C(4, 2)$. Thus the probability of the event E of picking two defective balls is

$$P(E) = \frac{n(E)}{n(S)} = \frac{C(4, 2)}{C(20, 2)} = \frac{6}{190} \approx 0.032$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17**

The Complement of an Event

The **complement** of an event E is the set of outcomes in the sample space that are not in E. We denote the complement of E by E'.

By solving this equation for $P(E)$, we also have

$$P(E) = 1 - P(E')$$

PROBABILITY OF THE COMPLEMENT OF AN EVENT

Let S be the sample space of an experiment, and let E be an event. Then the probability of E', the complement of E, is

$$P(E') = 1 - P(E)$$

PROOF We calculate the probability of E' using the definition of probability and the fact that $n(E') = n(S) - n(E)$.

$$P(E') = \frac{n(E')}{n(S)} = \frac{n(S) - n(E)}{n(S)} = \frac{n(S)}{n(S)} - \frac{n(E)}{n(S)} = 1 - P(E)$$

This is a very useful result, since it is often difficult to calculate the probability of an event E but easy to find the probability of E'.

EXAMPLE 5 | Finding a Probability Using the Complement of an Event

An urn contains 10 red balls and 15 blue balls. Six balls are drawn at random from the urn. What is the probability that at least one ball is red?

SOLUTION Let E be the event that at least one red ball is drawn. It is tedious to count all the possible ways in which one or more of the balls drawn are red. So let's consider E', the complement of this event—namely, that none of the balls that are chosen is red. The number of ways of choosing 6 blue balls from the 15 blue balls is $C(15, 6)$; the number of ways of choosing 6 balls from the 25 balls is $C(25, 6)$. Thus

$$P(E') = \frac{n(E')}{n(S)} = \frac{C(15, 6)}{C(25, 6)} = \frac{5005}{177{,}100} = \frac{13}{460}$$

By the formula for the complement of an event we have

$$P(E) = 1 - P(E') = 1 - \frac{13}{460} \approx 0.97$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **19**

The Union of Events

If E and F are events, what is the probability that E or F occurs? The word *or* indicates that we want the probability of the union of these events, that is, $E \cup F$.

PROBABILITY OF THE UNION OF EVENTS

If E and F are events in a sample space S, then the probability of E *or* F is

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

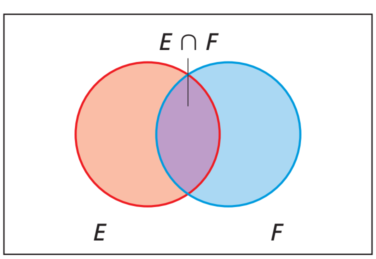


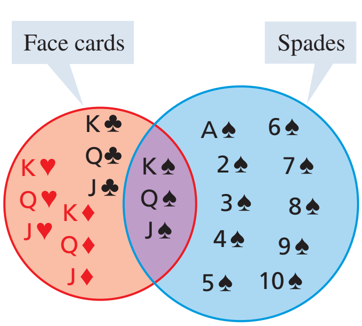
FIGURE 1

PROOF We need to find the number of elements in $E \cup F$. If we simply added the number of elements in E to the number of elements in F, we would be counting the elements in the overlap twice—once in E and once in F (see Figure 1). To get the correct total, we must subtract the number of elements in $E \cap F$. So $n(E \cup F) = n(E) + n(F) - n(E \cap F)$. Using the definition of probability we get

$$P(E \cup F) = \frac{n(E \cup F)}{n(S)} = \frac{n(E) + n(F) - n(E \cap F)}{n(S)} = P(E) + P(F) - P(E \cap F)$$

EXAMPLE 6 | Finding the Probability of the Union of Events

What is the probability that a card drawn at random from a standard 52-card deck is either a face card or a spade?



SOLUTION Let E denote the event "the card is a face card," and let F denote the event "the card is a spade." We want to find the probability of E or F, that is, $P(E \cup F)$.

There are 12 face cards and 13 spades in a 52-card deck, so

$$P(E) = \frac{12}{52} \qquad \text{and} \qquad P(F) = \frac{13}{52}$$

Since 3 cards are simultaneously face cards and spades, we have

$$P(E \cap F) = \frac{3}{52}$$

Now, by the formula for the probability of the union of two events we have

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$
$$= \frac{12}{52} + \frac{13}{52} - \frac{3}{52} = \frac{11}{26}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **21**

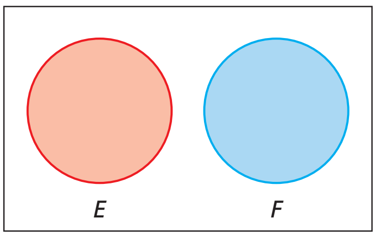


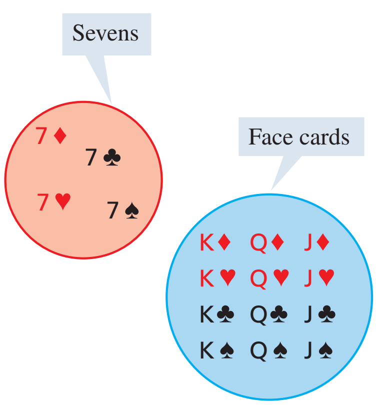
FIGURE 2

Two events that have no outcome in common are said to be **mutually exclusive** (see Figure 2). In other words, the events E and F are mutually exclusive if $E \cap F = \varnothing$. So if the events E and F are mutually exclusive, then $P(E \cap F) = 0$. The following result now follows from the formula for the union of two events.

PROBABILITY OF THE UNION OF MUTUALLY EXCLUSIVE EVENTS

If E and F are mutually exclusive events, then

$$P(E \cup F) = P(E) + P(F)$$



EXAMPLE 7 | Finding the Probability of the Union of Mutually Exclusive Events

What is the probability that a card drawn at random from a standard 52-card deck is either a seven or a face card?

SOLUTION Let E denote the event "the card is a seven," and let F denote the event "the card is a face card." These events are mutually exclusive because a card cannot be at the same time a seven and a face card. By the above formula we have

$$P(E \cup F) = P(E) + P(F) = \frac{4}{52} + \frac{12}{52} = \frac{4}{13}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **23** AND **25**

▼ Conditional Probability and the Intersection of Events

When we calculate probabilities, there sometimes is additional information that may alter the probability of an event. For example, suppose a person is chosen at random. What is the probability that the person has long hair? How does the probability change if we are given the additional information that the person chosen is a woman? In general, the probability of an event E given that another event F has occurred is expressed by writing

$$P(E \mid F) = \text{The probability of } E \text{ given } F$$

For example, suppose a die is rolled. Let E be the event of "getting a two," and let F be the event of "getting an even number." Then

$$P(E \mid F) = P(\text{The number is two given that the number is even})$$

Since we know that the number is even, the possible outcomes are the three numbers 2, 4, and 6. So in this case the probability of a "two" is $P(E \mid F) = \frac{1}{3}$.

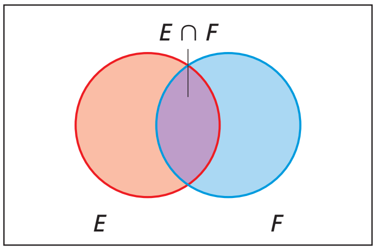


FIGURE 3

In general, if we know that F has occurred, then F serves as the sample space (see Figure 3). So $P(E \mid F)$ is determined by the number of outcomes in E that are also in F, that is, the number of outcomes in $E \cap F$.

CONDITIONAL PROBABILITY

Let E and F be events in a sample space S. The **conditional probability of E given that F occurs** is

$$P(E \mid F) = \frac{n(E \cap F)}{n(F)}$$

EXAMPLE 8 | Finding Conditional Probability

A mathematics class consists of 30 students; 12 of them study French, 8 study German, 3 study both of these languages, and the rest do not study a foreign language. If a student is chosen at random from this class, find the probability of each of the following events.

(a) The student studies French.

(b) The student studies French, given that he or she studies German.

(c) The student studies French, given that he or she studies a foreign language.

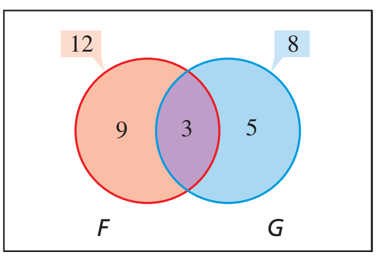


FIGURE 4

SOLUTION Let F denote the event "the student studies French," let G be the event "the student studies German," and let L be the event "the student studies a foreign language." It is helpful to organize the information in a Venn diagram, as in Figure 4.

(a) There are 30 students in the class, 12 of whom study French, so

$$P(F) = \frac{12}{30} = \frac{2}{5}$$

(b) We are asked to find $P(F \mid G)$, the probability that a student studies French given that the student studies German. Since eight students study German and three of these study French, it is clear that the required conditional probability is $\frac{3}{8}$. The formula for conditional probability confirms this:

$$P(F \mid G) = \frac{n(F \cap G)}{n(G)} = \frac{3}{8}$$

(c) From the Venn diagram in Figure 4 we see that the number of students who study a foreign language is $9 + 3 + 5 = 17$. Since 12 of these study French, we have

$$P(F \mid L) = \frac{n(F \cap L)}{n(L)} = \frac{12}{17}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **27** AND **29**

If we start with the expression for conditional probability and then divide numerator and denominator by $n(S)$, we get

$$P(E \mid F) = \frac{n(E \cap F)}{n(F)} = \frac{\dfrac{n(E \cap F)}{n(S)}}{\dfrac{n(F)}{n(S)}} = \frac{P(E \cap F)}{P(F)}$$

Multiplying both sides by $P(F)$ gives the following formula.

PROBABILITY OF THE INTERSECTION OF EVENTS

If E and F are events in a sample space S, then the probability of E *and* F is

$$P(E \cap F) = P(E)P(F \mid E)$$

EXAMPLE 9 | Finding the Probability of the Intersection of Events

Two cards are drawn, without replacement, from a 52-card deck. Find the probability of the following events.

(a) The first card drawn is an ace and the second is a king.

(b) The first card drawn is an ace and the second is also an ace.

SOLUTION Let E be the event "the first card is an ace," and let F be the event "the second card is a king."

(a) We are asked to find the probability of E *and* F, that is, $P(E \cap F)$. Now, $P(E) = \frac{4}{52}$. After an ace is drawn, 51 cards remain in the deck; of these, 4 are kings, so $P(F \mid E) = \frac{4}{51}$. By the above formula we have

$$P(E \cap F) = P(E)P(F \mid E) = \frac{4}{52} \times \frac{4}{51} \approx 0.0060$$

(b) Let E be the event "the first card is an ace," and let H be the event "the second card is an ace." The probability that the first card drawn is an ace is $P(E) = \frac{4}{52}$. After an ace is drawn, 51 cards remain; of these, 3 are aces, so $P(H \mid E) = \frac{3}{51}$. By the above formula we have

$$P(E \cap H) = P(E)P(H \mid E) = \frac{4}{52} \times \frac{3}{51} \approx 0.0045$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **33**

When the occurrence of one event does not affect the probability of the occurrence of another event, we say that the events are **independent**. This means that the events E and F are independent if $P(E \mid F) = P(E)$ and $P(F \mid E) = P(F)$. For instance, if a fair coin is tossed, the probability of showing heads on the second toss is $\frac{1}{2}$, regardless of what was obtained on the first toss. So any two tosses of a coin are independent.

PROBABILITY OF THE INTERSECTION OF INDEPENDENT EVENTS

If E and F are independent events in a sample space S, then the probability of E *and* F is

$$P(E \cap F) = P(E)P(F)$$

EXAMPLE 10 | Finding the Probability of Independent Events

A jar contains five red balls and four black balls. A ball is drawn at random from the jar and then replaced; then another ball is picked. What is the probability that both balls are red?

SOLUTION Let E be the event "the first ball drawn is red," and let F be the event "the second ball drawn is red." Since we replace the first ball before drawing the second, the events E and F are independent. Now, the probability that the first ball is red is $\frac{5}{9}$. The probability that the second is red is also $\frac{5}{9}$. Thus the probability that both balls are red is

$$P(E \cap F) = P(E)P(F) = \frac{5}{9} \times \frac{5}{9} \approx 0.31$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **37**

9.2 EXERCISES

CONCEPTS

1. The set of all possible outcomes of an experiment is called the ________ ________. A subset of the sample space is called an ________. The sample space for the experiment of tossing two coins is $S = \{HH, \_\_\_, \_\_\_, \_\_\_\}$. The event "getting at least one head" is $E = \{HH, \_\_\_, \_\_\_\}$. The probability of getting at least one head is

$$P(E) = \frac{n(\_\_\_)}{n(\_\_\_)} = \_\_\_\_.$$

2. Let E and F be events in a sample space S.

(a) The probability of E *or* F occurring is $P(E \cup F) =$ ____________.

(b) If the events E and F have no outcome in common (that is, the intersection of E and F is empty), then the events are called ________ ________. So in drawing a card from a deck, the event E, "getting a heart," and the event F, "getting a spade," are ________ ________.

(c) If E and F are mutually exclusive, then the probability of E *or* F is $P(E \cup F) =$ ____________.

3. The conditional probability of E given that F occurs is $P(E \mid F) =$ ________. So in rolling a die the conditional probability of the event E, "getting a six," given that the event F, "getting an even number," has occurred is $P(E \mid F) =$ ________.

4. Let E and F be events in a sample space S.

(a) The probability of E *and* F occurring is $P(E \cap F) =$ ____________.

(b) If the occurrence of E does not affect the probability of the occurrence F, then the events are called ________. So in tossing a coin twice, the event E, "getting heads on the first toss," and the event F, "getting heads on the second toss," are ________.

(c) If E and F are independent events, then the probability of E *and* F is $P(E \cap F) =$ ________.

SKILLS

5. An experiment consists of rolling a die. List the elements in the following sets.

(a) The sample space

(b) The event "getting an even number"

(c) The event "getting a number greater than 4"

6. An experiment consists of tossing a coin and drawing a card from a deck.

(a) How many elements does the sample space have?

(b) List the elements in the event "getting heads and an ace."

(c) List the elements in the event "getting tails and a face card."

(d) List the elements in the event "getting heads and a spade."

Exercises 7–20 are about finding probability by counting.

7. An experiment consists of tossing a coin twice.

(a) Find the sample space.

(b) Find the probability of getting heads exactly two times.

(c) Find the probability of getting heads at least one time.

(d) Find the probability of getting heads exactly one time.

8. An experiment consists of tossing a coin and rolling a die.

(a) Find the sample space.

(b) Find the probability of getting heads and an even number.

(c) Find the probability of getting heads and a number greater than 4.

(d) Find the probability of getting tails and an odd number.

9–10 ■ A die is rolled. Find the probability of the given event.

9. (a) The number showing is a six.

(b) The number showing is an even number.

(c) The number showing is greater than five.

10. (a) The number showing is a two or a three.

(b) The number showing is an odd number.

(c) The number showing is a number divisible by 3.

11–12 ■ A card is drawn randomly from a standard 52-card deck. Find the probability of the given event.

11. **(a)** The card drawn is a king.
(b) The card drawn is a face card.
(c) The card drawn is not a face card.

12. **(a)** The card drawn is a heart.
(b) The card drawn is either a heart or a spade.
(c) The card drawn is a heart, a diamond, or a spade.

13–14 ■ A ball is drawn randomly from a jar that contains five red balls, two white balls, and one yellow ball. Find the probability of the given event.

13. **(a)** A red ball is drawn.
(b) The ball drawn is not yellow.
(c) A black ball is drawn.

14. **(a)** Neither a white nor yellow ball is drawn.
(b) A red, white, or yellow ball is drawn.
(c) The ball that is drawn is not white.

15. A poker hand, consisting of five cards, is dealt from a standard deck of 52 cards. Find the probability that the hand contains the cards described.
(a) Five hearts
(b) Five cards of the same suit
(c) Five face cards
(d) An ace, king, queen, jack, and a ten, all of the same suit (royal flush)

16. Three CDs are picked at random from a collection of 12 CDs of which four are defective. Find the probability of the following.
(a) All three CDs are defective.
(b) All three CDs are functioning properly.

17. Two balls are picked at random from a jar that contains three red and five white balls. Find the probability of the following events.
(a) Both balls are red.
(b) Both balls are white.

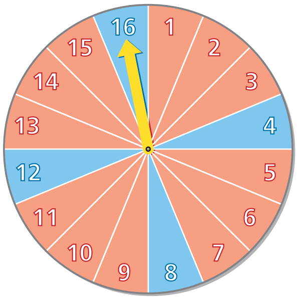
18. A letter is chosen at random from the word *EXTRATERRESTRIAL.* Find the probability of the given event.
(a) The letter T is chosen.
(b) The letter chosen is a vowel.
(c) The letter chosen is a consonant.

19. A five-card poker hand is drawn from a standard 52-card deck. Find the probability of the following events.
(a) At least one card is a spade.
(b) At least one card is a face card.

20. A pair of dice is rolled, and the numbers showing are observed.
(a) List the sample space of this experiment.
(b) Find the probability of getting a sum of 7.
(c) Find the probability of getting a sum of 9.
(d) Find the probability that the two dice show doubles (the same number).
(e) Find the probability that the two dice show different numbers.
(f) Find the probability of getting a sum of 9 or higher.

Exercises 21–26 are about the probability of the union of events.

21–22 ■ Refer to the spinner shown in the figure. Find the probability of the given event.



21. **(a)** The spinner stops on red.
(b) The spinner stops on an even number.
(c) The spinner stops on red or an even number.

22. **(a)** The spinner stops on blue.
(b) The spinner stops on an odd number.
(c) The spinner stops on blue or an odd number.

23–24 ■ A die is rolled, and the number showing is observed. Determine whether the events E and F are mutually exclusive. Then find the probability of the event $E \cup F$.

23. **(a)** E: The number is even.
F: The number is odd.
(b) E: The number is even.
F: The number is greater than 4.

24. **(a)** E: The number is greater than 3.
F: The number is less than 5.
(b) E: The number is divisible by 3.
F: The number is less than 3.

25–26 ■ A card is drawn at random from a standard 52-card deck. Determine whether the events E and F are mutually exclusive. Then find the probability of the event $E \cup F$.

25. **(a)** E: The card is a face card.
F: The card is a spade.
(b) E: The card is a heart.
F: The card is a spade.

26. **(a)** E: The card is a club.
F: The card is a king.
(b) E: The card is an ace.
F: The card is a spade.

Exercises 27–32 are about conditional probability.

27. A die is rolled. Find the given conditional probability.
(a) A "five" shows, given that the number showing is greater than 3.
(b) A "three" shows, given that the number showing is odd.

28. A card is drawn from a deck. Find the following conditional probability.
(a) The card is a queen, given that it is a face card.
(b) The card is a king, given that it is a spade.
(c) The card is a spade, given that it is a king.

29–30 ■ Refer to the spinner in Exercises 21–22.

29. Find the probability that the spinner has stopped on an even number, given that it has stopped on red.

30. Find the probability that the spinner has stopped on a number divisible by 3, given that it has stopped on blue.

31–32 ■ A jar contains five red balls numbered 1 to 5, and seven green balls numbered 1 to 7.

31. A ball is drawn at random from the jar. Find the following conditional probabilities.
(a) The ball is red, given that it is numbered 3.
(b) The ball is green, given that is numbered 7.
(c) The ball is red, given that it has an even number.
(d) The ball has an even number, given that it is red.

32. Two balls are drawn at random from the jar. Find the following conditional probabilities.
(a) The second ball drawn is red, given that the first is red.
(b) The second ball drawn is red, given that the first is green.
(c) The second ball drawn is even-numbered, given that the first is odd-numbered.
(d) The second ball drawn is even-numbered, given that the first is even-numbered.

Exercises 33–40 are about the probability of the intersection of events.

33. A jar contains seven black balls and three white balls. Two balls are drawn, without replacement, from the jar. Find the probability of the following events.
(a) The first ball drawn is black, and the second is white.
(b) The first ball drawn is black, and the second is black.

34. A drawer contains an unorganized collection of 18 socks. Three pairs are red, two pairs are white, and four pairs are black.
(a) If one sock is drawn at random from the drawer, what is the probability that it is red?
(b) Once a sock is drawn and discovered to be red, what is the probability of drawing another red sock to make a matching pair?
(c) If two socks are drawn from the drawer at the same time, what is the probability that both are red?

35. Two cards are drawn from a deck without replacement. Find the probability of the following events.
(a) The first is an ace and the second a king?
(b) Both cards are aces?

36. A die is rolled twice. Let E and F be the following events:

E: The first roll shows a "six."
F: The second roll shows a "six."

(a) Are the events E and F independent?
(b) Find the probability of showing a "six" on both rolls.

37. A die is rolled twice. What is the probability of getting a "one" on the first roll and an even number on the second roll?

38. A coin is tossed and a die is rolled.
(a) Are the events "tails" and "even number" independent?
(b) Find the probability of getting a tail and an even number.

39–40 ■ Spinners A and B shown in the figure are spun at the same time.

39. (a) Are the events "spinner A stops on red" and "spinner B stops on yellow" independent?
(b) Find the probability that spinner A stops on red and spinner B stops on yellow

40. (a) Find the probability that both spinners stop on purple.
(b) Find the probability that both spinners stop on blue.

APPLICATIONS

41. Four Siblings A couple intends to have four children. Assume that having a boy and having a girl are equally likely events.
(a) List the sample space of this experiment.
(b) Find the probability that the couple will have only boys.
(c) Find the probability that the couple will have two boys and two girls.
(d) Find the probability that the couple will have four children of the same gender
(e) Find the probability that the couple will have at least two girls.

42. Bridge Hands What is the probability that a 13-card bridge hand consists of all cards from the same suit?

43. Roulette An American roulette wheel has 38 slots; two slots are numbered 0 and 00, and the remaining slots are numbered from 1 to 36. Find the probability that the ball lands in an odd-numbered slot.

44. Making Words A toddler has wooden blocks showing the letters *C, E, F, H, N,* and *R*. Find the probability that the child arranges the letters in the indicated order.
(a) In the order *FRENCH*
(b) In alphabetical order

45. Lottery In the 6/49 lottery game, a player selects six numbers from 1 to 49. What is the probability of picking the six winning numbers?

46. An Unlikely Event The president of a large company selects six employees to receive a special bonus. He claims that the six employees are chosen randomly from among the 30 employees, of whom 19 are women and 11 are men. What is the probability that no woman is chosen?

47. Guessing on a Test An exam has ten true-false questions. A student who has not studied answers all ten questions by just guessing. Find the probability that the student correctly answers all ten questions.

48. Quality Control To control the quality of their product, the Bright-Light Company inspects three light bulbs out of each batch of ten bulbs manufactured. If a defective bulb is found, the batch is discarded. Suppose a batch contains two defective bulbs. What is the probability that the batch will be discarded?

49. Monkeys Typing Shakespeare An often-quoted example of an event of extremely low probability is that a monkey types Shakespeare's entire play *Hamlet* by randomly striking keys on a typewriter. Assume that the typewriter has 48 keys (including the space bar) and that the monkey is equally likely to hit any key.

(a) Find the probability that such a monkey will actually correctly type just the title of the play as his first word.

(b) What is the probability that the monkey will type the phrase "To be or not to be" as his first words?

50. Making Words A monkey is trained to arrange wooden blocks in a straight line. He is then given six blocks showing the letters *A, E, H, L, M, T.*

(a) What is the probability that he will arrange them to spell the word *HAMLET*?

(b) What is the probability that he will arrange them to spell the word *HAMLET* three consecutive times?

51. Making Words A toddler has eight wooden blocks showing the letters *A, E, I, G, L, N, T,* and *R.* What is the probability that the child will arrange the letters to spell one of the words *TRIANGLE* or *INTEGRAL*?

52. Horse Race Eight horses are entered in a race. You randomly predict a particular order for the horses to complete the race. What is the probability that your prediction is correct?

53. Genetics Many genetic traits are controlled by two genes, one dominant and one recessive. In Gregor Mendel's original experiments with peas, the genes controlling the height of the plant are denoted by T (tall) and t (short). The gene T is dominant, so a plant with the genotype (genetic makeup) TT or Tt is tall, whereas one with genotype tt is short. By a statistical analysis of the offspring in his experiments, Mendel concluded that offspring inherit one gene from each parent and that each possible combination of the two genes is equally likely. If each parent has the genotype Tt, then the following chart gives the possible genotypes of the offspring:

		Parent 2 T	Parent 2 t
Parent 1	T	TT	Tt
	t	Tt	tt

Find the probability that a given offspring of these parents will be

(a) tall **(b)** short

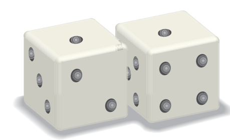
54. Genetics Refer to Exercise 53. Make a chart of the possible genotypes of the offspring if one parent has genotype Tt and the other has tt. Find the probability that a given offspring will be

(a) tall **(b)** short

55. Roulette An American roulette wheel has 38 slots. Two of the slots are numbered 0 and 00, and the rest are numbered from 1 to 36. A player places a bet on a number between 1 and 36 and wins if a ball thrown into the spinning roulette wheel lands in the slot with the same number. Find the probability of winning on two consecutive spins of the roulette wheel.

56. Choosing a Committee A committee of five is chosen randomly from a group of six males and eight females. What is the probability that the committee includes either all males or all females?

57. Snake Eyes What is the probability of rolling snake eyes ("double ones") three times in a row?



58. Lottery In the 6/49 lottery game a player selects six numbers from 1 to 49. What is the probability of selecting at least five of the six winning numbers?

59. Marbles in a Jar A jar contains six red marbles numbered 1 to 6 and ten blue marbles numbered 1 to 10. A marble is drawn at random from the jar. Find the probability that the given event occurs.

(a) The marble is red.

(b) The marble is odd-numbered.

(c) The marble is red or odd-numbered.

(d) The marble is blue or even-numbered.

60. Lottery In the 6/49 lottery game, a player selects six numbers from 1 to 49 and wins if he or she selects the winning six numbers. What is the probability of winning the lottery two times in a row?

61. Balls in a Jar Jar A contains three red balls and four white balls. Jar B contains five red balls and two white balls. Which one of the following ways of randomly selecting balls gives the greatest probability of drawing two red balls?

(i) Draw two balls from jar B.

(ii) Draw one ball from each jar.

(iii) Put all the balls in one jar, and then draw two balls.

62. Slot Machine A slot machine has three wheels. Each wheel has 11 positions: a bar and the digits 0, 1, 2, . . . , 9. When the handle is pulled, the three wheels spin independently before coming to rest. Find the probability that the wheels stop on the following positions.

(a) Three bars

(b) The same number on each wheel
(c) At least one bar

63. **Combination Lock** A student has locked her locker with a combination lock, showing numbers from 1 to 40, but she has forgotten the three-number combination that opens the lock. She remembers that all three numbers in the combination are different. To open the lock, she decides to try all possible combinations. If she can try ten different combinations every minute, what is the probability that she will open the lock within one hour?

64. **Committee Membership** A mathematics department consists of ten men and eight women. Six mathematics faculty members are to be selected at random for the curriculum committee.
(a) What is the probability that two women and four men are selected?
(b) What is the probability that two or fewer women are selected?
(c) What is the probability that more than two women are selected?

65. **Class Photo** Twenty students are arranged randomly in a row for a class picture. Paul wants to stand next to Phyllis. Find the probability that he gets his wish.

66. **Making Words** A monkey is trained to arrange wooden blocks in a row. The monkey is then given 6 blocks showing the letters *B, B, B, E, L, U*. What is the probability that the monkey will arrange the blocks to spell the word *BUBBLE*?

67. **Making Words** A monkey is trained to arrange wooden blocks in a row. The monkey is then given 11 blocks showing the letters *A, B, B, I, I, L, O, P, R, T, Y*. What is the probability that the monkey will arrange the blocks to spell the word *PROBABILITY*?

DISCOVERY ▪ DISCUSSION ▪ WRITING

68. **Oldest Son** A family with two children is randomly selected. Assume that the events of having a boy or a girl are equally likely. Find the following probabilities.
(a) The family has two boys given that the oldest child is a boy.
(b) The family has two boys given that one of the children is a boy.

DISCOVERY PROJECT **Small Samples, Big Results**

In this project we perform several experiments that show how we can obtain information about a big population from a small sample. You can find the project at the book companion website: **www.stewartmath.com**

9.3 Binomial Probability

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find binomial probabilities ▶ Make a table of a probability distribution

In this section we study a special kind of probability that plays a crucial role in modeling many real-world situations.

▼ Binomial Probability

Calculating the probability of independent events is studied on page 645.

A coin is weighted so that the probability of heads is 0.6. What is the probability of getting exactly two heads in five tosses of this coin? Since the tosses are independent, the probability of getting two heads followed by three tails is

$$\underbrace{0.6}_{\text{Heads}} \times \underbrace{0.6}_{\text{Heads}} \times \underbrace{0.4}_{\text{Tails}} \times \underbrace{0.4}_{\text{Tails}} \times \underbrace{0.4}_{\text{Tails}} = (0.6)^2(0.4)^3$$

But this is not the only way we can get exactly two heads. The two heads can occur, for example, on the second toss and the last toss. In this case the probability is

$$\underbrace{0.4}_{\text{Tails}} \times \underbrace{0.6}_{\text{Heads}} \times \underbrace{0.4}_{\text{Tails}} \times \underbrace{0.4}_{\text{Tails}} \times \underbrace{0.6}_{\text{Heads}} = (0.6)^2(0.4)^3$$

In fact, the two heads could occur on any two of the five tosses. Thus there are $C(5, 2)$ ways in which this can happen, each with probability $(0.6)^2(0.4)^3$. It follows that

$$P(\text{exactly 2 heads in 5 tosses}) = C(5, 2)(0.6)^2(0.4)^3 \approx 0.023$$

The probability that we have just calculated is an example of a binomial probability. In general, a **binomial experiment** is one in which there are two outcomes, which are called "success" and "failure." In the coin-tossing experiment described above, "success" is getting "heads," and "failure" is getting "tails." The following box tells us how to calculate the probabilities associated with binomial experiments when we perform them many times.

The name "binomial probability" is appropriate because $C(n, r)$ is the same as the binomial coefficient $\binom{n}{r}$ (see Exercise 94, page 637).

BINOMIAL PROBABILITY

An experiment has two possible outcomes called "success" and "failure," with $P(\text{success}) = p$ and $P(\text{failure}) = 1 - p$. The probability of getting exactly r successes in n independent trials of the experiment is

$$P(r \text{ successes in } n \text{ trials}) = C(n, r)p^r(1 - p)^{n-r}$$

EXAMPLE 1 | Binomial Probability

A fair die is rolled 10 times. Find the probability of each event.

(a) Exactly 2 sixes.

(b) At most 1 six.

(c) At least 2 sixes.

SOLUTION Let's call "getting a six" success and "not getting a six" failure. So $P(\text{success}) = \frac{1}{6}$ and $P(\text{failure}) = \frac{5}{6}$. Since each roll of the die is independent of the other rolls, we can use the formula for binomial probability with $n = 10$ and $p = \frac{1}{6}$.

(a) $P(\text{exactly 2 sixes}) = C(10, 2)(\frac{1}{6})^2(\frac{5}{6})^8 \approx 0.29$

(b) The statement "at most 1 six" means 0 sixes or 1 six. So

$$\begin{aligned}
&P(\text{at most one six}) \\
&\quad = P(0 \text{ sixes or } 1 \text{ six}) && \text{Meaning of "at most"} \\
&\quad = P(0 \text{ sixes}) + P(1 \text{ six}) && P(A \text{ or } B) = P(A) + P(B) \\
&\quad = C(10, 0)\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{10} + C(10, 1)\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^9 && \text{Binomial probability} \\
&\quad \approx 0.1615 + 0.3230 && \text{Calculator} \\
&\quad \approx 0.4845 && \text{Calculator}
\end{aligned}$$

(c) The statement "at least two sixes" means two or more sixes. Instead of adding the probabilities of getting 2, 3, 4, 5, 6, 7, 8, 9, or 10 sixes (which is a lot of work), it's easier to find the probability of the complement of this event. The complement of the event "two or more sixes" is "0 or 1 six." So

$$\begin{aligned}
P(\text{two or more sixes}) &= 1 - P(0 \text{ or } 1 \text{ six}) && P(E) = 1 - P(E') \\
&= 1 - 0.4845 && \text{From part (b)} \\
&= 0.5155 && \text{Calculator}
\end{aligned}$$

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISES **3** AND **21**

EXAMPLE 2 | Binomial Probability

Patients infected with a certain virus have a 40% chance of surviving. There are 10 patients in a hospital who have acquired this virus. Find the probability that 7 or more of the patients survive.

SOLUTION Let's call the event "patient survives" success and the event "patient dies" failure. We are given that the probability of success is $p = 0.4$, so the probability of failure is $1 - p = 1 - 0.4 = 0.6$. We need to calculate the probability of 7, 8, 9, or 10 successes in 10 trials:

$$\begin{aligned} P(7 \text{ out of } 10 \text{ recover}) &= C(10, 7)(0.4)^7(0.6)^3 \approx 0.04247 \\ P(8 \text{ out of } 10 \text{ recover}) &= C(10, 8)(0.4)^8(0.6)^2 \approx 0.01062 \\ P(9 \text{ out of } 10 \text{ recover}) &= C(10, 9)(0.4)^9(0.6)^1 \approx 0.00157 \\ P(10 \text{ out of } 10 \text{ recover}) &= C(10, 10)(0.4)^{10}(0.6)^0 \approx 0.00010 \end{aligned}$$

Adding the probabilities, we find that

$$P(7 \text{ or more recover}) \approx 0.05476$$

There is about a 1 in 20 chance that 7 or more patients recover.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **35**

The Binomial Distribution

We can describe how the probabilities of an experiment are "distributed" among all the outcomes of an experiment by making a table of values. The function that assigns to each outcome its corresponding probability is called a **probability distribution**. A bar graph of a probability distribution in which the width of each bar is 1 is called a **probability histogram**. The next example illustrates these concepts.

EXAMPLE 3 | Probability Distributions

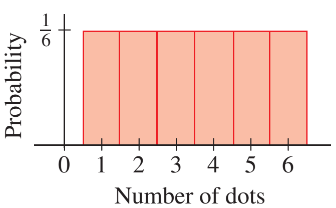
Make a table of the probability distribution for the experiment of rolling a fair die and observing the number of dots. Draw a histogram of the distribution.

SOLUTION When rolling a fair die each face has probability $1/6$ of showing. The probability distribution is shown in the following table. To draw a histogram, we draw bars of width 1 and height $\frac{1}{6}$ corresponding to each outcome.

Probability Distribution

Outcome (dots)	Probability
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

Probability Histogram



PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **15**

A probability distribution in which all outcomes have the same probability is called a **uniform distribution**. The rolling-a-die experiment in Example 3 is a uniform distribution. The probability distribution of a binomial experiment is called a **binomial distribution**.

EXAMPLE 4 | A Binomial Distribution

A fair coin is tossed eight times, and the number of heads is observed. Make a table of the probability distribution, and draw a histogram. What is the number of heads that is most likely to show up?

SOLUTION This is a binomial experiment with $n = 8$ and $p = \frac{1}{2}$, so $1 - p = \frac{1}{2}$ as well. We need to calculate the probability of getting 0 heads, 1 head, 2 heads, 3 heads, and so on. For example, to calculate the probability of 3 heads, we have

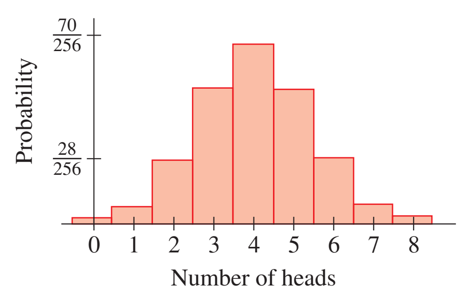
$$P(3\text{ heads}) = C(8, 3)\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^5 = \frac{28}{256}$$

The other entries in the following table are calculated similarly. We draw the histogram by making a bar for each outcome with width 1 and height equal to the corresponding probability.

Probability Distribution

Outcome (heads)	Probability
0	$\frac{1}{256}$
1	$\frac{8}{256}$
2	$\frac{28}{256}$
3	$\frac{56}{256}$
4	$\frac{70}{256}$
5	$\frac{56}{256}$
6	$\frac{28}{256}$
7	$\frac{8}{256}$
8	$\frac{1}{256}$

Probability Histogram



PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **17** ■

Notice that the sum of the probabilities in a probability distribution is 1, because the sum is the probability of the occurrence of *any* outcome in the sample space (this is the certain event).

9.3 EXERCISES

CONCEPTS

1. A binomial experiment is one in which there are exactly ________ outcomes. One outcome is called ________, and the other is called ________.

2. If a binomial experiment has probability p of success, then the probability of failure is ________. The probability of getting exactly r successes in n trials of this experiment is $C(\_\_\_, \_\_\_)p^{\square}(1 - p)^{\square}$.

SKILLS

3–14 ■ Five independent trials of a binomial experiment with probability of success $p = 0.7$ are performed. Find the probability of each event.

3. Exactly two successes

4. Exactly three successes

5. No successes

6. All successes

7. Exactly one success

8. Exactly one failure

9. At least four successes

10. At least three successes

11. At most one failure

12. At most two failures

13. At least two successes

14. At most three failures

15–16 ■ An experiment is described. **(a)** Complete the table of the probability distribution. **(b)** Draw a probability histogram.

15. A jar contains five balls numbered 1 to 5. A ball is drawn at random, and the number of the ball is observed.

Outcome	Probability
1	0.2
2	
3	
4	
5	

16. A jar contains five balls numbered 1, three balls numbered 2, one ball numbered 3, and one ball numbered 4. A ball is drawn at random and the number of the ball is observed.

Outcome	Probability
1	0.5
2	
3	
4	
5	

17–20 ■ A binomial experiment with probability of success p is performed n times. **(a)** Make a table of the probability distribution. **(b)** Draw a probability histogram.

17. $n = 4, \quad p = 0.5$

18. $n = 5, \quad p = 0.4$

19. $n = 7, \quad p = 0.2$

20. $n = 6, \quad p = 0.9$

APPLICATIONS

21. Rolling Dice Six dice are rolled. Find the probability that two of them show a four.

22. Archery An archer hits his target 80% of the time. If he shoots seven arrows, what is the probability of each event?
(a) He never hits the target.
(b) He hits the target each time.
(c) He hits the target more than once.
(d) He hits the target at least five times.

23. Television Ratings According to a ratings survey, 40% of the households in a certain city tune in to the local evening TV news. If ten households are visited at random, what is the probability that four of them will have their television tuned to the local news?

24. Spread of Disease Health authorities estimate that 10% of the raccoons in a certain rural county are carriers of rabies. A dog is bitten by four different raccoons in this county. What is the probability that he was bitten by at least one rabies carrier?

25. Blood Type About 45% of the populations of the United States and Canada have Type O blood.
(a) If a random sample of ten people is selected, what is the probability that exactly five have Type O blood?
(b) What is the probability that at least three of the random sample of ten have Type O blood?

26. Handedness A psychologist needs 12 left-handed subjects for an experiment, and she interviews 15 potential subjects. About 10% of the population is left-handed.
(a) What is the probability that exactly 12 of the potential subjects are left-handed?
(b) What is the probability that 12 or more are left-handed?

27. Germination Rates A certain brand of tomato seeds has a 0.75 probability of germinating. To increase the chance that at least one tomato plant per seed hill germinates, a gardener plants four seeds in each hill.
(a) What is the probability that at least one seed germinates in a given hill?
(b) What is the probability that two or more seeds will germinate in a given hill?
(c) What is the probability that all four seeds germinate in a given hill?

28. Genders of Children Assume that for any given live human birth, the chances that the child is a boy or a girl are equally likely.
(a) What is the probability that in a family of five children a majority are boys?
(b) What is the probability that in a family of seven children a majority are girls?

29. Genders of Children The ratio of male to female births is in fact not exactly one to one. The probability that a newborn turns out to be a male is about 0.52. A family has ten children.
(a) What is the probability that all ten children are boys?
(b) What is the probability all are girls?
(c) What is the probability that five are girls and five are boys?

30. Education Level In a certain county 20% of the population has a college degree. A jury consisting of 12 people is selected at random from this county.
(a) What is the probability that exactly two of the jurors have a college degree?
(b) What is the probability that three or more of the jurors have a college degree?

31. Defective Light Bulbs The DimBulb Lighting Company manufactures light bulbs for appliances such as ovens and refrigerators. Typically, 0.5% of their bulbs are defective. From a

crate with 100 bulbs, three are tested. Find the probability that the given event occurs.
(a) All three bulbs are defective.
(b) One or more bulbs is defective.

32. Quality Control An assembly line that manufactures fuses for automotive use is checked every hour to ensure the quality of the finished product. Ten fuses are selected randomly, and if any one of the ten is found to be defective, the process is halted and the machines are recalibrated. Suppose that at a certain time 5% of the fuses being produced are actually defective. What is the probability that the assembly line is halted at that hour's quality check?

33. Sick Leave The probability that a given worker at Dyno Nutrition will call in sick on a Monday is 0.04. The packaging department has eight workers. What is the probability that two or more packaging workers will call in sick next Monday?

34. Political Surveys In a certain county, 60% of the voters are in favor of an upcoming school bond initiative. If five voters are interviewed at random, what is the probability that exactly three of them will favor the initiative?

35. Pharmaceuticals A drug that is used to prevent motion sickness is found to be effective about 75% of the time. Six friends, all prone to seasickness, go on a sailing cruise, and all take the drug. Find the probability of each event.
(a) None of the friends gets seasick.
(b) All of the friends get seasick.
(c) Exactly three get seasick.
(d) At least two get seasick.

36. Reliability of a Machine A machine that is used in a manufacturing process has four separate components, each of which has a 0.01 probability of failing on any given day. If any component fails, the entire machine breaks down. Find the probability that on a given day the indicated event occurs.
(a) The machine breaks down.
(b) The machine does not break down.
(c) Only one component does not fail.

37. Genetics Huntington's disease is a hereditary ailment caused by a recessive gene. If both parents carry the gene but do not have the disease, there is a 0.25 probability that an offspring will fall victim to the condition. A newlywed couple find through genetic testing that they both carry the gene (but do not have the disease). If they intend to have four children, find the probability of each event.
(a) At least one child gets the disease.
(b) At least three of the children get the disease.

38. Selecting Cards Three cards are randomly selected from a standard 52-card deck, one at a time, with each card replaced in the deck before the next one is picked. Find the probability of each event.
(a) All three cards are hearts.
(b) Exactly two of the cards are spades.
(c) None of the cards is a diamond.
(d) At least one of the cards is a club.

39. Smokers and Nonsmokers The participants at a mathematics conference are housed dormitory-style, five to a room. Because of an oversight, conference organizers forget to ask whether the participants are smokers. In fact, it turns out that 30% are smokers. Find the probability that Fred, a nonsmoking conference participant, will be housed with:
(a) Exactly one smoker.
(b) One or more smokers.

40. Telephone Marketing A mortgage company advertises its rates by making unsolicited telephone calls to random numbers. About 2% of the calls reach consumers who are interested in the company's services. A telephone consultant can make 100 calls per evening shift.
(a) What is the probability that two or more calls will reach an interested party in one shift?
(b) How many calls does a consultant need to make to ensure at least a 0.5 probability of reaching one or more interested parties? [*Hint:* Use trial and error.]

41. Effectiveness of a Drug A certain disease has a mortality rate of 60%. A new drug is tested for its effectiveness against this disease. Ten patients are given the drug, and eight of them recover.
(a) Find the probability that eight or more of the patients would have recovered without the drug.
(b) Does the drug appear to be effective? (Consider the drug effective if the probability in part (a) is 0.05 or less.)

42. Hitting a Target An archer normally hits the target with probability of 0.6. She hires a new coach for a series of special lessons. After the lessons she hits the target in five out of eight attempts.
(a) Find the probability that she would have hit five or more out of the eight attempts before her lessons with the new coach.
(b) Did the new coaching appear to make a difference? (Consider the coaching effective if the probability in part (a) is 0.05 or less.)

DISCOVERY ■ DISCUSSION ■ WRITING

43. Most Likely Outcome for n Tosses of a Coin A balanced coin is tossed n times. In this exercise we investigate the following question: What is the number of heads that has the greatest probability of occurring? Note that for a balanced coin the probability of heads is $p = 0.5$.

(a) Suppose $n = 8$. Draw a probability histogram for the resulting binomial distribution. What number of heads has the greatest probability of occurring? If $n = 100$, what number of heads has the greatest probability of occurring?

(b) Suppose $n = 9$. Draw a probability histogram for the resulting binomial distribution. What number of heads has the greatest probability of occurring? If $n = 101$, what number of heads has the greatest probability of occurring?

9.4 EXPECTED VALUE

LEARNING OBJECTIVES **After completing this section, you will be able to:**

Find the expected value of a game ▶ Find the expected number of occurrences of an event

In this section we study an important application of probability called *expected value.*

▼ Expected Value

Suppose that a coin has probability 0.8 of showing heads. If the coin is tossed many times, we would *expect* to get heads about 80% of the time. Now, suppose that you get a payout of one dollar for each head. If you play this game many times, you would expect *on average* to gain \$0.80 per game:

$$\begin{pmatrix}\text{Expected payout}\\ \text{per game}\end{pmatrix} = \begin{pmatrix}\text{Amount of payout}\\ \text{per game}\end{pmatrix} \times \begin{pmatrix}\text{Probability of payout}\\ \text{per game}\end{pmatrix}$$

$$= \$1.00 \times 0.80 = \$0.80$$

The reasoning in this example motivates the following definition.

> **DEFINITION OF EXPECTED VALUE**
>
> A game gives payouts $a_1, a_2, \ldots, a_n$ with probabilities $p_1, p_2, \ldots, p_n$. The **expected value** (or **expectation**) E of this game is
>
> $$E = a_1p_1 + a_2p_2 + \cdots + a_np_n$$

EXAMPLE 1 | Finding Expected Value

A die is rolled, and you receive \$1 for each point that shows. What is your expectation?

SOLUTION Each face of the die has probability $\frac{1}{6}$ of showing. So you get \$1 with probability $\frac{1}{6}$, \$2 with probability $\frac{1}{6}$, \$3 with probability $\frac{1}{6}$, and so on. Thus the expected value is

$$E = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = 3.5$$

So if you play this game many times, you will make, on average, \$3.50 per game.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **3** ■

EXAMPLE 2 | Finding Expected Value

In Monte Carlo the game of roulette is played on a wheel with slots numbered 0, 1, 2, . . . , 36. The wheel is spun, and a ball dropped in the wheel is equally

likely to end up in any one of the slots. To play the game, you bet \$1 on any number. (For example, you may bet \$1 on number 23.) If the ball stops in your slot, you get \$36 (the \$1 you bet plus \$35). Find the expected value of this game.

SOLUTION The gambler can gain \$35 with probability $\frac{1}{37}$ and can lose \$1 with probability $\frac{36}{37}$. So the gambler's expected value is

$$E = (35)\frac{1}{37} + (-1)\frac{36}{37} \approx -0.027$$

In other words, if you play this game many times, you would expect to lose 2.7 cents on every dollar you bet (on average). Consequently, the house expects to gain 2.7 cents on every dollar that is bet.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **13** ■

EXAMPLE 3 | Expected Number

At any given time, the express checkout lane at a small supermarket has three shoppers in line with probability 0.2, two shoppers with probability 0.5, one shopper with probability 0.2, and no shoppers with probability 0.1. If you go to this market, how many shoppers would you expect to find waiting in the express checkout lane?

SOLUTION The "payouts" here are the number of shoppers waiting in line. To find the expected number of shoppers waiting in line, we multiply each "payout" by its probability and add the results.

$$E = 3(0.2) + 2(0.5) + 1(0.2) + 0(0.1) = 1.8$$

So on average, you would expect 1.8 shoppers waiting in the express lane.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **21** ■

▼ What Is a Fair Game?

A **fair game** is game with expected value zero. So if you play a fair game many times, you would expect, on average, to break even.

EXAMPLE 4 | A Fair Game?

Suppose that you play the following game. A card is drawn from a deck. If the card is an ace, you get a payout of \$10. If the card is not an ace, you have to pay \$1.

(a) Is this a fair game?

(b) If the game is not fair, find the payout amount that would make this game a fair game.

MATHEMATICS IN THE MODERN WORLD

Fair Voting Methods

The methods of mathematics have recently been applied to problems in the social sciences. For example, how do we find fair voting methods? You may ask, "What is the problem with how we vote in elections?" Well, suppose candidates A, B, and C are running for president. The final vote tally is as follows: A gets 40%, B gets 39%, and C gets 21%. So candidate A wins. But 60% of the voters *didn't* want A. Moreover, suppose you voted for C, but you dislike A so much that you would have been willing to change your vote to B to avoid having A win. Suppose most of the voters who voted for C feel the same way you do. Then we have a situation in which most of the voters prefer B over A, but A wins. Is that fair?

In the 1950s Kenneth Arrow showed mathematically that no democratic method of voting can be completely fair; he later won a Nobel Prize for his work. Mathematicians continue to work on finding fairer voting systems. The system that is most often used in federal, state, and local elections is called *plurality voting* (the candidate with the most votes wins). Other systems include *majority voting* (if no candidate gets a majority, a runoff is held between the top two vote-getters), *approval voting* (each voter can vote for as many candidates as he or she approves of), *preference voting* (each voter orders the candidates according to his or her preference), and *cumulative voting* (each voter gets as many votes as there are candidates and can give all of his or her votes to one candidate or distribute them among the candidates as he or she sees fit). This last system is often used to select corporate boards of directors. Each system of voting has both advantages and disadvantages.

SOLUTION

(a) In this game you get a payout of \$10 if an ace is drawn (probability $\frac{4}{52}$), and you lose \$1 if any other card is drawn (probability $\frac{48}{52}$). So the expected value is

$$E = 10\left(\frac{4}{52}\right) - 1\left(\frac{48}{52}\right) = -\frac{8}{52}$$

Since the expected value is not zero, the game is not fair. If you play this game many times, you would expect to lose, on average, $\frac{8}{52} \approx \$0.15$ per game.

(b) We want to find the payout x that makes the expected value 0:

$$E = x\left(\frac{4}{52}\right) - 1\left(\frac{48}{52}\right) = 0$$

Solving this equation, we get $x = 12$. So a payout of \$12 for an ace would make this a fair game.

PRACTICE WHAT YOU'VE LEARNED: DO EXERCISE **25**

Games of chance in casinos are never fair; the gambler always has a negative expected value (as in Examples 2 and 4(a)). This makes gambling profitable for the casino and unprofitable for the gambler.

9.4 EXERCISES

CONCEPTS

1. If a game gives payoffs of \$10 and \$100 with probabilities 0.9 and 0.1, respectively, then the expected value of this game is

$$E = \_\_\_\_\_ \times 0.9 + \_\_\_\_\_ \times 0.1 = \_\_\_\_\_\_$$

2. If you played the game in Exercise 1 many times, then you would expect your average payoff per game to be about

\$___________.

SKILLS

3–12 ■ Find the expected value (or expectation) of the games described.

3. Mike wins \$2 if a coin toss shows heads and \$1 if it shows tails.

4. Jane wins \$10 if a die roll shows a six, and she loses \$1 otherwise.

5. The game consists of drawing a card from a deck. You win \$100 if you draw the ace of spades or lose \$1 if you draw any other card.

6. Tim wins \$3 if a coin toss shows heads or \$2 if it shows tails.

7. Carol wins \$3 if a die roll shows a six, and she wins \$0.50 otherwise.

8. A coin is tossed twice. Albert wins \$2 for each heads and must pay \$1 for each tails.

9. A die is rolled. Tom wins \$2 if the die shows an even number, and he pays \$2 otherwise.

10. A card is drawn from a deck. You win \$104 if the card is an ace, \$26 if it is a face card, and \$13 if it is the 8 of clubs.

11. A bag contains two silver dollars and eight slugs. You pay 50 cents to reach into the bag and take a coin, which you get to keep.

12. A bag contains eight white balls and two black balls. John picks two balls at random from the bag, and he wins \$5 if he does not pick a black ball.

APPLICATIONS

13. Roulette In the game of roulette as played in Las Vegas, the wheel has 38 slots. Two slots are numbered 0 and 00, and the rest are numbered 1 to 36. A \$1 bet on any number wins \$36 (\$35 plus the \$1 bet). Find the expected value of this game.

14. Sweepstakes A sweepstakes offers a first prize of \$1,000,000, second prize of \$100,000, and third prize of \$10,000. Suppose that two million people enter the contest and three names are drawn randomly for the three prizes.

(a) Find the expected winnings for a person participating in this contest.

(b) Is it worth paying a dollar to enter this sweepstakes?

15. A Game of Chance A box contains 100 envelopes. Ten envelopes contain \$10 each, ten contain \$5 each, two are "unlucky," and the rest are empty. A player draws an envelope from the box and keeps whatever is in it. If a person draws an unlucky envelope, however, he must pay \$100. What is the expectation of a person playing this game?

16. Combination Lock A safe containing $1,000,000 is locked with a combination lock. You pay $1 for one guess at the six-digit combination. If you open the lock, you get to keep the million dollars. What is your expectation?

17. Gambling on Stocks An investor buys 1000 shares of a risky stock for $5 a share. She estimates that the probability that the stock will rise in value to $20 a share is 0.1 and the probability that it will fall to $1 a share is 0.9. If the only criterion for her decision to buy this stock was the expected value of her profit, did she make a wise investment?

18. Slot Machine A slot machine has three wheels, and each wheel has 11 positions: the digits 0, 1, 2, . . . , 9 and the picture of a watermelon. When a quarter is placed in the machine and the handle is pulled, the three wheels spin independently and come to rest. When three watermelons show, the machine pays the player $5; otherwise, nothing is paid. What is the expected value of this game?

19. Lottery In a 6/49 lottery game, a player pays $1 and selects six numbers from 1 to 49. Any player who has chosen the six winning numbers wins $1,000,000. Assuming that this is the only way to win, what is the expected value of this game?

20. Lightning Insurance An insurance company has determined that in a certain region the probability of lightning striking a house in a given year is about 0.0003, and the average cost of repairs of lightning damage is $7500 per incident. The company charges $25 per year for lightning insurance.

(a) Find the company's expected value for each lightning insurance policy.

(b) If the company has 450,000 lightning damage policies, what is the company's expected yearly income from lightning insurance?

21. Expected Number During the school year, a college student watches TV for two hours a week with probability 0.15, three hours with probability 0.45, four hours with probability 0.30, and five hours with probability 0.10. Find the expected number of hours of TV that he watches per week.

22. Expected Number In a large liberal arts college 5% of the students are studying three foreign languages, 15% are studying two foreign languages, 45% are studying one foreign language, and 35% are not studying a foreign language. If a student is selected at random, find the expected number of foreign languages that he or she is studying.

23. Expected Number A student goes to swim practice several times a week. In any given week the probability that he swims three times is 0.30, two times is 0.45, one time is 0.15, and no times is 0.10. Find the expected number of times the student goes to practice in any given week.

24. Expected Number Consider families with three children, and assume that the probability of having a girl is $\frac{1}{2}$.

(a) Complete the table for the probabilities of having 0, 1, 2, or 3 girls in a family of three children.

(b) Find the expected number of girls in a family of three children.

Number of girls	Probability
0	$\frac{1}{8}$
1	
2	
3	

25–30 ■ A game of chance is described. **(a)** Is the game fair? **(b)** If the game is not fair, find the payout amount that would make the game fair.

25. A Fair Game? A card is drawn from a deck. If the card is the ace of spades you get a payout of $12. If the card is not an ace you have to pay $0.50.

26. A Fair Game? A die is rolled. You get $20 if a one or a six shows; otherwise, you pay $10.

27. A Fair Game? A pair of dice is rolled. You get $30 if two ones show; otherwise, you pay $2.

28. A Fair Game? A die is rolled and a coin is tossed. If the result is a "six" and "heads," you get $10. For any other result you pay $1.

29. A Fair Game? A card is drawn from a deck, a die is rolled, and a coin is tossed. If the result is the "ace of spades," a "six," and "heads," you get $600. For any other result you pay $1.

30. A Fair Game? A bag contains two silver dollars and six slugs. A game consists of reaching into the bag and drawing a coin, which you get to keep. If you draw a slug, you pay $0.50.

DISCOVERY ■ DISCUSSION ■ WRITING

31. The Expected Value of a Sweepstakes Contest A magazine clearinghouse holds a sweepstakes contest to sell subscriptions. If you return the winning number, you win $1,000,000. You have a 1-in-20-million chance of winning, but your only cost to enter the contest is a first-class stamp to mail the entry. Use the current price of a first-class stamp to calculate your expected net winnings if you enter this contest. Is it worth entering the contest?

CHAPTER 9 | REVIEW

■ PROPERTIES AND FORMULAS

Fundamental Counting Principle (p. 626)

If $E_1, E_2, \ldots, E_k$ are events that occur in order and if event E_i can occur in n_i ways $(i = 1, 2, \ldots, k)$, then the sequence of events can occur in order in $n_1 \times n_2 \times \cdots \times n_k$ ways.

Permutations (p. 628)

A **permutation** of a set of objects is an ordering of these objects. If the set has n objects, then there are $n!$ permutations of the objects.

If a set has n objects, then the number of ways of ordering the r-element subsets of the set is denoted $\boldsymbol{P(n, r)}$ and is called the **number of permutations of *n* objects taken *r* at a time**:

$$P(n, r) = \frac{n!}{(n - r)!}$$

Distinguishable Permutations (p. 630)

Suppose that a set has n objects of k kinds (where the objects in each kind cannot be distinguished from each other), and suppose that there are n_1 objects of the first kind, n_2 of the second kind, and so on (so $n_1 + n_2 + \cdots + n_k = n$). Two permutations of the set are **distinguishable** from each other if one cannot be obtained from the other simply by interchanging the positions of elements of the same kind. (In other words, the permutations "look" different.)

The number of distinguishable permutations of these objects is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Combinations (pp. 630–631)

A **combination** of r objects from a set is any subset of the set that contains r elements (without regard to order).

If a set has n objects, then the number of combinations of r elements from the set is denoted $\boldsymbol{C(n, r)}$ and is called the **number of combinations of *n* objects taken *r* at a time**:

$$C(n, r) = \frac{n!}{r!(n - r)!}$$

Permutations or Combinations? (p. 632)

When solving a problem that involves counting the number of ways of picking r objects from a set of n objects, we ask, "Does the order in which the objects are picked make a difference?"

If the order matters, use permutations.

If the order doesn't matter, use combinations.

Sample Spaces and Events (pp. 638–639)

An **experiment** is a process that gives definite results, called the **outcomes**. (For example, rolling a die results in the outcomes 1, 2, 3, 4, 5, or 6.) The **sample space** of an experiment is the set of all possible outcomes.

An **event** is any subset of the sample space. (For example, in rolling a die, the event "get an even number" is the subset $\{2, 4, 6\}$.)

Probability (pp. 639–640)

Suppose that S is the sample space of an experiment in which all outcomes are equally likely and that E is an event in this experiment. The **probability** of E, denoted $P(E)$, is

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}$$

The probability of any event E satisfies

$$0 \le P(E) \le 1$$

If $P(E) = 0$, then E is **impossible** (will never happen). If $P(E) = 1$, then E is **certain** (will definitely happen).

The Complement of an Event (p. 641)

If S is the sample space of an experiment and E is an event, then the **complement** of E (denoted E') is the set of all outcomes in S that are not in E. The probability of E' is given by

$$P(E') = 1 - P(E)$$

The Union of Events (pp. 642–643)

Suppose E and F are events in a sample space S.

The **union** of E and F is the set of all outcomes in S that are in either E *or* F (or both). The union of E and F is denoted $E \cup F$.

For any events E and F the probability of their union is

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

The events E and F are **mutually exclusive** if $E \cap F = \varnothing$. For mutually exclusive events E and F the probability of their union is

$$P(E \cup F) = P(E) + P(F)$$

Conditional Probability (pp. 643–644)

Suppose E and F are events in a sample space S.

The **conditional probability of *E* given that *F* occurs** is denoted by $P(E|F)$ and is given by

$$P(E|F) = \frac{n(E \cap F)}{n(F)}$$

The Intersection of Events (p. 645)

Suppose E and F are events in a sample space S.

The **intersection** of E and F is the set of all outcomes in S that are in both E *and* F. The intersection of E and F is denoted by $E \cap F$.

The probability of the intersection of E and F is

$$P(E \cap F) = P(E)P(F|E)$$

The events E and F are **independent** if the occurrence of one of them does not affect the probability of the occurrence of the other. For independent events E and F the probability of their intersection is

$$P(E \cap F) = P(E)P(F)$$

Binomial Probabilities (p. 651)

A **binomial experiment** is one that has two possible outcomes, S and F ("success" and "failure"). If

$$P(S) = p$$

and

$$P(F) = q = 1 - p$$

then the probability of getting exactly r successes in n trials of the experiment is

$$P(r \text{ successes in } n \text{ trials}) = C(n, r)p^r q^{n-r}$$

Expected Value (p. 656)

If a game gives payoffs $a_1, a_2, \ldots, a_n$ with probabilities $p_1, p_2, \ldots, p_n$, then the **expected value** (or **expectation**) E of this game is

$$E = a_1 p_1 + a_2 p_2 + \cdots + a_n p_n$$

LEARNING OBJECTIVES SUMMARY

Section	After completing this chapter, you should be able to . . .	Review Exercises
9.1	▪ Use the Fundamental Counting Principle	1, 2, 6, 9, 12, 14, 15, 18, 37–40
	▪ Count permutations	2, 3, 10, 17
	▪ Count distinguishable permutations	16, 19–22
	▪ Count combinations	3–5, 7–8, 11, 13
	▪ Solve counting problems involving both permutations and combinations	17, 23–24
9.2	▪ Find the probability of an event by counting	18, 25–40
	▪ Find the probability of the complement of an event	26, 32, 36–37, 42
	▪ Find the probability of the union of events	25–27, 30
	▪ Find conditional probabilities	41
	▪ Find the probability of the intersection of events	26, 29–32
9.3	▪ Find binomial probabilities	27, 42–43
	▪ Make a table of a probability distribution	45
9.4	▪ Find the expected value of a game	44, 46–47
	▪ Find the expected number of occurrences of an event	48

EXERCISES

1. A coin is tossed, a die is rolled, and a card is drawn from a deck. How many possible outcomes does this experiment have?

2. How many three-digit numbers can be formed by using the digits 1, 2, 3, 4, 5, and 6 if repetition of digits
(a) is allowed?
(b) is not allowed?

3. (a) How many different two-element subsets does the set $\{A, E, I, O, U\}$ have?
(b) How many different two-letter "words" can be made by using the letters from the set in part (a)?

4. An airline company has overbooked a particular flight, and seven passengers must be "bumped" from the flight. If 120 passengers are booked on this flight, in how many ways can the airline choose the seven passengers to be bumped?

5. A quiz has ten true-false questions. In how many different ways can a student earn a score of exactly 70% on this quiz?

6. A test has ten true-false questions and five multiple-choice questions with four choices for each. In how many ways can this test be completed?

7. If you must answer only eight of ten questions on a test, how many ways do you have of choosing the questions you will omit?

8. An ice-cream store offers 15 flavors of ice cream. The specialty is a banana split with four scoops of ice cream. If each scoop must be a different flavor, how many different banana splits may be ordered?

9. A company uses a different three-letter security code for each of its employees. What is the maximum number of codes this security system can generate?

10. A group of students determines that they can stand in a row for their class picture in 120 different ways. How many students are in this class?

11. A coin is tossed ten times. In how many different ways can the result be three heads and seven tails?

12. The Yukon Territory in Canada uses a license-plate system for automobiles that consists of two letters followed by three numbers. Explain how we can know that fewer than 700,000 autos are licensed in the Yukon.

13. A group of friends have reserved a tennis court. They find that there are ten different ways in which two of them can play a singles game on this court. How many friends are in this group?

14. A pizza parlor advertises that they prepare 2048 different types of pizza. How many toppings does this parlor offer?

15. In Morse code, each letter is represented by a sequence of dots and dashes, with repetition allowed. How many letters can be represented by using Morse code if three or fewer symbols are used?

16. The genetic code is based on the four nucleotides adenine (A), cytosine (C), guanine (G), and thymine (T). These are connected in long strings to form DNA molecules. For example, a sequence in the DNA may look like CAGTGGTACC . . . . The code uses "words," all the same length, that are composed of the nucleotides A, C, G, and T. It is known that at least 20 different words exist. What is the minimum word length necessary to generate 20 words?

17. Given 16 subjects from which to choose, in how many ways can a student select fields of study as follows?
(a) A major and a minor
(b) A major, a first minor, and a second minor
(c) A major and two minors

18. **(a)** How many three-digit numbers can be formed by using the digits 0, 1, . . . , 9? (Remember, a three-digit number cannot have 0 as the leftmost digit.)
(b) If a number is chosen randomly from the set {0, 1, 2, . . . , 1000}, what is the probability that the number chosen is a three-digit number?

19–22 ■ An **anagram** of a word is a permutation of the letters of that word. For example, anagrams of the word *triangle* include *griantle, integral,* and *tenalgir.* Find the number of different anagrams of the given word.

19. RANDOM **20.** BLOB

21. BUBBLE **22.** MISSISSIPPI

23. A committee of seven is to be chosen from a group of ten men and eight women. In how many ways can the committee be chosen using each of the following selection requirements?
(a) No restriction is placed on the number of men and women on the committee.
(b) The committee must have exactly four men and three women.
(c) Susie refuses to serve on the committee.
(d) At least five women must serve on the committee.
(e) At most two men can serve on the committee.
(f) The committee is to have a chairman, a vice chairman, a secretary, and four other members.

24. The U.S. Senate has two senators from each of the 50 states. In how many ways can a committee of five senators be chosen if no state is to have two members on the committee?

25. A jar contains ten red balls labeled 0, 1, 2, . . . , 9 and five white balls labeled 0, 1, 2, 3, 4. If a ball is drawn from the jar, find the probability of the given event.
(a) The ball is red.
(b) The ball is even-numbered.
(c) The ball is white and odd-numbered.
(d) The ball is red or odd-numbered.

26. If two balls are drawn from the jar in Exercise 23, find the probability of the given event.
(a) Both balls are red.
(b) One ball is white, and the other is red.
(c) At least one ball is red.
(d) Both balls are red and even-numbered.
(e) Both balls are white and odd-numbered.

27. A coin is tossed three times in a row, and the outcomes of each toss are observed.
(a) Find the sample space for this experiment.
(b) Find the probability of getting three heads.
(c) Find the probability of getting two or more heads.
(d) Find the probability of getting tails on the first toss.

28. A shelf has ten books: two mysteries, four romance novels, and four mathematics textbooks. If you select a book at random to take to the beach, what is the probability that it turns out to be a mathematics text?

29. A die is rolled, and a card is selected from a standard 52-card deck. What is the probability that both the die and the card show a six?

30. Find the probability that the indicated card is drawn at random from a 52-card deck.
(a) An ace **(b)** An ace or a jack
(c) An ace or a spade **(d)** A red ace

31. A card is drawn from a 52-card deck, a die is rolled, and a coin is tossed. Find the probability of each outcome.
(a) The ace of spades, a six, and heads
(b) A spade, a six, and heads
(c) A face card, a number greater than 3, and heads

32. Two dice are rolled. Find the probability of each outcome.
(a) The dice show the same number.
(b) The dice show different numbers.

33. Four cards are dealt from a standard 52-card deck. Find the probability that the cards are
(a) all kings **(b)** all spades
(c) all the same color

34. In the "numbers game" lottery a player picks a three-digit number (from 000 to 999), and if the number is selected in the drawing, the player wins \$500. If another number with the same digits (in any order) is drawn, the player wins \$50. John plays the number 159.
(a) What is the probability that he will win \$500?
(b) What is the probability that he will win \$50?

35. In a TV game show, a contestant is given five cards with a different digit on each and is asked to arrange them to match the price of a brand-new car. If she gets the price right, she wins the car. What is the probability that she wins, assuming that she knows the first digit but must guess the remaining four?

36. A pizza parlor offers 12 different toppings, one of which is anchovies. If a a pizza is ordered at random (that is, any number of the toppings from 0 to all 12 may be ordered), what is the probability that anchovies is one of the toppings selected?

37. A drawer contains an unorganized collection of 50 socks; 20 are red and 30 are blue. Suppose the lights go out, so Kathy can't distinguish the color of the socks.
(a) What is the minimum number of socks Kathy must take out of the drawer to be sure of getting a matching pair?
(b) If two socks are taken at random from the drawer, what is the probability that they make a matching pair?

38. Zip codes consist of five digits.
(a) How many different zip codes are possible?
(b) How many different zip codes can be read when the envelope is turned upside down? (An upside-down 9 is a 6; and 0, 1, and 8 are the same when read upside down.)
(c) What is the probability that a randomly chosen zip code can be read upside down?
(d) How many zip codes read the same upside down as right side up?

39. In the Zip+4 postal code system, zip codes consist of nine digits.
(a) How many different Zip+4 codes are possible?
(b) How many different Zip+4 codes are palindromes? (A palindrome is a number that reads the same from left to right as right to left.)
(c) What is the probability that a randomly chosen Zip+4 code is a palindrome?

40. Let $N = 3{,}600{,}000$. (Note that $N = 2^7 3^2 5^5$.)
(a) How many divisors does N have?
(b) How many even divisors does N have?
(c) How many divisors of N are multiples of 6?
(d) What is the probability that a randomly chosen divisor of N is even?

41. A card is drawn at random from a standard 52-card deck. Find the probability of each event.
(a) The card is a king.
(b) The card is a king or an ace.
(c) The card is a king given that it is a face card.
(d) The card is a king given that it is not an ace.

42. A fair die is rolled eight times. Find the probability of each event.
(a) A six occurs four times.
(b) An even number occurs two or more times.

43. Pacific Chinook salmon occur in two varieties: white-fleshed and red-fleshed. It is impossible to tell without cutting the fish open whether it is the white or red variety. About 30% of Chinooks have white flesh. An angler catches five Chinooks. Find the probability of each event.
(a) All are white.
(b) All are red.
(c) Exactly two are white.
(d) Three or more are red.

44. Two dice are rolled. John gets \$5 if they show the same number; he pays \$1 if they show different numbers. What is the expected value of this game?

45. An unbalanced coin has probability 0.7 of showing a "heads." The coin is tossed four times. Make a table of the probability distribution for the number of heads.

Number of heads	Probability
0	
1	
2	
3	
4	

46. Three dice are rolled. John gets \$5 if they all show the same number; he pays \$1 otherwise. What is the expected value of this game?

47. Mary will win \$1,000,000 if she can name the 13 original states in the order in which they ratified the U.S. Constitution. Mary has no knowledge of this order, so she makes a guess. What is her expectation?

48. Liam goes jogging several times a week. In any given week the probability that he jogs three times is 0.4, that he jogs two times is 0.1, that he jogs once is 0.2, and that he doesn't go jogging is 0.3. Find the expected number of times he goes jogging in any given week.

CHAPTER 9 TEST

1. Alice and Bill have four grandchildren, and they have three framed pictures of each grandchild. They wish to choose one picture of each grandchild to display on the piano in their living room, arranged from oldest to youngest. In how many ways can they do this?

2. A hospital cafeteria offers a fixed-price lunch consisting of a main course, a dessert, and a drink. If there are four main courses, three desserts, and six drinks to pick from, in how many ways can a customer select a meal consisting of one choice from each category?

3. An Internet service provider requires its customers to select a password consisting of four letters followed by three digits. Find how many such passwords are possible in each of the following cases:
   - **(a)** Repetition of letters and digits is allowed.
   - **(b)** Repetition of letters and digits is not allowed.

4. Over the past year, John has purchased 30 books.
   - **(a)** In how many ways can he pick four of these books and arrange them, in order, on his nightstand bookshelf?
   - **(b)** In how many ways can he choose four of these books to take with him on his vacation at the shore?

5. A commuter must travel from Ajax to Barrie and back every day. Four roads join the two cities. The commuter likes to vary the trip as much as possible, so she always leaves and returns by different roads. In how many different ways can she make the round-trip?

6. A pizza parlor offers four sizes of pizza and 14 different toppings. A customer may choose any number of toppings (or no topping at all). How many different pizzas does this parlor offer?

7. An *anagram* of a word is a rearrangement of the letters of the word.
   - **(a)** How many anagrams of the word *LOVE* are possible?
   - **(b)** How many different anagrams of the word *KISSES* are possible?

8. A board of directors consisting of eight members is to be chosen from a pool of 30 candidates. The board is to have a chairman, a treasurer, a secretary, and five other members. In how many ways can the board of directors be chosen?

9. One card is drawn from a deck. Find the probability of each event.
   - **(a)** The card is red.
   - **(b)** The card is a king.
   - **(c)** The card is a red king.

10. A jar contains five red balls, numbered 1 to 5, and eight white balls, numbered 1 to 8. A ball is chosen at random from the jar. Find the probability of each event.
    - **(a)** The ball is red.
    - **(b)** The ball is even-numbered.
    - **(c)** The ball is red or even-numbered.

11. Three people are chosen at random from a group of five men and ten women. What is the probability that all three are men?

12. Two dice are rolled. What is the probability of getting doubles?

13. In a group of four students, what is the probability that at least two have the same astrological sign?

14. An unbalanced coin is weighted so that the probability of heads is 0.55. The coin is tossed ten times.
    - **(a)** What is the probability of getting exactly 6 heads?
    - **(b)** What is the probability of getting less than 3 heads?

15. You are to draw one card from a deck. If it is an ace, you win \$10; if it is a face card, you win \$1; otherwise, you lose \$0.50. What is the expected value of this game?

The Monte Carlo Method

A good way to familiarize ourselves with a fact is to experiment with it. For instance, to convince ourselves that the earth is a sphere (which was considered a major paradox at one time), we could go up in a space shuttle to see that it is so; to see whether a given equation is an identity, we might try some special cases to make sure there are no obvious counterexamples. In problems involving probability, we can perform an experiment many times and use the results to estimate the probability in question. In fact, we often model the experiment on a computer, thereby making it feasible to perform the experiment a large number of times. This technique is called the **Monte Carlo method**, named after the famous gambling casino in Monaco.

EXAMPLE 1 | The Contestant's Dilemma

In a TV game show, a contestant chooses one of three doors. Behind one of them is a valuable prize; the other two doors have nothing behind them. After the contestant has made her choice, the host opens one of the other two doors—one that he knows does not conceal a prize—and then gives her the opportunity to change her choice.

Should the contestant switch or stay, or does it matter? In other words, by switching doors, does she increase, decrease, or leave unchanged her probability of winning? At first, it may seem that switching doors doesn't make any difference. After all, two doors are left—one with the prize and one without—so it seems reasonable that the contestant has an equal chance of winning or losing. But if you play this game many times, you will find that by switching doors, you actually win about $\frac{2}{3}$ of the time.

Contestant: "I choose door number 2."

Contestant: "Oh no, what should I do?"

The authors modeled this game on a computer and found that in one million games the simulated contestant (who always switches) won 667,049 times—very close to $\frac{2}{3}$ of the time. Thus it seems that switching doors does make a difference: Switching increases the contestant's chances of winning. This experiment forces us to reexamine our reasoning. Here is why switching doors is the correct strategy:

1. When the contestant first made her choice, she had a $\frac{1}{3}$ chance of winning. If she doesn't switch, no matter what the host does, her probability of winning remains $\frac{1}{3}$.
2. If the contestant decides to switch, she will switch to the winning door if she had initially chosen a losing one or to a losing door if she had initially chosen the winning one. Since the probability of having initially selected a losing door is $\frac{2}{3}$, by switching the probability of winning then becomes $\frac{2}{3}$.

We conclude that the contestant should switch, because her probability of winning is $\frac{2}{3}$ if she switches and $\frac{1}{3}$ if she doesn't. Put simply, there is a much greater chance that she initially chose a losing door (since there are more of these), so she should switch. ■

An experiment can be modeled by using any computer language or programmable calculator that has a random-number generator. This is a command or function (usually called `Rnd` or `Rand`) that returns a randomly chosen number x with $0 \le x < 1$. In the next example we see how to use this to model a simple experiment.

EXAMPLE 2 | Monte Carlo Model of a Coin Toss

When a balanced coin is tossed, each outcome—"heads" or "tails"—has probability $\frac{1}{2}$. This doesn't mean that if we toss a coin several times, we will necessarily get exactly half heads and half tails. We would expect, however, the proportion of heads and of tails to get closer and closer to $\frac{1}{2}$ as the number of tosses increases. To test this hypothesis, we could toss a coin a very large number of times and keep track of the results. But this is a very tedious process, so we will use the Monte Carlo method to model this process.

To model a coin toss with a calculator or computer, we use the random-number generator to get a random number x such that $0 \le x < 1$. Because the number is chosen randomly, the probability that it lies in the first half of this interval $\left(0 \le x < \frac{1}{2}\right)$ is the

```
PROGRAM:HEADTAIL
:0→J:0→K
:For(N,1,100)
:rand→X
:int(2X)→Y
:J+(1-Y)→J
:K+Y→K
:END
:Disp"HEADS=",J
:Disp"TAILS=",K
```

same as the probability that it lies in the second half $\left(\frac{1}{2} \le x < 1\right)$. Thus we could model the outcome "heads" by the event that $0 \le x < \frac{1}{2}$ and the outcome "tails" by the event that $\frac{1}{2} \le x < 1$.

An easier way to keep track of heads and tails is to note that if $0 \le x < 1$, then $0 \le 2x < 2$, and so $[\![2x]\!]$, the integer part of $2x$, is either 0 or 1, each with probability $\frac{1}{2}$. (On most programmable calculators, the function `Int` gives the integer part of a number.) Thus we could model "heads" with the outcome "0" and "tails" with the outcome "1" when we take the integer part of $2x$. The program in the margin models 100 tosses of a coin on the TI-83 calculator. The graph in Figure 1 shows what proportion p of the tosses have come up "heads" after n tosses. As you can see, this proportion settles down near 0.5 as the number n of tosses increases—just as we hypothesized.

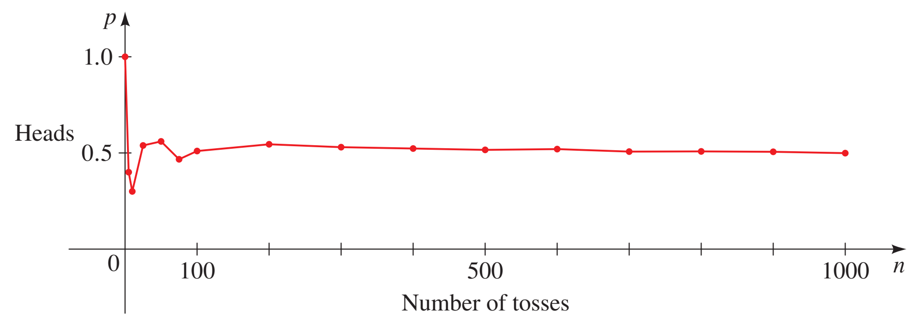


FIGURE 1 Relative frequency of "heads"

In general, if a process has n equally likely outcomes, then we can model the process using a random-number generator as follows: If our program or calculator produces the random number x, with $0 \le x < 1$, then the integer part of nx will be a random choice from the n integers $0, 1, 2, \ldots, n-1$. Thus we can use the outcomes $0, 1, 2, \ldots, n-1$ as models for the outcomes of the actual experiment.

PROBLEMS

1. **Winning Strategy** In a game show like the one described in Example 1, a prize is concealed behind one of ten doors. After the contestant chooses a door, the host opens eight losing doors and then gives the contestant the opportunity to switch to the other unopened door.
   - **(a)** Play this game with a friend 30 or more times, using the strategy of switching doors each time. Count the number of times you win, and estimate the probability of winning with this strategy.
   - **(b)** Calculate the probability of winning with the "switching" strategy using reasoning similar to that in Example 1. Compare with your result from part (a).

2. **Family Planning** A couple intend to have two children. What is the probability that they will have one child of each sex? The French mathematician D'Alembert analyzed this problem (incorrectly) by reasoning that three outcomes are possible: two boys, or two girls, or one child of each sex. He concluded that the probability of having one of each sex is $\frac{1}{3}$, mistakenly assuming that the three outcomes are equally likely.
   - **(a)** Model this problem with a pair of coins (using "heads" for boys and "tails" for girls), or write a program to model the problem. Perform the experiment 40 or more times, counting the number of boy-girl combinations. Estimate the probability of having one child of each sex.
   - **(b)** Calculate the correct probability of having one child of each sex, and compare this with your result from part (a).

3. **Dividing a Jackpot** A game between two players consists of tossing a coin. Player A gets a point if the coin shows heads, and player B gets a point if it shows tails. The first player to get six points wins an $8000 jackpot. As it happens, the police raid the place when player A has five points and B has three points. After everyone has calmed down, how should the

jackpot be divided between the two players? In other words, what is the probability of A winning (and that of B winning) if the game were to continue?

The French mathematicians Pascal and Fermat corresponded about this problem, and both came to the same correct conclusion (though by very different reasonings). Their friend Roberval disagreed with both of them. He argued that player A has probability $\frac{3}{4}$ of winning, because the game can end in the four ways H, TH, TTH, TTT, and in three of these, A wins. Roberval's reasoning was wrong.

(a) Continue the game from the point at which it was interrupted, using either a coin or a modeling program. Perform this experiment 80 or more times, and estimate the probability that player A wins.

(b) Calculate the probability that player A wins. Compare with your estimate from part (a).

4. Long or Short World Series? In the World Series the top teams in the National League and the American League play a best-of-seven series; that is, they play until one team has won four games. (No tie is allowed, so this results in a maximum of seven games.) Suppose the teams are evenly matched, so the probability that either team wins a given game is $\frac{1}{2}$.

(a) Use a coin or a modeling program to model a World Series, in which "heads" represents a win by Team A and "tails" represents a win by Team B. Perform this experiment at least 80 times, keeping track of how many games are needed to decide each series. Estimate the probability that an evenly matched series will end in four games. Do the same for five, six, and seven games.

(b) What is the probability that the series will end in four games? Five games? Six games? Seven games? Compare with your estimates from part (a).

(c) Find the expected value for the number of games until the series ends. [*Hint:* This will be $P(\text{four games}) \times 4 + P(\text{five}) \times 5 + P(\text{six}) \times 6 + P(\text{seven}) \times 7$.]

5. Estimating π In this problem we use the Monte Carlo method to estimate the value of π. The circle in the figure has radius 1, so its area is π, and the square has area 4. If we choose a point at random from the square, the probability that it lies inside the circle will be

$$\frac{\text{area of circle}}{\text{area of square}} = \frac{\pi}{4}$$

The Monte Carlo method involves choosing many points inside the square. Then we have

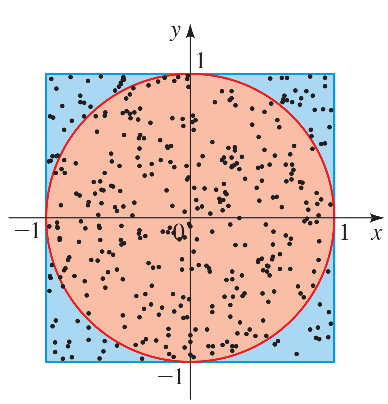
$$\frac{\text{number of hits inside circle}}{\text{number of hits inside square}} \approx \frac{\pi}{4}$$

Thus 4 times this ratio will give us an approximation for π.

To implement this method, we use a random-number generator to obtain the coordinates (x, y) of a random point in the square and then check to see whether it lies inside the circle (that is, we check if $x^2 + y^2 < 1$). Note that we need to use only points in the first quadrant, since the ratio of areas is the same in each quadrant. The program in the margin shows a way of doing this on the TI-83 calculator for 1000 randomly selected points.

```
PROGRAM:PI
:0→P
:For(N,1,1000)
:rand→X:rand→Y
:P+((X²+Y²)<1)→P
:End
:Disp "PI IS
APPROX",4*P/N
```

Carry out this Monte Carlo simulation for as many points as you can. How do your results compare with the actual value of π? Do you think this is a reasonable way to get a good approximation for π?

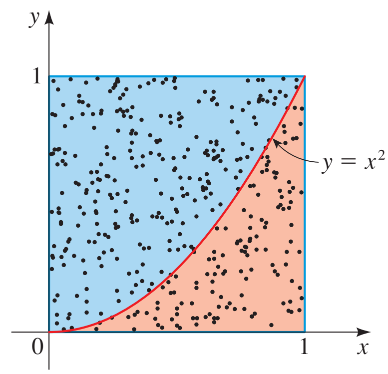


The **"contestant's dilemma"** problem discussed on page 665 is an example of how subtle probability can be. This problem was posed in a nationally syndicated column in *Parade* magazine in 1990. The correct solution was presented in the column, but it generated considerable controversy, with thousands of letters arguing that the solution was wrong. This shows how problems in probability can be quite tricky. Without a lot of experience in probabilistic thinking, it's easy to make a mistake. Even great mathematicians such as D'Alembert and Roberval (see Problems 2 and 3) made mistakes in probability. Professor David Burton writes in his book *The History of Mathematics,* "Probability theory abounds in paradoxes that wrench the common sense and trip the unwary."

6. Areas of Curved Regions The Monte Carlo method can be used to estimate the area under the graph of a function. The figure below shows the region under the graph of $f(x) = x^2$, above the x-axis, between $x = 0$ and $x = 1$. If we choose a point in the square at random, the probability that it lies under the graph of $f(x) = x^2$ is the area under the graph divided by the area of the square. So if we randomly select a large number of points in the square, we have

$$\frac{\text{number of hits under graph}}{\text{number of hits in square}} \approx \frac{\text{area under graph}}{\text{area of square}}$$

Modify the program from Problem 5 to carry out this Monte Carlo simulation and approximate the required area.



7. Random Numbers Choose two numbers at random from the interval $[0, 1)$. What is the probability that the sum of the two numbers is less than 1?

(a) Use a Monte Carlo model to estimate the probability.

(b) Calculate the exact value of the probability. [*Hint:* Call the numbers x and y. Choosing these numbers is the same as choosing an ordered pair (x, y) in the unit square $\{(x, y) \mid 0 \le x < 1, 0 \le y < 1\}$. What proportion of the points in this square corresponds to $x + y$ being less than 1?]

CUMULATIVE REVIEW TEST CHAPTERS 8 and 9

1. For each sequence, find the seventh term and the 20th term.

(a) $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \ldots$

(b) $a_n = \dfrac{2n^2 + 1}{n^3 - n + 4}$

(c) The arithmetic sequence with initial term $a = \frac{1}{2}$ and common difference $d = 3$.

(d) The geometric sequence with initial term $a = 12$ and common ratio $r = \frac{5}{6}$.

(e) The sequence defined recursively by $a_1 = 0.01$ and $a_n = -2a_{n-1}$.

2. Calculate the sum.

(a) $\frac{3}{5} + \frac{4}{5} + 1 + \frac{6}{5} + \frac{7}{5} + \frac{8}{5} + \cdots + \frac{19}{5} + 4$

(b) $3 + 9 + 27 + 81 + \cdots + 3^{10}$

(c) $\displaystyle\sum_{n=0}^{9} \frac{5}{2^n}$

(d) $6 + 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \cdots$

3. Mary and Kevin buy a vacation home for \$350,000. They pay \$35,000 down and take out a 15-year mortgage for the remainder. If their annual interest rate is 6%, how much will their monthly mortgage payment be?

4. A sequence is defined inductively by $a_1 = 1$ and $a_n = a_{n-1} + 2n - 1$. Use mathematical induction to prove that $a_n = n^2$.

5. **(a)** Use the Binomial Theorem to expand the expression $\left(2x - \frac{1}{2}\right)^5$.

(b) Find the term containing x^4 in the binomial expansion of $\left(2x - \frac{1}{2}\right)^{12}$.

6. When students receive their e-mail accounts at Oldenburg University they are assigned a randomly selected password, which consists of three letters followed by four digits (for example, ABC1234).

(a) How many such passwords are possible?

(b) How many passwords consist of three different letters followed by four different digits?

(c) The system administrator decides that in the interest of security, no two passwords can contain the same set of letters and digits (regardless of the order), and no character can be repeated in a password. What is the maximum number of users the system can accommodate under these rules?

7. Toftree is a game in which players roll three dice and receive points based on the outcome. Find the probability of each of the following outcomes.

(a) All three dice show the same number.

(b) All three dice show an even number.

(c) The sum of the numbers showing is 15.

8. An alumni association holds a "Vegas night" at its annual homecoming event. At one booth, participants play the following dice game: The player pays a fee of \$5, rolls a pair of dice, and then gets back \$15 if both dice show the same number or \$7 if the dice show numbers that differ by one (such as 2 and 3, or 5 and 4). What is the expected value of this game?

9. A weighted coin has probability p of showing heads and $q = 1 - p$ of showing tails when tossed.

(a) Find the binomial expansion of $(p + q)^5$. If this coin is tossed five times in a row, what event has the probability represented by the term in this binomial expansion that contains p^3?

(b) If the probability of heads is $\frac{2}{3}$, find the probability that in five tosses of the coin there are 2 heads and 3 tails.

10. An insect species has white wings that, when closed, cover the insect's back, like the wings of a ladybug. Some individual insects have black spots on their wings, arranged randomly, with a total of one to five spots. The probability that a randomly selected insect has n spots is $\left(\frac{1}{4}\right)^n$, where $n = 1, 2, 3, 4$, or 5.

(a) What event has probability $\sum_{n=1}^{5} \left(\frac{1}{4}\right)^n$? Calculate this sum.

(b) What is the probability that a randomly selected insect has no spots?

APPENDIX A Calculations and Significant Figures

Most of the applied examples and exercises in this book involve approximate values. For example, one exercise states that the moon has a radius of 1074 miles. This does not mean that the moon's radius is exactly 1074 miles but simply that this is the radius rounded to the nearest mile.

One simple method for specifying the accuracy of a number is to state how many **significant digits** it has. The significant digits in a number are the ones from the first nonzero digit to the last nonzero digit (reading from left to right). Thus 1074 has four significant digits, 1070 has three, 1100 has two, and 1000 has one significant digit. This rule may sometimes lead to ambiguities. For example, if a distance is 200 km to the nearest kilometer, then the number 200 really has three significant digits, not just one. This ambiguity is avoided if we use scientific notation—that is, if we express the number as a multiple of a power of 10:

$$2.00 \times 10^2$$

When working with approximate values, students often make the mistake of giving a final answer with *more* significant digits than the original data. This is incorrect because you cannot "create" precision by using a calculator. The final result can be no more accurate than the measurements given in the problem. For example, suppose we are told that the two shorter sides of a right triangle are measured to be 1.25 and 2.33 inches long. By the Pythagorean Theorem, we find, using a calculator, that the hypotenuse has length

$$\sqrt{1.25^2 + 2.33^2} \approx 2.644125564 \text{ in.}$$

But since the given lengths were expressed to three significant digits, the answer cannot be any more accurate. We can therefore say only that the hypotenuse is 2.64 in. long, rounding to the nearest hundredth.

In general, the final answer should be expressed with the same accuracy as the *least*-accurate measurement given in the statement of the problem. The following rules make this principle more precise.

RULES FOR WORKING WITH APPROXIMATE DATA

1. When multiplying or dividing, round off the final result so that it has as many *significant digits* as the given value with the fewest number of significant digits.
2. When adding or subtracting, round off the final result so that it has its last significant digit in the *decimal place* in which the least-accurate given value has its last significant digit.
3. When taking powers or roots, round off the final result so that it has the same number of *significant digits* as the given value.

EXAMPLE 1 | Working with Approximate Data

A rectangular table top is measured to be 122.64 in. by 37.3 in. Find the area and perimeter.

SOLUTION Using the formulas for area and perimeter, we get the following.

$$\text{Area} = \text{length} \times \text{width} = 122.64 \times 37.3 \approx 4570 \text{ in}^2$$ Three significant digits

$$\text{Perimeter} = 2(\text{length} + \text{width}) = 2(122.64 + 37.3) \approx 319.9 \text{ in.}$$ Tenths digit

So the area is approximately 4570 in^2, and the perimeter is approximately 319.9 in. ■

Note that in the formula for the perimeter, the value 2 is an exact value, not an approximate measurement. It therefore does not affect the accuracy of the final result. In general, if a problem involves only exact values, we may express the final answer with as many significant digits as we wish.

Note also that to make the final result as accurate as possible, *you should wait until the last step to round off your answer*. If necessary, use the memory feature of your calculator to retain the results of intermediate calculations.

A EXERCISES

1–10 ■ Evaluate the expression. Round your final answer to the appropriate number of decimal places or significant figures.

1. $3.27 - 0.1834$

2. $102.68 + 26.7$

3. 28.36×501.375

4. $\dfrac{201{,}186}{5238}$

5. $(1.36)^3$

6. $\sqrt{427.3}$

7. $3.3(642.75 + 66.787)$

8. $\dfrac{701}{1.27 - 10.5}$

9. $(5.10 \times 10^{-3})(12.4 \times 10^{7})(6.007 \times 10^{-6})$

10. $\dfrac{(1.361 \times 10^{7})(4.7717 \times 10^{-5})}{1.281876}$

11–12 ■ Use the geometric formulas on the inside back cover of the book to solve these problems.

11. The Measures of a Circle Find the circumference and area of a circle whose radius is 5.27 ft.

12. Volume of a Cone Find the volume of a cone whose height is 52.3 cm and whose radius is 4.267 cm.

13–14 ■ **Newton's Law of Gravity** The gravitational force F (in newtons) between two objects with masses m_1 and m_2 (in kilograms), separated by a distance r (in meters), is given by Newton's Law of Gravity:

$$F = G\frac{m_1 m_2}{r^2}$$

where $G \approx 6.67428 \times 10^{-11}\ \text{Nm}^2/\text{kg}^2$.

13. Find the gravitational force between two satellites in stationary earth orbit, 57.2 km apart, each with a mass of 11,426 kg.

14. The sun and the earth are 1.50×10^{11} m apart, with masses 1.9891×10^{30} kg, and 5.972×10^{24} kg, respectively.

(a) Find the gravitational force between the sun and the earth.

(b) Convert your answer in part (a) from newtons to pounds, using the fact that 1 N ≈ 0.225 lb.

APPENDIX B Graphing with a Graphing Calculator

A graphing calculator is a powerful tool for graphing equations and functions. In this appendix we give general guidelines to follow and common pitfalls to avoid when graphing with a graphing calculator. See Appendix C for specific guidelines on graphing with the TI-83/84 graphing calculators.

▼ Selecting the Viewing Rectangle

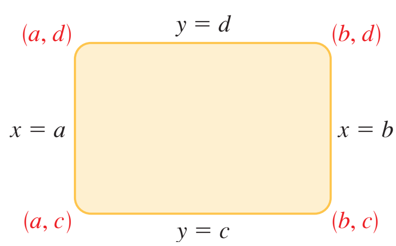


FIGURE 1 The viewing rectangle $[a, b]$ by $[c, d]$

A graphing calculator or computer displays a rectangular portion of the graph of an equation in a display window or viewing screen, which we call a **viewing rectangle**. The default screen often gives an incomplete or misleading picture, so it is important to choose the viewing rectangle with care. If we choose the x-values to range from a minimum value of `Xmin` $= a$ to a maximum value of `Xmax` $= b$ and the y-values to range from a minimum value of `Ymin` $= c$ to a maximum value of `Ymax` $= d$, then the displayed portion of the graph lies in the rectangle

$$[a, b] \times [c, d] = \{(x, y) \mid a \le x \le b, c \le y \le d\}$$

as shown in Figure 1. We refer to this as the $[a, b]$ by $[c, d]$ viewing rectangle.

The graphing device draws the graph of an equation much as you would. It plots points of the form (x, y) for a certain number of values of x, equally spaced between a and b. If the equation is not defined for an x-value or if the corresponding y-value lies outside the viewing rectangle, the device ignores this value and moves on to the next x-value. The machine connects each point to the preceding plotted point to form a representation of the graph of the equation.

EXAMPLE 1 | Choosing an Appropriate Viewing Rectangle

Graph the equation $y = x^2 + 3$ in an appropriate viewing rectangle.

SOLUTION Let's experiment with different viewing rectangles. We start with the viewing rectangle $[-2, 2]$ by $[-2, 2]$, so we set

`Xmin` $= -2$	`Ymin` $= -2$
`Xmax` $= 2$	`Ymax` $= 2$

The resulting graph in Figure 2(a) (on the next page) is blank! This is because $x^2 \ge 0$, so $x^2 + 3 \ge 3$ for all x. Thus the graph lies entirely above the viewing rectangle, so this viewing rectangle is not appropriate. If we enlarge the viewing rectangle to $[-4, 4]$ by $[-4, 4]$, as in Figure 2(b), we begin to see a portion of the graph.

Now let's try the viewing rectangle $[-10, 10]$ by $[-5, 30]$. The graph in Figure 2(c) seems to give a more complete view of the graph. If we enlarge the viewing rectangle even further, as in Figure 2(d), the graph doesn't show clearly that the y-intercept is 3.

So the viewing rectangle $[-10, 10]$ by $[-5, 30]$ gives an appropriate representation of the graph.

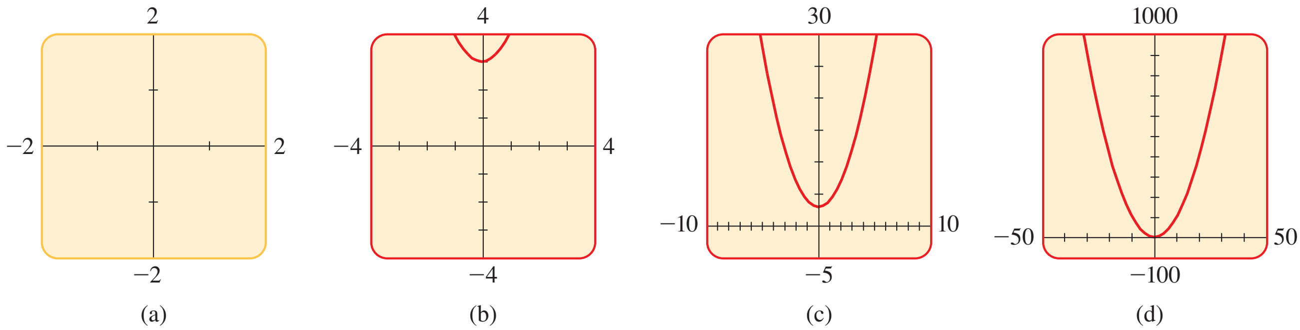


FIGURE 2 Graphs of $y = x^2 + 3$

EXAMPLE 2 | Graphing a Cubic Equation

Graph the equation $y = x^3 - 49x$.

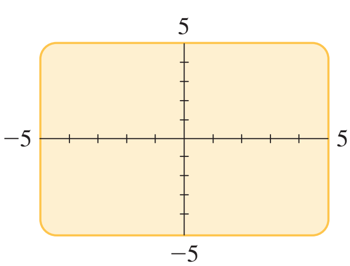


FIGURE 3

SOLUTION Let's experiment with different viewing rectangles. If we start with the viewing rectangle

$$[-5, 5] \text{ by } [-5, 5]$$

we get the graph in Figure 3. On most graphing calculators the screen appears to be blank, but it is not quite blank because the point $(0, 0)$ has been plotted. It turns out that for all other x-values that the calculator chooses, the corresponding y-value is greater than 5 or less than -5, so the resulting point on the graph lies outside the viewing rectangle.

Let's use the zoom-out feature of a graphing calculator to change the viewing rectangle to the larger rectangle

$$[-10, 10] \text{ by } [-10, 10]$$

In this case we get the graph shown in Figure 4(a), which appears to consist of vertical lines, but we know that cannot be true. If we look carefully while the graph is being drawn, we see that the graph leaves the screen and reappears during the graphing process. That indicates that we need to see more of the graph in the vertical direction, so we change the viewing rectangle to

$$[-10, 10] \text{ by } [-100, 100]$$

The resulting graph is shown in Figure 4(b). It still doesn't reveal all the main features of the equation. It appears that we need to see still more in the vertical direction. So we try the viewing rectangle

$$[-10, 10] \text{ by } [-200, 200]$$

The resulting graph is shown in Figure 4(c). Now we are more confident that we have arrived at an appropriate viewing rectangle. In Chapter 3, where third-degree polynomials are discussed, we learn that the graph shown in Figure 4(c) does indeed reveal all the main features of the equation.

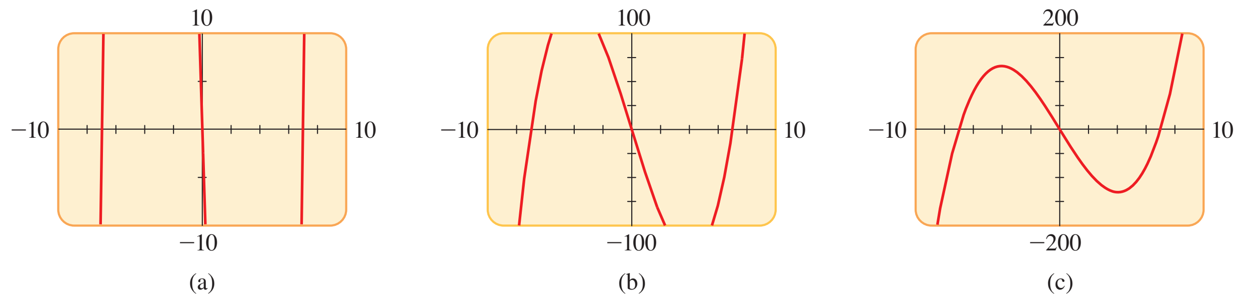


FIGURE 4 Graphing of $y = x^3 - 49x$

Interpreting the Screen Image

Once a graph of an equation has been obtained by using a graphing calculator, we sometimes need to interpret what the graph means in terms of the equation. Certain limitations of the calculator can cause it to produce graphs that are inaccurate or need further modifications. Here are two examples.

EXAMPLE 3 | Two Graphs on the Same Screen

Graph the equations $y = 3x^2 - 6x + 1$ and $y = 0.23x - 2.25$ together in the viewing rectangle $[-1, 3]$ by $[-2.5, 1.5]$. Do the graphs intersect in this viewing rectangle?

SOLUTION Figure 5(a) shows the essential features of both graphs. One is a parabola, and the other is a line. It looks as if the graphs intersect near the point $(1, -2)$. However, if we zoom in on the area around this point as shown in Figure 5(b), we see that although the graphs almost touch, they do not actually intersect.

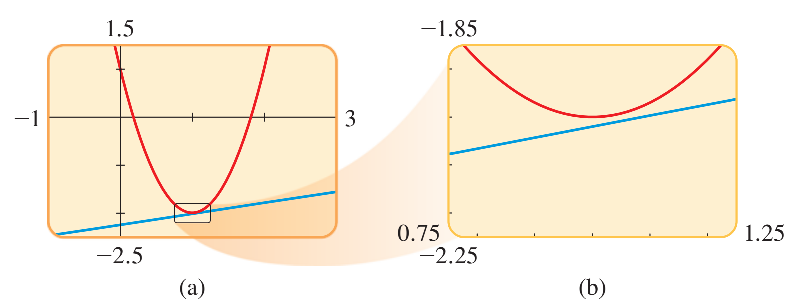


FIGURE 5

You can see from Examples 1, 2, and 3 that the choice of a viewing rectangle makes a big difference in the appearance of a graph. If you want an overview of the essential features of a graph, you must choose a relatively large viewing rectangle to obtain a global view of the graph. If you want to investigate the details of a graph, you must zoom in to a small viewing rectangle that shows just the feature of interest.

EXAMPLE 4 | Avoiding Extraneous Lines in Graphs

Graph the equation $y = \dfrac{1}{1 - x}$.

SOLUTION Figure 6(a) shows the graph produced by a graphing calculator with viewing rectangle

$$[-5, 5] \text{ by } [-5, 5]$$

In connecting successive points on the graph, the calculator produced a steep line segment from the top to the bottom of the screen. That line segment should not be part of the graph. The right side of the equation is not defined for $x = 1$, so the calculator connects points on the graph to the left and right of $x = 1$, and this produces the extraneous line segment. We can get rid of the extraneous near-vertical line by changing the graphing mode on the calculator. If we choose the DOT mode, in which points on the graph are not connected, we get the better graph in Figure 6(b). The graph in Figure 6(b) has gaps so we have to interpret it as having the points connected but without creating the extraneous line segment.

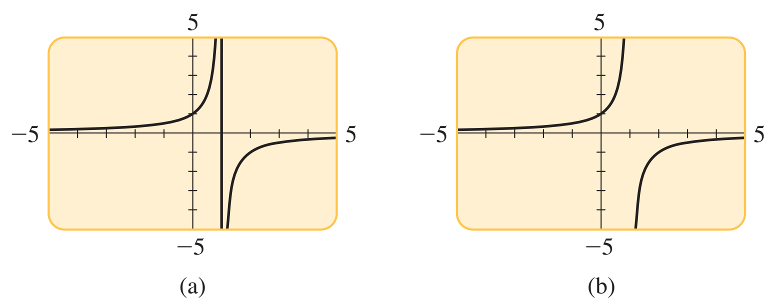


FIGURE 6 Graphing $y = \dfrac{1}{1 - x}$

▼ Graphing Equations That Are Not Functions

Most graphing calculators can only graph equations in which y is isolated on one side of the equal sign. Such equations are ones that represent functions (see page 185). The next example shows how to graph equations that don't have this property.

EXAMPLE 5 | Graphing a Circle

Graph the circle $x^2 + y^2 = 1$.

SOLUTION We first solve for y, to isolate it on one side of the equal sign:

$$y^2 = 1 - x^2 \qquad \text{Subtract } x^2$$

$$y = \pm\sqrt{1 - x^2} \qquad \text{Take square roots}$$

The graph in Figure 7(c) looks somewhat flattened. Most graphing calculators allow you to set the scales on the axes so that circles really look like circles. On the TI-83, from the `ZOOM` menu, choose `ZSquare` to set the scales appropriately. (On the TI-86 the command is `Zsq`.)

Therefore the circle is described by the graphs of *two* equations:

$$y = \sqrt{1 - x^2} \qquad \text{and} \qquad y = -\sqrt{1 - x^2}$$

The first equation represents the top half of the circle (because $y \geq 0$), and the second represents the bottom half of the circle (because $y \leq 0$). If we graph the first equation in the viewing rectangle $[-2, 2]$ by $[-2, 2]$, we get the semicircle shown in Figure 7(a). The graph of the second equation is the semicircle in Figure 7(b). Graphing these semicircles together on the same viewing screen, we get the full circle in Figure 7(c).

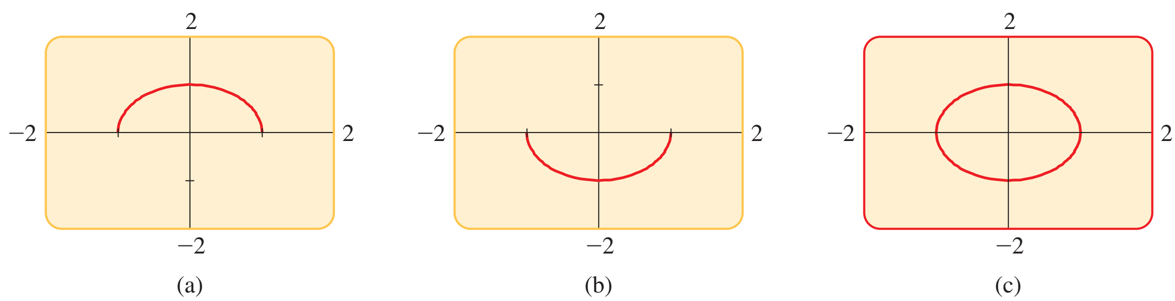


FIGURE 7 Graphing the equation $x^2 + y^2 = 1$

B EXERCISES

1–6 ■ Use a graphing calculator or computer to decide which viewing rectangle (a)–(d) produces the most appropriate graph of the equation.

1. $y = x^4 + 2$
(a) $[-2, 2]$ by $[-2, 2]$
(b) $[0, 4]$ by $[0, 4]$
(c) $[-8, 8]$ by $[-4, 40]$
(d) $[-40, 40]$ by $[-80, 800]$

2. $y = x^2 + 7x + 6$
(a) $[-5, 5]$ by $[-5, 5]$
(b) $[0, 10]$ by $[-20, 100]$
(c) $[-15, 8]$ by $[-20, 100]$
(d) $[-10, 3]$ by $[-100, 20]$

3. $y = 100 - x^2$
(a) $[-4, 4]$ by $[-4, 4]$
(b) $[-10, 10]$ by $[-10, 10]$
(c) $[-15, 15]$ by $[-30, 110]$
(d) $[-4, 4]$ by $[-30, 110]$

4. $y = 2x^2 - 1000$
(a) $[-10, 10]$ by $[-10, 10]$
(b) $[-10, 10]$ by $[-100, 100]$
(c) $[-10, 10]$ by $[-1000, 1000]$
(d) $[-25, 25]$ by $[-1200, 200]$

5. $y = 10 + 25x - x^3$
(a) $[-4, 4]$ by $[-4, 4]$
(b) $[-10, 10]$ by $[-10, 10]$
(c) $[-20, 20]$ by $[-100, 100]$
(d) $[-100, 100]$ by $[-200, 200]$

6. $y = \sqrt{8x - x^2}$
(a) $[-4, 4]$ by $[-4, 4]$
(b) $[-5, 5]$ by $[0, 100]$
(c) $[-10, 10]$ by $[-10, 40]$
(d) $[-2, 10]$ by $[-2, 6]$

7–18 ■ Determine an appropriate viewing rectangle for the equation, and use it to draw the graph.

7. $y = 100x^2$
8. $y = -100x^2$
9. $y = 4 + 6x - x^2$
10. $y = 0.3x^2 + 1.7x - 3$
11. $y = \sqrt[4]{256 - x^2}$
12. $y = \sqrt{12x - 17}$
13. $y = 0.01x^3 - x^2 + 5$
14. $y = x(x + 6)(x - 9)$
15. $y = \dfrac{1}{x^2 - 2x}$
16. $y = \dfrac{x}{x^2 + 25}$
17. $y = 1 + |x - 1|$
18. $y = 2x - |x^2 - 5|$

19–26 ■ Do the graphs intersect in the given viewing rectangle? If they do, how many points of intersection are there?

19. $y = -3x^2 + 6x - \frac{1}{2}$, $y = \sqrt{7 - \frac{7}{12}x^2}$; $[-4, 4]$ by $[-1, 3]$

20. $y = \sqrt{49 - x^2}$, $y = \frac{1}{5}(41 - 3x)$; $[-8, 8]$ by $[-1, 8]$

21. $y = 6 - 4x - x^2$, $y = 3x + 18$; $[-6, 2]$ by $[-5, 20]$

22. $y = x^3 - 4x$, $y = x + 5$; $[-4, 4]$ by $[-15, 15]$

23. Graph the circle $x^2 + y^2 = 9$ by solving for y and graphing two equations as in Example 3.

24. Graph the circle $(y - 1)^2 + x^2 = 1$ by solving for y and graphing two equations as in Example 3.

25. Graph the equation $4x^2 + 2y^2 = 1$ by solving for y and graphing two equations corresponding to the negative and positive square roots. (This graph is called an *ellipse*.)

26. Graph the equation $y^2 - 9x^2 = 1$ by solving for y and graphing the two equations corresponding to the positive and negative square roots. (This graph is called a *hyperbola*.)

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APPENDIX C Using the TI-83/84 Graphing Calculator

A TI-83 or TI-84 graphing calculator is a powerful tool that can draw graphs as well as do many of the other calculations that we study in this book. Here we give some of the basic calculator operations. When you master these you'll be able to easily use the calculator to do many other tasks.

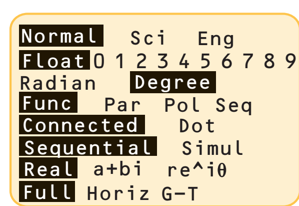
1. Set the Mode

Make sure the calculator is in the "mode" that you want.

STEP 1 Find the Mode Menu To get the mode menu, press the `MODE` key.

STEP 2 Make the Appropriate Selections Use the cursor to highlight a selection, then press `ENTER` to make the selection. For example, choose `Normal` for decimal notation, `Sci` for scientific notation, `Func` to graph functions in rectangular coordinates, `Real` to work with real numbers, or `a+bi` to work with complex numbers. The standard choices are shown here.

NOTE: Press `2nd` `QUIT` to exit this menu (or to exit any other menu).



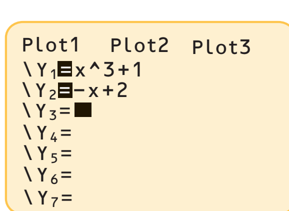
2. Graph an Equation

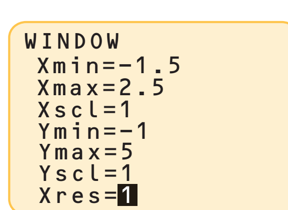
To graph one or more equations on the same screen, first express each equation in function form, with y on one side of the equation (see page 185). Let's graph $y = x^3 + 1$ and $y = -x + 2$.

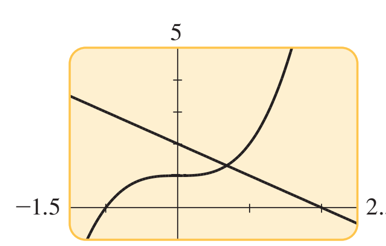
STEP 1 Enter the Equation Press the `Y=` key, and then enter the equations as shown.

STEP 2 Choose the Window Press the `WINDOW` key, and then enter the values for `Xmin`, `Xmax`, `Ymin`, and `Ymax` that you want.

STEP 3 Get the Graph To get the graph, press the `GRAPH` key.







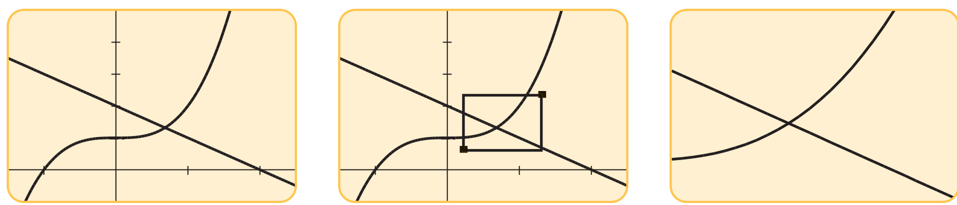
3. Zoom in on a Graph

To zoom in on a portion of a graph, first draw the graph(s) by following the steps in Part 2. With the graph(s) on the screen, follow these steps.

STEP 1 Choose the Zoom Menu Press the ZOOM key to obtain the zoom menu. Choose `ZBox`, and press ENTER. (You can experiment with other choices also.)

STEP 2 Draw the Zoom Box Move the cursor to the location of the bottom left corner of the rectangle (or box) that you want to zoom in on, then press ENTER. Then move the cursor to the location of the top right corner of the zoom box.

STEP 3 Zoom In Press ENTER to zoom in on the portion of the graph that is in the zoom box.



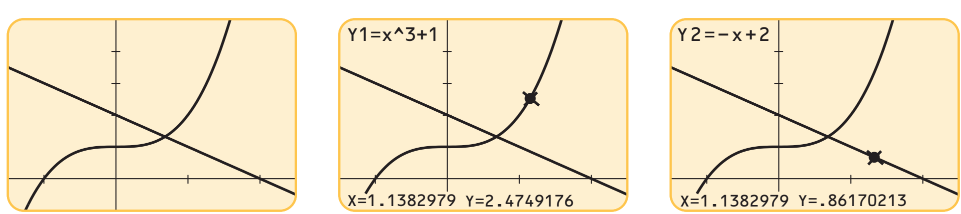
4. Trace a Graph

Once a graph has been drawn on the calculator screen, you can find the coordinates of any point on the graph.

STEP 1 Graph an Equation Graph an equation (or several equations) as in Part 2. Keep the graph(s) on the calculator screen.

STEP 2 Choose the Command Go to the trace command by pressing the TRACE key. A cursor (✱) appears on the screen.

STEP 3 Trace the Graph Move the cursor along the curve by using the left or right arrow keys. You can jump from one curve to another by using the up or down arrow keys. The numbers at the bottom of the screen give the coordinates of the location of the cursor.



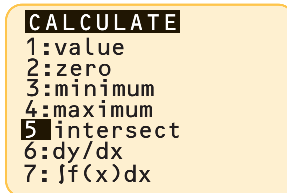
5. Find Points of Intersection of Two Graphs

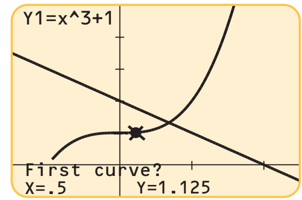
To find the point of intersection of the graphs of two equations, first graph the two equations on the same screen as in Part 2.

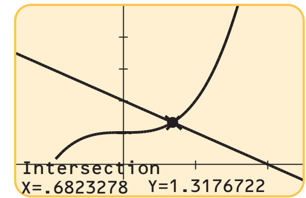
STEP 1 Choose the Calc Menu Press 2nd CALC to obtain the menu. Choose the `intersect` command, and press ENTER. (You can also experiment with the other commands on this menu.)

STEP 2 Choose the Two Curves Use the up and down keys to display the equations you have entered (they appear at the top of the screen). Select the first equation you want by pressing ENTER. Use the up and down keys again, and select the second equation. A cursor appears on one of the graphs. The numbers at the bottom of the screen give the coordinates of the cursor.

STEP 3 Get the Intersection Point Now `Guess?` appears on the screen. Move the cursor to a point near the point of intersection that you want to find (this is your guess). Press ENTER. The point of intersection is displayed at the bottom of the screen.







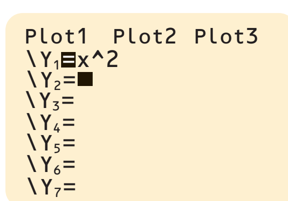
6. Make a Table of Values of a Function

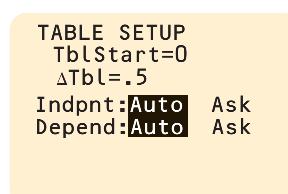
To make a table of values of a function, first enter the function. Let's work with the function $y = x^2$.

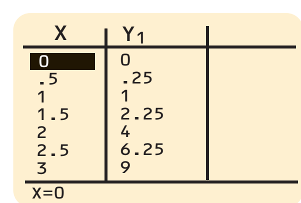
STEP 1 Enter the Function Press the Y= key, and then enter the definition of the function as shown.

STEP 2 Set the Table Properties Press 2nd TBLSET, and then select the value at which you want the table to start (`TblStart`) and the step size (`ΔTbl`).

STEP 3 Get the Table Press 2nd TABLE to obtain the table. Scroll up or down to see more of the table.







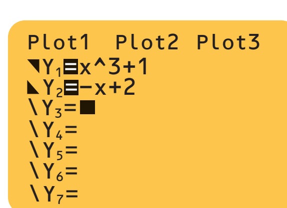
7. Graph an Inequality

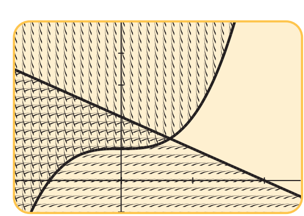
To graph an inequality in two variables, first enter the corresponding equation as in Part 2. We illustrate the process with the inequalities $y \geq x^3 + 1$ and $y \leq -x + 2$.

STEP 1 Enter the Equation(s) Enter the equation(s) as in Part 2 and set the window.

STEP 2 Choose the Inequalities For each equation, use the left arrow key to move the cursor to the very left of the equation. Press ENTER repeatedly to cycle through the inequality options (◥ and ◣). When the desired inequality appears, move on to the next equation.

STEP 3 Get the Graph To get the graph, press the GRAPH key.



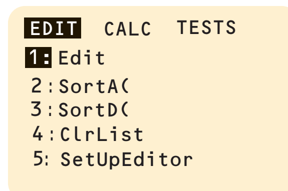


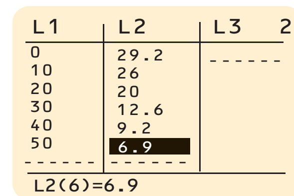
8. Enter Data

To enter data such as a list of one-variable data or a list of two-variable data into the calculator, we use the STAT menu.

STEP 1 Go to the Statistics Menu Press the STAT key. From the top menu choose EDIT, then 1:Edit, and then press ENTER.

STEP 2 Enter the Data Enter the data in one or more of the columns labeled L1, L2, L3, ... For example, for two-variable data enter the *x*-coordinates of the data points in L1 and the *y*-coordinates in L2.





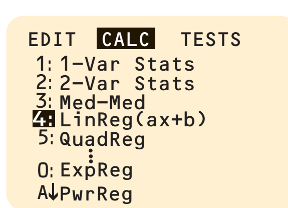
9. Find the Curve of Best Fit

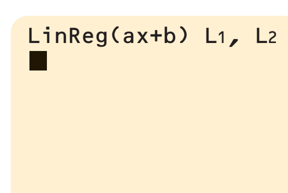
To find the curve that best fits a given set of two-variable data, we first enter the data.

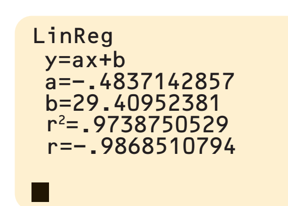
STEP 1 Enter the Data Enter the two-variable data in two columns, say L1 and L2, as in Part 8.

STEP 2 Choose the Regression Command Press the STAT key again. From the top menu choose CALC, then select the type of curve you want (LinReg(ax+b), QuadReg, ExpReg, PwrReg, . . .) and press ENTER.

STEP 3 Obtain the Regression Line Now select the columns in which you stored the data. For example, enter L1 and L2 separated by a comma, as in the middle graph. Note that the column names are located at 2nd L1 and 2nd L2. Press ENTER again. The regression equation with the values of the coefficients appears on the screen.







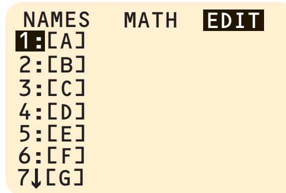
10. Enter a Matrix

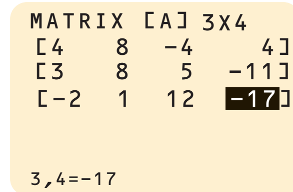
To enter a matrix into the calculator, we start with the MATRIX menu.

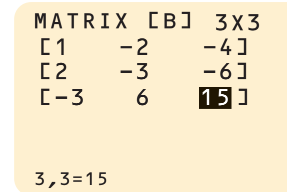
STEP 1 Go to the Matrix Menu Press the 2nd MATRIX key to obtain the matrix menu. From the top menu choose EDIT, then select a matrix name (e.g., [A]), and press ENTER.

STEP 2 Enter the Matrix Now enter the dimension of the matrix you want, (e.g., 3×4), and press ENTER. A matrix with the desired dimension appears. Key in the entries of the matrix, pressing ENTER after inputting each entry. Press 2nd QUIT when you have completed entering the matrix.

STEP 3 Enter Another Matrix Press the 2nd MATRIX key again, and repeat the process in Step 2 to enter another matrix [B].







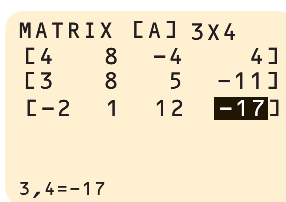
11. Find the (Reduced) Row-Echelon Form of a Matrix

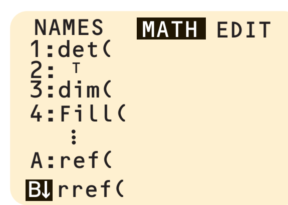
To find the row-echelon form or the reduced row echelon form of a matrix, we first enter the matrix.

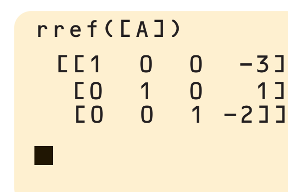
STEP 1 Enter the Matrix Enter a matrix as in Part 10.

STEP 2 Choose the Form Press the `2nd` `MATRIX` key again. From the top menu choose `MATH`, then select `rref` (or `ref`) and press `ENTER`. (You can also experiment with the other commands on this menu.) Press the `2nd` `MATRIX` key yet again. From the top menu choose `NAMES`, then select the name of the matrix you want (e.g., `[A]`).

STEP 3 Obtain the (Reduced) Row-Echelon Form You now have `rref([A])` on the screen. Press `ENTER` to obtain the reduced row-echelon form of the matrix you stored in `[A]`.







12. Perform Algebraic Operations on Matrices

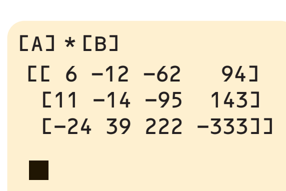
Before performing operations on matrices, store the matrices in the memory of the calculator with the names `[A]`, `[B]`, . . . as in Part 10.

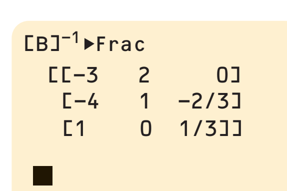
STEP 1 Select a Matrix by Name To enter the name of a matrix on the screen, go to `2nd` `MATRIX`. From the top menu choose `NAMES`, then select the name of the matrix you want (`[A]`, `[B]`, . . .) and press `ENTER`.

STEP 2 Choose the Operation To do algebraic operations on matrices, use the ordinary arithmetic operation keys `+`, `X` , or `X⁻¹`. To multiply or add matrices, enter `[A]*[B]` or `[A]+[B]`. For the inverse use the `X⁻¹` key to enter $[B]^{-1}$.

STEP 3 Obtain the Result On the screen you now have `[A]*[B]`, `[A]+[B]`, or $[B]^{-1}$. Press `ENTER` to obtain the result.

NOTE: To obtain the result of any calculation as a fraction (as opposed to a decimal), go to `MATH` and select ▶`Frac` (see the second screen below).





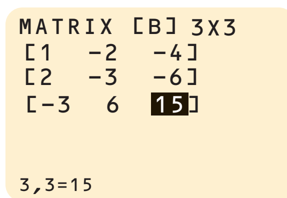
13. Find the Determinant of a Matrix

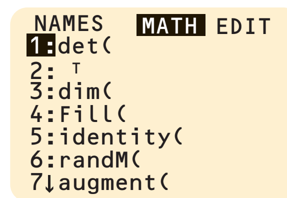
To find the determinant of a matrix, we must first store the matrix in the memory of the calculator with a name `[A]`, `[B]`, . . . as in Part 10.

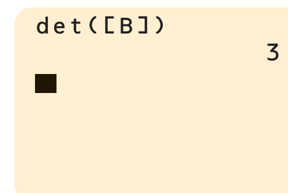
STEP 1 Select the Determinant Command Press `2nd` `MATRIX` to go to the matrix menu. From the top menu select `MATH`, then choose `det(`, and then press `ENTER`. The symbol `det(` appears on the screen.

STEP 2 Choose the Name of a Matrix To find the determinant of the matrix B, press `2nd` `MATRIX`. From the top menu choose `NAMES`, and then select `[B]`.

STEP 3 Obtain the Result On the screen you now have `det([B])`. Press `ENTER` to obtain the value of the determinant.







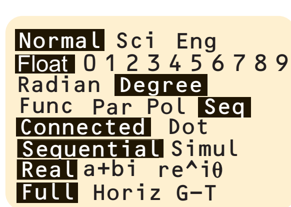
14. Find a Term of a Sequence

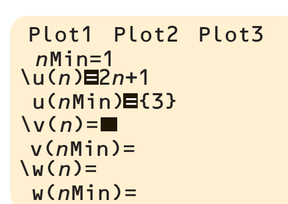
We can work with sequences on the calculator, but we must first put the calculator in the proper mode by following the instructions in Part 1.

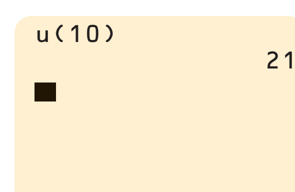
STEP 1 Select the Sequence Mode Press `MODE`, then select `seq` and press `ENTER`. This puts the calculator in sequence mode. Press `2nd` `QUIT` to exit the mode menu.

STEP 2 Enter the Sequence Press the `Y=` key, and then enter the definition of the sequence. For the sequence $a_n = 2n + 1$, enter `u(n)=2n+1`, as shown. We must also enter the minimum value of `n` (in this case `nMin=1`) and the first term of the sequence (in this case `u(nMin)={3}`).

STEP 3 Obtain Results To find a term of the sequence, say a_{10}, use the keypad to enter `u(10)`. Note that `u` is located at `2nd` `u` on the keypad.







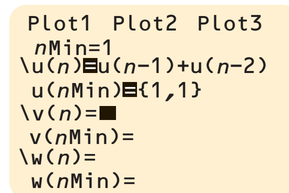
15. Find a Term of a Recursive Sequence

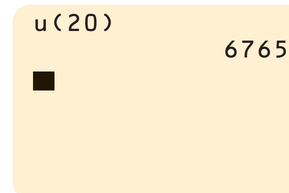
To find a term of a recursively defined sequence, first put the calculator in sequence mode. We find the 20th term of the Fibonacci sequence.

STEP 1 Select the Sequence Mode Put the calculator in sequence mode as in Part 14. Press `2nd` `QUIT` to exit the mode menu.

STEP 2 Enter the Sequence Press the `Y=` key, and then enter the definition of the sequence. For the Fibonacci sequence, enter `u(n)=u(n-1)+u(n-2)`, as shown. We must also enter the minimum value of `n` (in this case `nMin=1`) and the first two terms of the sequence (in this case `u(nMin)={1,1}`).

STEP 3 Obtain Results To find a term of the sequence, say, F_{20}, use the keypad to enter `u(20)`. Note that `u` is located at 2nd u on the keypad.





16. List Terms of a Sequence

To list the terms of a sequence, we use the LIST menu. We illustrate the process with the sequence $a_n = 1/n$ from $n = 1$ to $n = 5$.

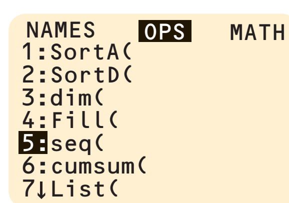
STEP 1 Get the Sequence Command Press 2nd LIST. From the top menu choose `OPS`, select `seq(`, and then press ENTER.

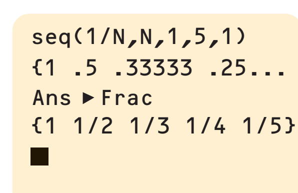
STEP 2 Define the Sequence Complete the `seq(` command as `seq(1/N,N,1,5,1)`. The entries have the following meaning: The formula is $1/N$, the variable is N, the starting point is 1, the ending point is 5, and the step size is 1.

NOTE: Get the letter `N` by pressing ALPHA N.

STEP 3 Obtain the List of Terms of the Sequence Press ENTER to obtain a list of the terms of the sequence.

NOTE: Use the ▶`Frac` command to obtain the result in fractions. (See the note in Part 12.)





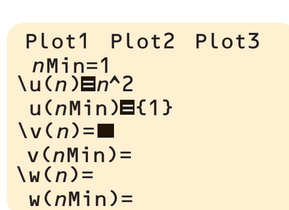
17. Make a Table of Values of a Sequence

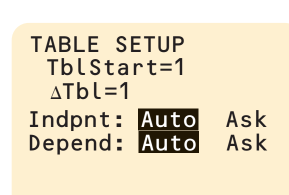
To make a table of values of a sequence, first put the calculator in sequence mode (see Part 1). Let's work with the sequence $u(n) = n^2$.

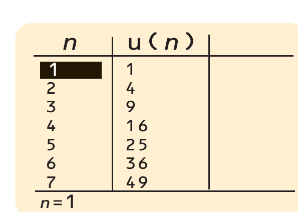
STEP 1 Enter the Sequence Press the Y= key, and then enter the definition of the sequence as shown.

STEP 2 Set the Table Properties Press 2nd TBLSET, and then select the value of n at which you want the table to start (`TblStart`) and the step size (`ΔTbl`) to be 1.

STEP 3 Get the Table Press 2nd TABLE to obtain the table. Scroll up or down to see more of the table.







18. Graph a Sequence

To graph a sequence, first put the calculator in sequence mode (see Part 1). Let's work with the sequence $u(n) = n/(n + 1)$.

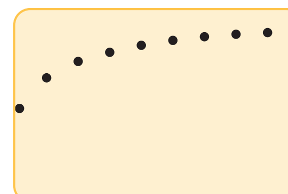
STEP 1 Enter the Sequence Press the `Y=` key, and then enter the definition of the sequence. To obtain a sequence graph where the dots are not connected, use the left arrow key to move the cursor to the very left of the equation. Press `ENTER` repeatedly to obtain the dots (∵) to the left of the equation, as shown.

STEP 2 Choose the Window Press the `WINDOW` key, and then enter the required values. Make sure you scroll down far enough to enter the values for `Xmin`, `Xmax`, `Ymin`, and `Ymax` that you want.

STEP 3 Get the Graph Press `GRAPH` to obtain the graph.

```
Plot1  Plot2  Plot3
 nMin=1
∵u(n)=n/(n+1)
 u(nMin)={.5}
\v(n)=■
 v(nMin)=
\w(n)=
 w(nMin)=
```

```
WINDOW
nMin=1
nMax=10
PlotStart=1
PlotStep=1
Xmin=1
Xmax=10
Xscl=1
↓Ymin=0■
```



19. Find a Partial Sum of a Sequence

To find a partial sum of a sequence, we use the LIST menu. We work with the sequence of odd numbers $a_n = 2n - 1$ from $n = 1$ to $n = 5$.

STEP 1 Find a Sum of a Sequence Press `2nd` `LIST`. From the top menu choose `MATH`, select `sum(`, and then press `ENTER`. Key in the sequence as in Part 16: `sum(seq(2N-1,N,1,5,1))`. Press `ENTER` to get the sum.

STEP 2 Find the Partial Sums Press `2nd` `LIST`. From the top menu choose `OPS`, select `cumsum(`, and then press `ENTER`. Key in the sequence as in Part 16: `cumsum(seq(2N-1,N,1,5,1))`. Press `ENTER` to get the sequence of partial sums.

```
NAMES    OPS    MATH
1:min(
2:max(
3:mean(
4:median(
5:sum(
6:prod(
7↓stdDev(
```

```
NAMES    OPS    MATH
1:SortA(
2:SortD(
3:dim(
4:Fill(
5:seq(
6:cumsum(
7↓List(
```

```
sum(seq(2N-1,N,1,5,1))
                    25
cumsum(seq(2N-1,N,1,
5,1))
    {1  4  9  16  25}
■
```

ANSWERS to Selected Exercises and Chapter Tests

PROLOGUE ■ PAGE P4

1. It can't go fast enough. **2.** 40% discount **3.** 427, $3n + 1$ **4.** 57 min **5.** No, not necessarily **6.** The same amount **7.** 2π **8.** The North Pole is one such point; there are infinitely many others near the South Pole.

CHAPTER P

SECTION P.1 ■ PAGE 5

1. 48 **2.** $C = 3.5x$ **3.** $T = \$7.20$ **5.** \$90 **7.** 245 mi **9.** **(a)** 30 mi/gal **(b)** 7 gal **11.** **(a)** 38 km^3 **(b)** 2 km^3 **13.** **(a)**

Depth (ft)	Pressure (lb/in^2)
0	14.7
10	19.2
20	23.7
30	28.2
40	32.7
50	37.2
60	41.7

(b) 34 ft

15. $N = 7w$ **17.** $A = \dfrac{a + b}{2}$ **19.** $C = 3.50x$ **21.** $d = 60t$ **23.** **(a)** \$15 **(b)** $C = 12 + n$ **(c)** 4 **25.** **(a)** **(i)** $C = 0.04x$ **(ii)** $C = 0.12x$ **(b)** **(i)** \$400 **(ii)** \$1200 **27.** **(a)** \$2 **(b)** $C = 1.00 + 0.10t$ **(c)** 12 min **(d)** $C = F + rt$

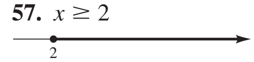
SECTION P.2 ■ PAGE 15

1. Answers may vary. Examples: **(a)** 2 **(b)** -3 **(c)** $\frac{3}{2}$ **(d)** $\sqrt{2}$ **2.** **(a)** ba; Commutative **(b)** $(a + b) + c$; Associative **(c)** $ab + ac$; Distributive **3.** **(a)** $\{x \mid 2 < x < 7\}$ **(b)** $(2, 7)$ **4.** A includes -2 and 5; B does not **5.** absolute value; positive **6.** distance **7.** **(a)** None **(b)** $-3, 0, -1000$ **(c)** $-3, 0, \frac{22}{7}, 3.14, 2.7\overline{6}, -1000, -\frac{2}{5}$ **(d)** $\sqrt{7}, -\pi$ **9.** Commutative Property for addition **11.** Associative Property for multiplication **13.** Distributive Property **15.** Associative Property for addition **17.** Distributive Property **19.** Commutative Property for multiplication **21.** $3 + x$ **23.** $4A + 4B$ **25.** $3x + 3y$ **27.** $8m$ **29.** $-5x + 10y$ **31.** **(a)** $\frac{17}{30}$ **(b)** $\frac{9}{20}$ **33.** **(a)** 3 **(b)** $\frac{25}{72}$ **35.** **(a)** $\frac{8}{3}$ **(b)** 6 **37.** **(a)** $<$ **(b)** $>$ **(c)** $=$ **39.** **(a)** True **(b)** False **41.** **(a)** False **(b)** True **43.** **(a)** False **(b)** False **45.** **(a)** $x > 0$ **(b)** $t < 4$ **(c)** $a \geq \pi$ **(d)** $-5 < x < \frac{1}{3}$ **(e)** $|p - 3| \leq 5$ **47.** **(a)** $\{1, 2, 3, 4, 5, 6, 7, 8\}$ **(b)** $\{2, 4, 6\}$ **49.** **(a)** $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ **(b)** $\{7\}$ **51.** **(a)** $\{x \mid x \leq 5\}$ **(b)** $\{x \mid -1 < x < 4\}$

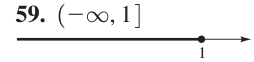
53. $-3 < x < 0$

55. $2 \leq x < 8$

57. $x \geq 2$



59. $(-\infty, 1]$



61. $(-2, 1]$

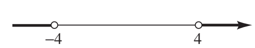
63. $(-1, \infty)$

65. **(a)** $[-3, 5]$ **(b)** $(-3, 5]$ **67.** **(a)** $[-4, -1)$ **(b)** $[1, 4)$ **69.** **(a)** $[-5, \infty)$ **(b)** $(-\infty, 5)$

71.

73.

75.



77. **(a)** 100 **(b)** 73 **79.** **(a)** 2 **(b)** -1 **81.** **(a)** 12 **(b)** 5 **83.** 5 **85.** **(a)** 15 **(b)** 24 **(c)** $\frac{67}{40}$ **87.** **(a)** $\frac{7}{9}$ **(b)** $\frac{13}{45}$ **(c)** $\frac{19}{33}$ **89.** Distributive Property **91.** **(a)** Yes, no **(b)** 6 ft

SECTION P.3 ■ PAGE 23

1. 5^6 **2.** Yes **3.** Base, exponent **4.** Add, 3^9 **5.** Subtract, 3^3 **6.** Multiply, 3^8 **7.** **(a)** $\frac{1}{2}$ **(b)** $\frac{1}{8}$ **(c)** 2 **(d)** 8 **8.** Scientific; 8.3×10^6; 3.27×10^{-5} **9.** **(a)** -9 **(b)** 9 **(c)** -8 **11.** **(a)** $\frac{1}{2}$ **(b)** $\frac{1}{8}$ **(c)** 16 **13.** **(a)** 625 **(b)** 9 **(c)** 64 **15.** **(a)** 1 **(b)** 1 **(c)** -1 **17.** **(a)** 25 **(b)** 1000 **(c)** $\frac{1}{9}$ **19.** **(a)** x^7 **(b)** $8y^6$ **(c)** y^5 **21.** **(a)** $\dfrac{1}{x^2}$ **(b)** $\dfrac{1}{w}$ **(c)** y^3 **23.** **(a)** a^6 **(b)** a^{18} **(c)** $20x^8$ **25.** **(a)** $6x^3y^5$ **(b)** $3a^7b^5$ **27.** **(a)** $\dfrac{25}{z}$ **(b)** $\dfrac{27a^{14}}{b^7}$ **29.** **(a)** $\dfrac{x^7}{y}$ **(b)** $\dfrac{y^3}{x}$ **31.** **(a)** $\dfrac{a^9}{8b^6}$ **(b)** $\dfrac{1}{2x^4y}$ **33.** **(a)** $\dfrac{a^{19}b}{c^9}$ **(b)** $\dfrac{v^{10}}{u^{11}}$ **35.** **(a)** $\dfrac{4a^8}{b^9}$ **(b)** $\dfrac{125}{x^6y^3}$ **37.** **(a)** $\dfrac{b^3}{3a}$ **(b)** $\dfrac{s^3}{q^7r^4}$ **39.** **(a)** 6.93×10^7 **(b)** 7.2×10^{12} **(c)** 2.8536×10^{-5} **(d)** 1.213×10^{-4} **41.** **(a)** 319,000 **(b)** 272,100,000 **(c)** 0.00000002670 **(d)** 0.000000009999

43. **(a)** 5.9×10^{12} mi **(b)** 4×10^{-13} cm **(c)** 3.3×10^{19} molecules **45.** 1.3×10^{-20} **47.** 1.429×10^{19} **49.** 7.4×10^{-14} **53.** 2.5×10^{13} mi **55.** 1.3×10^{21} L **57.** 4.03×10^{27} molecules **59.** \$470.26, \$636.64, \$808.08

SECTION P.4 ■ PAGE 29

1. $5^{1/3}$ **2.** $\sqrt{5}$ **3.** No **4.** $(4^{1/2})^3 = 8, (4^3)^{1/2} = 8$ **5.** $\dfrac{1}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ **6.** $\frac{2}{3}$ **7.** $7^{-1/2}$ **9.** $\sqrt[3]{4^2}$ **11.** $5^{3/5}$ **13.** $\sqrt[5]{a^2}$ **15.** $y^{4/3}$ **17.** **(a)** 4 **(b)** 2 **(c)** $\frac{1}{2}$ **19.** **(a)** $6\sqrt[3]{2}$ **(b)** $\dfrac{\sqrt{2}}{3}$ **(c)** $\dfrac{3\sqrt{3}}{2}$ **21.** **(a)** 14 **(b)** 4 **(c)** 6 **23.** **(a)** 6 **(b)** 4 **(c)** $\frac{1}{4}$ **25.** $|x|$ **27.** $2y\sqrt[5]{y}$ **29.** $2x^2$ **31.** $x\sqrt[4]{y}$ **33.** $6|r|t^2$ **35.** $ab\sqrt[5]{ab^2}$ **37.** $2|x|$ **39.** $7\sqrt{2}$ **41.** $2\sqrt{5}$ **43.** $\sqrt[3]{4}$ **45.** $(3a - 1)\sqrt{a}$ **47.** $(x + 2)\sqrt[3]{x}$ **49.** $(2a - 3)\sqrt[3]{2a^2}$ **51.** **(a)** 2 **(b)** -5 **(c)** $\frac{1}{3}$ **53.** **(a)** 4 **(b)** $\frac{3}{2}$ **(c)** $\frac{8}{27}$ **55.** **(a)** 5 **(b)** $\sqrt[5]{3}$ **(c)** 4 **57.** 5 **59.** 14 **61.** **(a)** x^2 **(b)** y^2 **63.** **(a)** $16b^{3/4}$ **(b)** $45a^2$ **65.** **(a)** $w^{5/3}$ **(b)** $4s^{9/2}$ **67.** **(a)** $4a^4b$ **(b)** $8a^9b^{12}$ **69.** **(a)** $\dfrac{1}{4y^2}$ **(b)** $\dfrac{1}{u^{4/3}v^2}$ **71.** **(a)** $\dfrac{1}{x}$ **(b)** $\dfrac{8y^8}{x^2}$ **73.** $x^{5/9}$ **75.** $y^{3/2}$ **77.** $10x^{7/12}$ **79.** $2st^{11/6}$ **81.** x **83.** $\dfrac{x^{1/4}y^{1/4}}{2}$ **85.** $\dfrac{4u}{v^2}$ **87.** $y^{1/2}$ **89.** **(a)** $\dfrac{\sqrt{6}}{6}$ **(b)** $\dfrac{3\sqrt{2}}{2}$ **(c)** $3\sqrt{3}$ **91.** **(a)** $\dfrac{\sqrt[3]{2}}{2}$ **(b)** $\dfrac{\sqrt[4]{27}}{3}$ **(c)** $4\sqrt[5]{16}$ **93.** **(a)** $\dfrac{\sqrt[3]{x^2}}{x}$ **(b)** $\dfrac{\sqrt[6]{x}}{x}$ **(c)** $\dfrac{\sqrt[7]{x^4}}{x}$ **95.** 41.3 mi **97.** **(a)** Yes **(b)** 3292 ft^2

SECTION P.5 ■ PAGE 35

1. (a), (d), (f) **2.** like, $11x^2 + x + 5$ **3.** like, $x^3 + 8x^2 - 5x + 2$ **4.** FOIL, $x^2 + 5x + 6$ **5.** $A^2 + 2AB + B^2$, $4x^2 + 12x + 9$ **6.** $A^2 - B^2$, $25 - x^2$ **7.** Trinomial; $x^2, -3x, 7$; 2 **9.** Monomial; -8; 0 **11.** Four terms; $-x^4, x^3, -x^2, x$; 4 **13.** $9x + 4$ **15.** $x^2 + 3x - 3$ **17.** $7x + 5$ **19.** $x^3 + 3x^2 - 6x + 11$ **21.** $2x^2 - 2x$ **23.** $x^3 + 3x^2$ **25.** $t^2 + 4$ **27.** $7r^3 - 3r^2 - 9r$ **29.** $2x^4 - x^3 + x^2$ **31.** $x^2 + 2x - 15$ **33.** $2s^2 + 15s + 18$ **35.** $21t^2 - 26t + 8$ **37.** $6x^2 + 7x - 5$ **39.** $2x^2 + 5xy - 3y^2$ **41.** $6r^2 - 19rs + 10s^2$ **43.** $x^2 + 10x + 25$ **45.** $9y^2 - 6y + 1$ **47.** $4u^2 + 4uv + v^2$ **49.** $4x^2 + 12xy + 9y^2$ **51.** $x^4 + 2x^2 + 1$ **53.** $x^2 - 25$ **55.** $9x^2 - 16$ **57.** $x^2 - 9y^2$ **59.** $x - 4$ **61.** $y^3 + 6y^2 + 12y + 8$ **63.** $1 - 6r + 12r^2 - 8r^3$ **65.** $x^3 + 4x^2 + 7x + 6$ **67.** $2x^3 - 7x^2 + 7x - 5$ **69.** $x^{3/2} - x$ **71.** $y + y^2$ **73.** $x^4 + 2x^2y^2 + y^4$ **75.** $x^4 - a^4$ **77.** $a - b^2$ **79.** $1 - x^{4/3}$ **81.** $-x^4 + x^2 - 2x + 1$ **83.** $4x^2 + 4xy + y^2 - 9$ **85.** **(b)** $4x^3 - 32x^2 + 60x$; 3 **(c)** 32, 24 **87.** **(a)** $2000r^3 + 6000r^2 + 6000r + 2000$; 3 **(b)** \$2122.42, \$2185.45, \$2282.33, \$2382.03, \$2662.00

SECTION P.6 ■ PAGE 41

1. **(a)** 3; $2x^5, 6x^4, 4x^3$ **(b)** $2x^3$; $2x^3(x^2 + 3x + 2)$ **2.** 10, 7; 2, 5; $(x + 2)(x + 5)$ **3.** $(A + B)(A - B)$; $(2x + 5)(2x - 5)$ **4.** $(A + B)^2$; $(x + 5)^2$ **5.** $5(a - 4)$ **7.** $2x(-x^2 + 8)$ **9.** $xy(2x - 6y + 3)$ **11.** $(y - 6)(y + 9)$ **13.** $(x - 1)(x + 3)$ **15.** $(x + 5)(x - 3)$ **17.** $(3x - 1)(x - 5)$ **19.** $(3x + 4)(3x + 8)$ **21.** $(x - 5)(x + 5)$ **23.** $(7 - 2z)(7 + 2z)$ **25.** $(4y - z)(4y + z)$ **27.** $(x + 3 - y)(x + 3 + y)$ **29.** $(x + 5)^2$ **31.** $(z - 6)^2$ **33.** $(2t - 5)^2$ **35.** $(3u - v)^2$ **37.** $(x + 3)(x^2 - 3x + 9)$ **39.** $(2a - 1)(4a^2 + 2a + 1)$ **41.** $(3x + y)(9x^2 - 3xy + y^2)$ **43.** $(u - v^2)(u^2 + uv^2 + v^4)$ **45.** $(x + 4)(x^2 + 1)$ **47.** $(2x + 1)(x^2 - 3)$ **49.** $(x + 1)(x^2 + 1)$ **51.** $x^{1/2}(x + 1)(x - 1)$ **53.** $x^{-3/2}(x + 1)^2$ **55.** $(x^2 + 3)(x^2 + 1)^{-1/2}$ **57.** $x^{1/3}(x - 2)^{-1/3}(-3x - 4)$ **59.** $6x(2x^2 + 3)$ **61.** $3y^3(2y - 5)$ **63.** $(x - 4)(x + 2)$ **65.** $(y - 3)(y - 5)$ **67.** $(2x + 3)(x + 1)$ **69.** $9(x - 5)(x + 1)$ **71.** $(3x + 2)(2x - 3)$ **73.** $(x + 6)(x - 6)$ **75.** $(7 + 2y)(7 - 2y)$ **77.** $(t - 3)^2$ **79.** $(2x + y)^2$ **81.** $(t + 1)(t^2 - t + 1)$ **83.** $(2x - 5)(4x^2 + 10x + 25)$ **85.** $x(x + 1)^2$ **87.** $x^2(x + 3)(x - 1)$ **89.** $x^2y^3(x + y)(x - y)$ **91.** $(x^2 - 2y)(x^4 + 2x^2y + 4y^2)$ **93.** $(y + 2)(y - 2)(y - 3)$ **95.** $(2x^2 + 1)(x + 2)$ **97.** $4ab$ **99.** $(x + 3)(x - 3)(x + 1)(x - 1)$ **101.** $3(x - 1)(x + 2)$ **103.** $y^4(y + 2)^3(y + 1)^2$ **105.** $(a + 2)(a - 2)(a + 1)(a - 1)$ **107.** $16x^2(x - 3)(5x - 9)$ **109.** $(2x - 1)^2(x + 3)^{-1/2}(7x + \frac{35}{2})$ **111.** $(x^2 + 3)^{-4/3}(\frac{1}{3}x^2 + 3)$ **113.** **(d)** $(a + b + c)(a + b - c)(a - b + c)(b - a + c)$

SECTION P.7 ■ PAGE 50

1. (a), (c) **2.** numerator; denominator; $\dfrac{x + 1}{x + 3}$ **3.** **(a)** False **(b)** True **4.** **(a)** numerators; denominators; $\dfrac{2x}{x^2 + 4x + 3}$ **(b)** invert; $\dfrac{3x + 6}{x^2 + 5x}$ **5.** **(a)** 3 **(b)** $x(x + 1)^2$ **(c)** $\dfrac{-2x^2 + 1}{x(x + 1)^2}$ **6.** **(a)** False **(b)** True **7.** $\mathbb{R}$ **9.** $\{x \mid x \neq 0\}$ **11.** $\{x \mid x \neq 4\}$ **13.** $\{x \mid x \neq -1, x \neq 2\}$ **15.** $\{x \mid x \geq -3\}$ **17.** $\dfrac{2}{x}$ **19.** $\dfrac{5y}{10 + y}$ **21.** $\dfrac{x + 2}{2(x - 1)}$ **23.** $\dfrac{1}{x + 2}$ **25.** $\dfrac{x + 2}{x + 1}$ **27.** $\dfrac{y}{y - 1}$ **29.** $\dfrac{x(2x + 3)}{2x - 3}$ **31.** $\dfrac{1}{4(x - 2)}$ **33.** $\dfrac{x + 3}{x - 3}$ **35.** $\dfrac{1}{t^2 + 9}$ **37.** $\dfrac{x + 4}{x + 1}$ **39.** $\dfrac{x + 5}{(2x + 3)(x + 4)}$ **41.** $\dfrac{(2x + 1)(2x - 1)}{(x + 5)^2}$ **43.** $x^2(x + 1)$ **45.** $\dfrac{x}{yz}$ **47.** $\dfrac{3(x + 2)}{x + 3}$ **49.** $\dfrac{3x + 7}{(x - 3)(x + 5)}$ **51.** $\dfrac{1}{(x + 1)(x + 2)}$ **53.** $\dfrac{3x + 2}{(x + 1)^2}$ **55.** $\dfrac{u^2 + 3u + 1}{u + 1}$ **57.** $\dfrac{2x + 1}{x^2(x + 1)}$ **59.** $\dfrac{2x + 7}{(x + 3)(x + 4)}$ **61.** $\dfrac{x - 2}{(x + 3)(x - 3)}$ **63.** $\dfrac{5x - 6}{x(x - 1)}$ **65.** $\dfrac{-5}{(x + 1)(x + 2)(x - 3)}$ **67.** $\dfrac{2x}{x - 1}$ **69.** $\dfrac{x - 1}{x + 1}$ **71.** $\dfrac{x^2(y - 1)}{y^2(x - 1)}$ **73.** $\dfrac{(x + 1)^2}{x^2 + 2x - 1}$

75. $\dfrac{4x-7}{(x-1)(x-2)(x+2)}$ **77.** $-xy$ **79.** $\dfrac{x^2+y^2}{xy(x+y)}$

81. $\dfrac{1}{1-x}$ **83.** $-\dfrac{1}{(1+x+h)(1+x)}$ **85.** $\dfrac{-2x-h}{x^2(x+h)^2}$

87. $\dfrac{1}{\sqrt{1-x^2}}$ **89.** $\dfrac{(x+2)^2(x-13)}{(x-3)^3}$ **91.** $\dfrac{x+2}{(x+1)^{3/2}}$

93. $\dfrac{2x+3}{(x+1)^{4/3}}$ **95.** $2+\sqrt{3}$ **97.** $\dfrac{2(\sqrt{7}-\sqrt{2})}{5}$

99. $\dfrac{y\sqrt{3}-y\sqrt{y}}{3-y}$ **101.** $\dfrac{-4}{3(1+\sqrt{5})}$ **103.** $\dfrac{r-2}{5(\sqrt{r}-\sqrt{2})}$

105. $\dfrac{1}{\sqrt{x^2+1}+x}$ **107.** True **109.** False **111.** False

113. True **115. (a)** $\dfrac{R_1R_2}{R_1+R_2}$ **(b)** $\frac{20}{3}\approx 6.7$ ohms

SECTION P.8 ■ PAGE 59

1. Solution **2.** $3x=6$; $x=2$; 2 **3.** (a), (c) **4. (a)** Equation contains a square of the variable. **(b)** Equation contains a square root of the variable. **(c)** Equation contains a square of the variable.
5. (a) True **(b)** False (because quantity could be 0) **(c)** False
6. cube, 5 **7. (a)** No **(b)** Yes **9. (a)** Yes **(b)** No
11. (a) No **(b)** Yes **13. (a)** Yes **(b)** No **15.** $-\frac{7}{3}$ **17.** -4
19. -8 **21.** -9 **23.** -3 **25.** 12 **27.** $-\frac{3}{4}$ **29.** $\frac{32}{9}$ **31.** 30
33. $\frac{1}{17}$ **35.** $\frac{4}{3}$ **37.** $\frac{14}{13}$ **39.** $-\frac{1}{3}$ **41.** $\frac{13}{6}$ **43.** -20 **45.** $\frac{13}{3}$
47. $\frac{3}{97}$ **49.** No solution **51.** No solution **53.** ± 7 **55.** $\pm 2\sqrt{6}$
57. $\pm 2\sqrt{2}$ **59.** No solution **61.** $-4, 0$ **63.** 3 **65.** ± 2
67. No solution **69.** $-5, 1$ **71.** 8 **73.** 125 **75.** -8

77. 3.13 **79.** 5.06 **81.** 43.66 **83.** 1.60 **85.** $M=\dfrac{12}{r}$

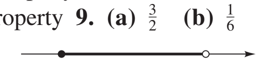
87. $R=\dfrac{PV}{nT}$ **89.** $w=\dfrac{(P-2l)}{2}$ **91.** $r=\pm\sqrt{\dfrac{3V}{\pi h}}$

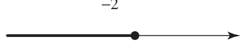
93. $r=\sqrt[3]{\dfrac{3V}{4\pi}}$ **95.** $i=-100\pm 100\sqrt{\dfrac{A}{P}}$ **97.** $x=\dfrac{2d-b}{a-2c}$

99. (a) 0.00055, 12.018 m **(b)** 234.375 kg/m^3
101. (a) 8.6 km/h **(b)** 14.7 km/h

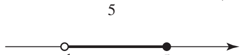
CHAPTER P REVIEW ■ PAGE 63

1. (a) $T=250-2x$ **(b)** 190 **(c)** 125
3. (a) rational, natural number, integer
(b) rational, integer **(c)** rational, natural number, integer
(d) irrational **(e)** rational, neither **(f)** rational, integer
5. Commutative Property for addition
7. Distributive Property **9. (a)** $\frac{3}{2}$ **(b)** $\frac{1}{6}$ **11. (a)** $\frac{9}{2}$ **(b)** $\frac{25}{32}$

13. $-2\le x<6$ 

15. $x\le 4$ 

17. $[5,\infty)$ 

19. $(-1,5]$ 

21. (a) $\{-1, 0, \frac{1}{2}, 1, 2, 3, 4\}$ **(b)** $\{1\}$ **23. (a)** $\{1, 2\}$ **(b)** $\{\frac{1}{2}, 1\}$
25. 3 **27.** 6 **29.** $\frac{1}{72}$ **31.** $\frac{1}{6}$ **33.** 11 **35.** -5
37. (a) $|3-5|=2$ **(b)** $|3-(-5)|=8$

39. (a) $7^{1/3}$ **(b)** $7^{4/5}$ **41. (a)** $x^{5/6}$ **(b)** $x^{9/2}$ **43.** $12x^5y^4$

45. $9x^3$ **47.** x^2y^2 **49.** $\dfrac{4r^{5/2}}{s^7}$ **51.** 7.825×10^{10}

53. 1.65×10^{-32} **55.** $2xy(x-3y)$ **57.** $(x-6)(x-3)$
59. $(3x+1)(x-1)$ **61.** $(4t+3)(t-4)$
63. $(5-4t)(5+4t)$ **65.** $ab^2(a+b)(a^2-ab+b^2)$
67. $(2x+y^2)(4x^2-2xy^2+y^4)$ **69.** $(x-2)(4x^2+3)$ **71.** x^2
73. $6x^2-21x+3$ **75.** $4a^4-4a^2b+b^2$
77. $8x^3+12x^2+6x+1$ **79.** $2x^3-6x^2+4x$

81. $\dfrac{x-3}{2x+3}$ **83.** $\dfrac{3(x+3)}{x+4}$ **85.** $\dfrac{x+1}{x-4}$ **87.** $\dfrac{x+1}{(x-1)(x^2+1)}$

89. $\dfrac{1}{x+1}$ **91.** $-\dfrac{1}{2x}$ **93.** $6x+3h-5$ **95.** $\dfrac{\sqrt{7}}{7}$

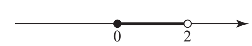
97. $6+6\sqrt{3}$ **99.** $\dfrac{x(\sqrt{x}-2)}{x-4}$

101. $\{x \mid x\ne -10\}$ **103.** $\{x \mid x\ge 0 \text{ and } x\ne 4\}$ **105.** No
107. Yes **109.** No **111.** 4 **113.** 5 **115.** $\frac{15}{2}$ **117.** -6
119. 0 **121.** No solution **123.** ± 12 **125.** 3 **127.** -5
129. -27 **131.** 625 **133.** $x=2A-y$ **135.** $t=\dfrac{11}{6J}$

CHAPTER P TEST ■ PAGE 66

1. (a) $C=9+1.50x$ **(b)** \$15 **2. (a)** Rational, natural number, integer **(b)** Irrational **(c)** Rational, integer
(d) Rational, integer **3. (a)** $\{0, 1, 5\}$ **(b)** $\{-2, 0, \frac{1}{2}, 1, 3, 5, 7\}$

4. (a) 

(b) Intersection $[0, 2)$ 

Union $[-4, 3]$ 

(c) $|-4-2|=6$

5. (a) -64 **(b)** 64 **(c)** $\frac{1}{64}$ **(d)** $\frac{1}{49}$ **(e)** $\frac{4}{9}$ **(f)** $\frac{1}{2}$ **(g)** $\frac{9}{16}$

(h) $\frac{1}{27}$ **6. (a)** 1.86×10^{11} **(b)** 3.965×10^{-7} **7. (a)** $\dfrac{a^3}{b^4}$

(b) $27y^{3/2}$ **(c)** $48a^5b^7$ **(d)** $6\sqrt{2}$ **(e)** $4x^2y^2\sqrt{3y}$ **(f)** $5x^3$

(g) $\dfrac{x}{9y^7}$ **8. (a)** $11x-2$ **(b)** $4x^2+7x-15$ **(c)** $a-b$

(d) $4x^2+12x+9$ **(e)** $x^3+6x^2+12x+8$ **(f)** x^4-9x^2
9. (a) $(2x-5)(2x+5)$ **(b)** $(2x-3)(x+4)$
(c) $(x-3)(x-2)(x+2)$ **(d)** $x(x+3)(x^2-3x+9)$

(e) $(2x-y-5)^2$ **(f)** $xy(x-2)(x+2)$ **10. (a)** $\dfrac{x+2}{x-2}$

(b) $\dfrac{x-1}{x-3}$ **(c)** $\dfrac{1}{x-2}$ **(d)** $-(x+y)$ **11. (a)** $3\sqrt[3]{2}$

(b) $2\sqrt{6}-3\sqrt{2}$ **12. (a)** 5 **(b)** $-\frac{5}{2}$ **(c)** 512 **(d)** $\frac{15}{2}$

(e) $\pm\sqrt{6}-1$ **13.** $c=\sqrt{\dfrac{E}{m}}$

FOCUS ON MODELING ■ PAGE 71

1. (a) $C=5800+265n$ **(b)** $C=575n$
(c)

n	Purchase	Rent
12	8,980	6,900
24	12,160	13,800
36	15,340	20,700
48	18,520	27,600
60	21,700	34,500
72	24,880	41,400

(d) 19 months **3. (a)** $C = 8000 + 22x$ **(b)** $R = 49x$
(c) $P = 27x - 8000$ **(d)** 297
5. (a) Design 2 **(b)** Design 1
7. (a)

Minutes used	Plan A	Plan B	Plan C
500	\$30	\$40	\$60
600	\$80	\$70	\$70
700	\$130	\$100	\$80
800	\$180	\$130	\$90
900	\$230	\$160	\$100
1000	\$280	\$190	\$110
1100	\$330	\$220	\$120

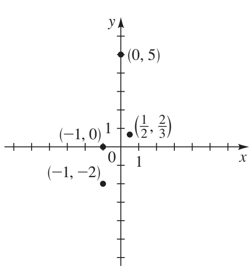
(b) $A = 30 + 0.50(x - 500)$, $B = 40 + 0.30(x - 500)$, $C = 60 + 0.10(x - 500)$ **(c)** 550 minutes: A = \$55, B = \$55, C = \$65; 975 minutes: A = \$267.50, B = \$182.50, C = \$107.50; 1200 minutes: A = \$380, B = \$250, C = \$130
(d) (i) 550 minutes (ii) 575 minutes

CHAPTER 1

SECTION 1.1 ■ PAGE 77

1. $(-2, 4)$ **2.** IV **3.** $\sqrt{(c-a)^2 + (d-b)^2}$; 10
4. $\left(\dfrac{a+c}{2}, \dfrac{b+d}{2}\right)$; $(4, 6)$
5. $A(5, 1)$, $B(1, 2)$, $C(-2, 6)$, $D(-6, 2)$, $E(-4, -1)$, $F(-2, 0)$, $G(-1, -3)$, $H(2, -2)$

7.



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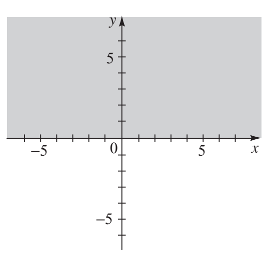

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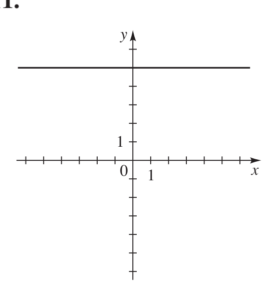

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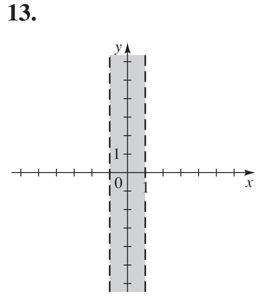

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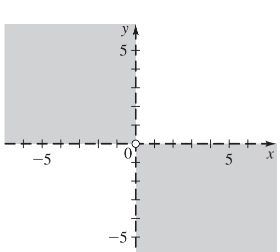

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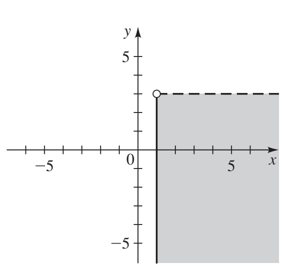

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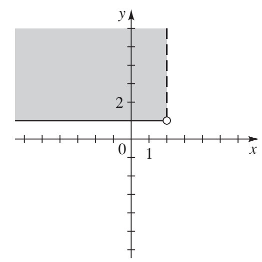

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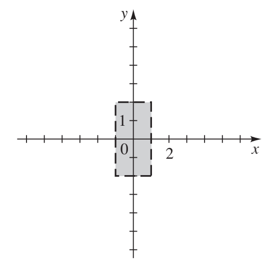
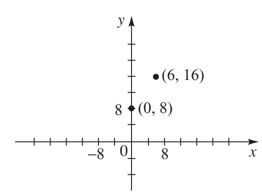
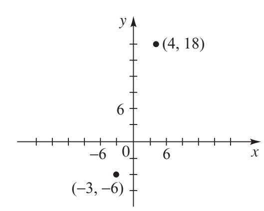


23. (a) $\sqrt{13}$ **(b)** $\left(\frac{3}{2}, 1\right)$ **25. (a)** 10 **(b)** $(1, 0)$
27. (a)



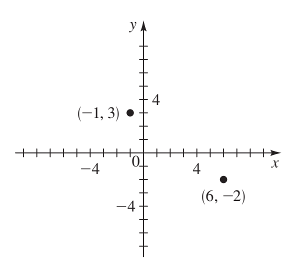
(b) 10 **(c)** $(3, 12)$

29. (a)



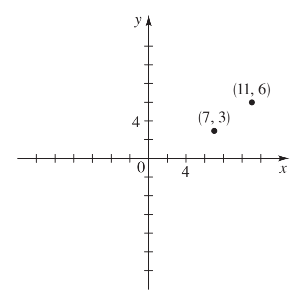
(b) 25 **(c)** $\left(\frac{1}{2}, 6\right)$

31. (a)



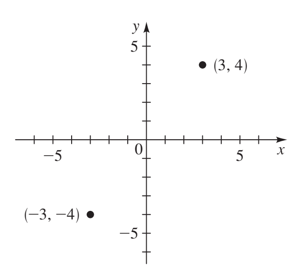
(b) $\sqrt{74}$ **(c)** $\left(\frac{5}{2}, \frac{1}{2}\right)$

33. (a)



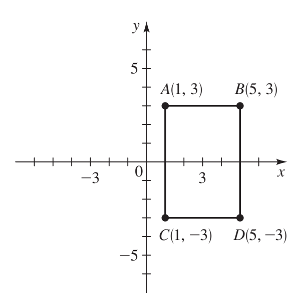
(b) 5 **(c)** $\left(9, \frac{9}{2}\right)$

35. (a)

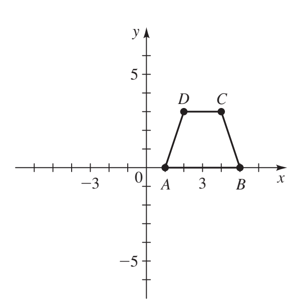


(b) 10 **(c)** $(0, 0)$

37. 24

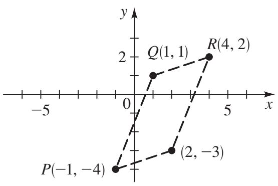


39. Trapezoid, area = 9

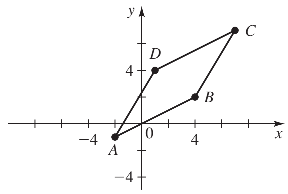


41. $A(6, 7)$ **43.** $Q(-1, 3)$ **47. (b)** 10 **51.** $(0, -4)$ **53.** $\left(1, \frac{7}{2}\right)$

55. $(2, -3)$



57. (a)



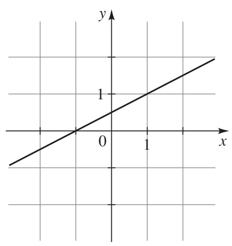
(b) $\left(\frac{5}{2}, 3\right), \left(\frac{5}{2}, 3\right)$

59. (a) 5 **(b)** 31; 25 **(c)** Points P and Q must be on either the same street or the same avenue. **61.** $(66, 45)$; the y-value of the midpoint is the pressure experienced by the diver at a depth of 66 ft.

SECTION 1.2 ■ PAGE 87

1. 2; 3; No

x	y
-2	$-\frac{1}{2}$
-1	0
0	$\frac{1}{2}$
1	1
2	$\frac{3}{2}$



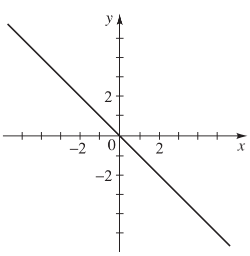
2. y; x; -1 **3.** x; y; $\frac{1}{2}$ **4.** $(1, 2)$; 3

5. (a) $(a, -b)$ **(b)** $(-a, b)$ **(c)** $(-a, -b)$

6. (a) -3 and 3; -1 and 2 **(b)** y-axis **7.** Yes, no, yes

9. No, yes, yes **11.** Yes, yes, yes

13.



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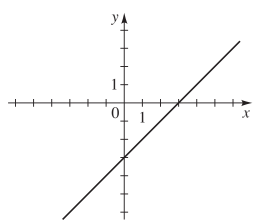

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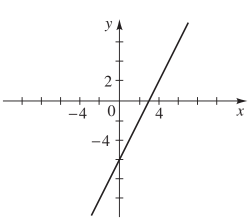

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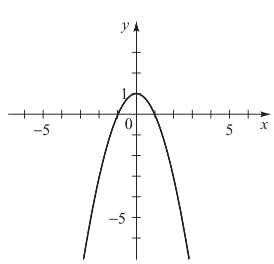

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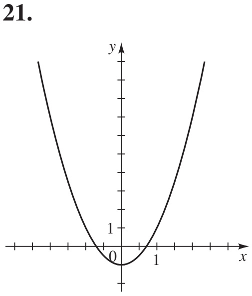

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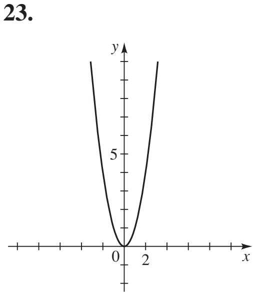

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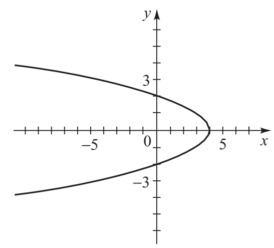

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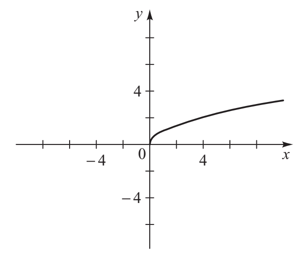

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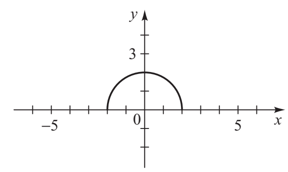

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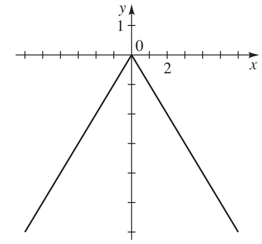

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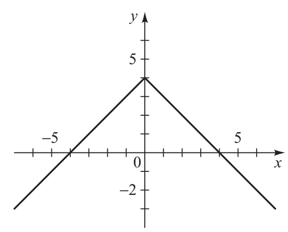

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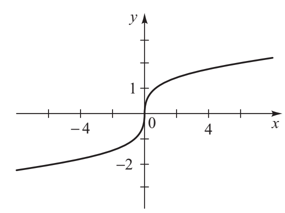

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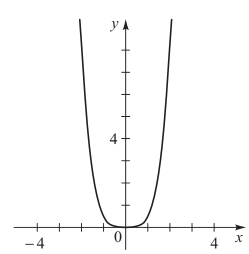

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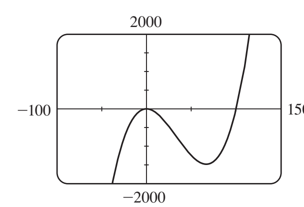

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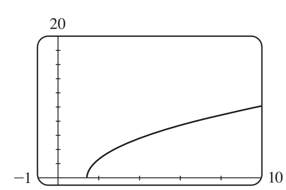
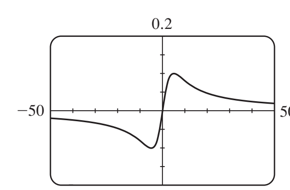


43.



45. x-intercept 3; y-intercept -3

47. x-intercepts ± 3; y-intercept -9

49. x-intercept $\frac{1}{2}$; y-intercept 1

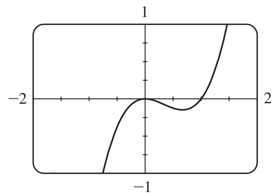
51. x-intercept -1; y-intercept 1

53. x-intercepts ± 2; y-intercepts ± 5

55. x-intercepts 0, 4; y-intercept 0

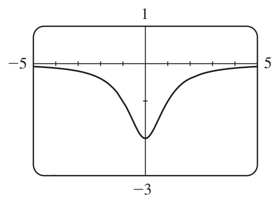
57. x-intercepts ± 2; y-intercepts ± 4

59. **(a)**



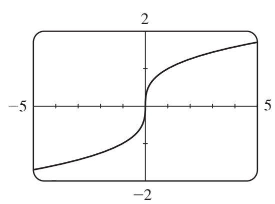
(b) x-intercepts 0, 1; y-intercept 0

61. **(a)**



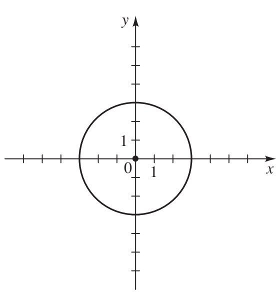
(b) No x-intercept; y-intercept -2

63. **(a)**

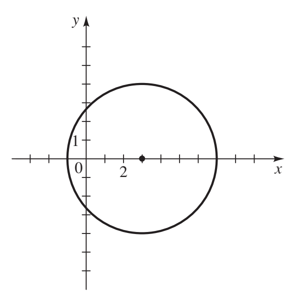


(b) x-intercept 0; y-intercept 0

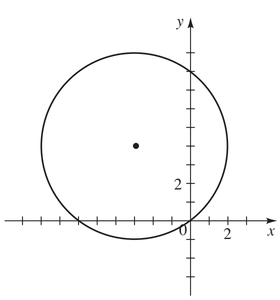
65. $(0, 0)$, 3



67. $(3, 0)$, 4



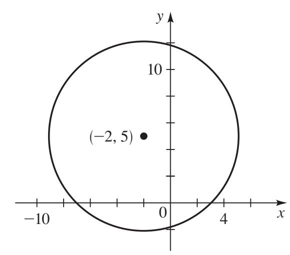
69. $(-3, 4)$, 5



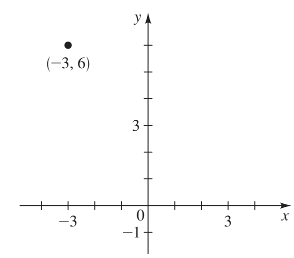
71. $(x - 2)^2 + (y + 1)^2 = 9$
73. $x^2 + y^2 = 65$

75. $(x - 2)^2 + (y - 5)^2 = 25$
77. $(x - 7)^2 + (y + 3)^2 = 9$
79. $(x + 2)^2 + (y - 2)^2 = 4$
81. $(1, -2)$, 2 **83.** $(2, -5)$, 4
85. $\left(-\frac{1}{2}, 0\right), \frac{1}{2}$ **87.** $\left(\frac{1}{4}, -\frac{1}{4}\right), \frac{1}{2}$

89.

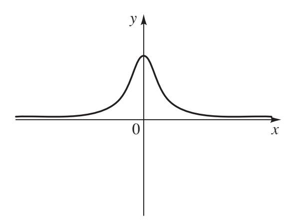


91.

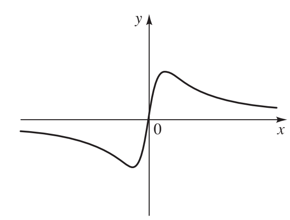


93. Symmetry about y-axis
95. Symmetry about origin
97. Symmetry about x-axis, y-axis, and origin

99.



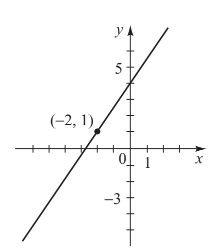
101.



103. **(a)** 14%, 6%, 2% **(b)** 1975–1976, 1978–1982
(c) Decrease, increase **(d)** 14%, 1%

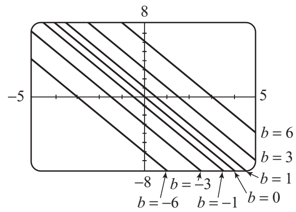
SECTION 1.3 ■ PAGE 100

1. y; x; 2 **2.** **(a)** 3 **(b)** 3 **(c)** $-\frac{1}{3}$ **3.** $y - 2 = 3(x - 1)$
4. 0; $y = 3$ **5.** Undefined; $x = 2$ **6.** 6; 4; $-\frac{2}{3}x + 4$
7. $-\frac{4}{3}$ **9.** $-\frac{1}{3}$ **11.** $-\frac{1}{2}$ **13.** $-\frac{9}{2}$ **15.** $-2, \frac{1}{2}, 3, -\frac{1}{4}$
17. $x + y - 4 = 0$ **19.** $3x - 2y - 6 = 0$
21. $3x - y - 2 = 0$ **23.** $5x - y - 7 = 0$
25. $2x - 3y + 19 = 0$ **27.** $5x + y - 11 = 0$
29. $8x + y + 11 = 0$ **31.** $3x - y - 3 = 0$ **33.** $y = 3$
35. $x = 2$ **37.** $3x - y - 1 = 0$ **39.** $y = 5$
41. $x + 2y + 11 = 0$ **43.** $x = -1$
45. $5x - 2y + 1 = 0$ **47.** $x - y + 6 = 0$
49. **(a)**

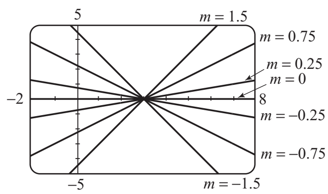


(b) $3x - 2y + 8 = 0$

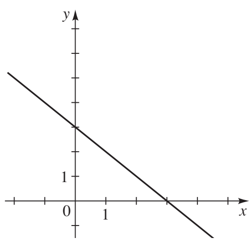
51. They all have the same slope.



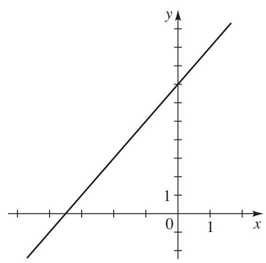
53. They all have the same x-intercept.



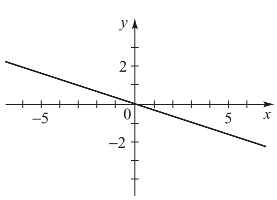
55. $-1, 3$



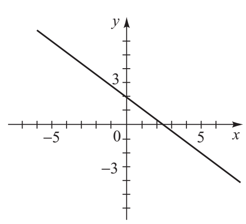
57. $2, 7$



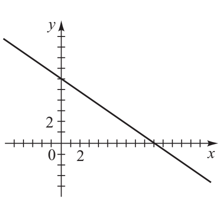
59. $-\frac{1}{3}, 0$



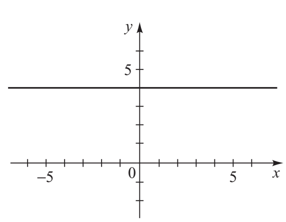
61. $-\frac{4}{5}, 2$



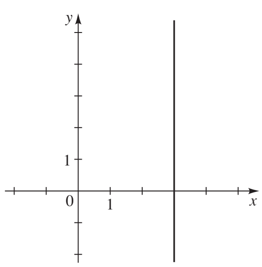
63. $-\frac{3}{5}, 6$



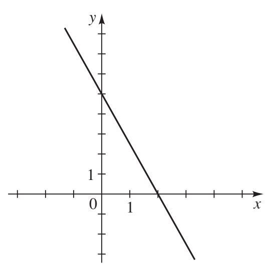
65. $0, 4$



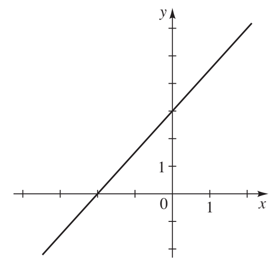
67. Undefined, none



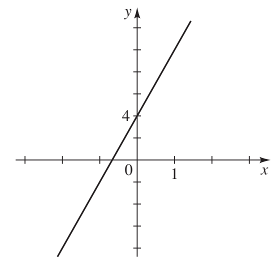
69. $2, 5$



71. $-2, 3$

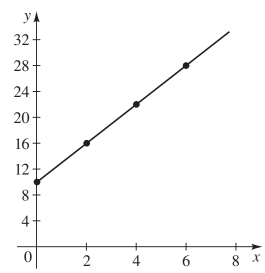


73. $-\frac{2}{3}, 4$



75. Parallel **77.** Perpendicular **79.** Neither

81. **(a)**



(b) $y = 3x + 10$

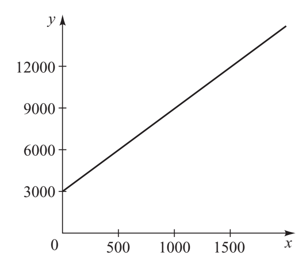
87. $x - y - 3 = 0$

89. **(b)** $4x - 3y - 24 = 0$

91. 16,667 ft

93. **(a)** 8.34; the slope represents the increase in dosage for a one-year increase in age. **(b)** 8.34 mg

95. **(a)**

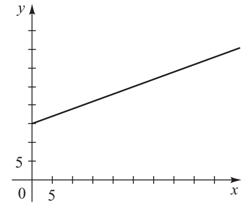


(b) The slope represents production cost per toaster; the y-intercept represents monthly fixed cost.

97. **(a)** $t = \frac{5}{24}n + 45$ **(b)** 76°F

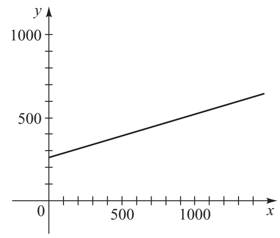
99. **(a)** $P = 0.434d + 15$, where P is pressure in lb/in^2 and d is depth in feet

(b)

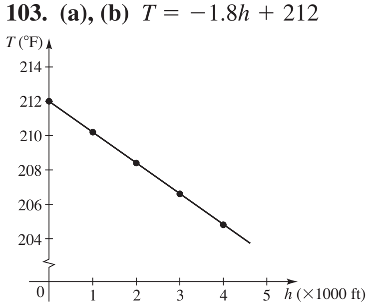


(c) The slope is the rate of increase in water pressure, and the y-intercept is the air pressure at the surface. **(d)** 196 ft

101. (a) $C = \frac{1}{4}d + 260$
(b) \$635
(c) The slope represents cost per mile.
(d) The y-intercept represents monthly fixed cost.



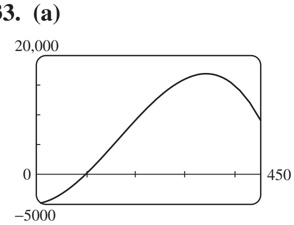
103. (a), (b) $T = -1.8h + 212$



(c) 177.2°F **(d)** The rate at which the boiling point of water changes as the elevation above sea level increases

SECTION 1.4 ■ PAGE 108

1. (a) x **(b)** $-1, 0, 1, 3$ **2.** $x = 1, 4$ **3.** -4 **5.** $\frac{5}{14}$
7. $\pm 4\sqrt{2} \approx \pm 5.7$ **9.** No solution **11.** 2.5, −2.5
13. $5 + 2\sqrt[4]{5} \approx 7.99, 5 - 2\sqrt[4]{5} \approx 2.01$
15. 3.00, 4.00 **17.** 1.00, 2.00, 3.00 **19.** 4 **21.** 1.62
23. 4, 9 **25.** −1.00, 0.00, 1.00 **27.** 2.55 **29.** −2.05, 0, 1.05
31. 2.27
33. (a)



(b) 101 cooktops

SECTION 1.5 ■ PAGE 116

2. principal; interest rate; time in years **3. (a)** x^2 **(b)** lw
(c) πr^2 **4.** 1.6 **5.** $\frac{1}{x}$ **6.** $r = \frac{d}{t}; t = \frac{d}{r}$ **7.** $3n + 3$
9. $3n + 6$ **11.** $\frac{160 + s}{3}$ **13.** $0.025x$ **15.** $5w^2$ **17.** $\frac{d}{55}$
19. $\frac{25}{x + 3}$ **21.** 400 miles **23.** 86%
25. \$9000 at $4\frac{1}{2}$% and \$3000 at 4%
27. 7.5% **29.** \$7400 **31.** 8 h **33.** 40 years old
35. 9 pennies, 9 nickels, 9 dimes **37.** 45 ft
39. 120 ft by 120 ft **41.** 8.94 in. **43.** 4 in. **45.** 5 m
47. 200 mL **49.** 18 g **51.** 0.6 L **53.** 35% **55.** 37 min 20 s
57. 3 h **59.** 4 h **61.** 500 mi/h **63.** 6.4 ft from the fulcrum
65. 120 ft **67.** 18 ft

SECTION 1.6 ■ PAGE 129

1. (a) $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ **(b)** $\frac{1}{2}, -1, -4; 4, -2$
2. (a) Factor into $(x + 1)(x - 5)$ and use the Zero-Product Property. **(b)** Add 5 to each side, then complete the square by adding 4 to both sides. **(c)** Insert coefficients into the Quadratic Formula. **3.** $b^2 - 4ac$; two distinct real; exactly one real; no real
5. $-3, 4$ **7.** $3, 4$ **9.** $-\frac{1}{3}, 2$ **11.** $-\frac{1}{2}, 3$ **13.** $-\frac{5}{6}, \frac{9}{2}$
15. $-\frac{4}{3}, \frac{1}{2}$ **17.** $-20, 25$ **19.** $-1 \pm \sqrt{6}$ **21.** $3 \pm 2\sqrt{5}$
23. $\frac{1}{2}, -\frac{3}{2}$ **25.** $-21, -1$ **27.** $-2 \pm \frac{\sqrt{14}}{2}$ **29.** $-\frac{7}{4} \pm \frac{\sqrt{17}}{4}$
31. $-3, 5$ **33.** $2, 5$ **35.** $-\frac{3}{2}, 1$ **37.** $-6 \pm 3\sqrt{7}$
39. $\frac{-3 \pm 2\sqrt{6}}{3}$ **41.** $\frac{3}{4}$ **43.** $-\frac{9}{2}, \frac{1}{2}$ **45.** No real solution
47. $\frac{\sqrt{5} \pm 1}{2}$ **49.** $\frac{8 \pm \sqrt{14}}{10}$ **51.** No real solution
53. −0.248, 0.259 **55.** No real solution
57. $t = \frac{-v_0 \pm \sqrt{v_0^2 + 2gh}}{g}$ **59.** $x = \frac{-2h \pm \sqrt{4h^2 + 2A}}{2}$
61. $s = \frac{-(a + b - 2c) \pm \sqrt{a^2 + b^2 + 4c^2 - 2ab}}{2}$
63. 2 **65.** 1 **67.** No real solution **69.** $\frac{-1}{a}$
71. $-2, 3$ **73.** -3 **75.** No solution **77.** $k = \pm 20$
79. 19 and 36 **81.** 25 ft by 35 ft **83.** 60 ft by 40 ft
85. 48 cm **87.** 13 in. by 13 in. **89.** 120 ft by 126 ft
91. 50 mi/h (or 240 mi/h) **93.** 6 km/h **95.** 4.24 s
97. (a) After 1 s and $1\frac{1}{2}$ s **(b)** Never **(c)** 25 ft **(d)** After $1\frac{1}{4}$ s
(e) After $2\frac{1}{2}$ s **99. (a)** After 17 yr, on Jan. 1, 2019
(b) After 18.612 yr, on Aug. 12, 2020 **101.** 30 ft; 120 ft by 180 ft
103. Irene 3 h, Henry $4\frac{1}{2}$ h **105.** 215,000 mi

SECTION 1.7 ■ PAGE 139

1. (a) factor **(b)** 0, 4 **2. (a)** $\sqrt{2x} = -x$ **(b)** $2x = x^2$
(c) 0, 2 **(d)** 0 **3.** quadratic; $x + 1$; $W^2 - 5W + 6 = 0$
4. quadratic; x^3; $W^2 + 7W - 8 = 0$ **5.** $-8, 0, 8$ **7.** $0, \pm 3$
9. $0, \sqrt[3]{\frac{5}{2}}$ **11.** $-2, 0$ **13.** $0, 2, 3$ **15.** $0, -2 \pm \sqrt{2}$
17. $-\frac{5}{2}, -\frac{1}{2}, \frac{3}{2}$ **19.** $\pm\sqrt{2}, 5$ **21.** 2 **23.** 1
25. $-\frac{7}{5}, 2$ **27.** $-50, 100$ **29.** -4 **31.** $-4, -\frac{7}{3}$
33. $\frac{-5 \pm 4\sqrt{2}}{7}$ **35.** 7 **37.** 4 **39.** 2 **41.** 4 **43.** 5
45. 6 **47.** $-7, 0$ **49.** $-\frac{3}{2}, -\frac{3}{4}$ **51.** $\pm 2\sqrt{2}, \pm\sqrt{5}$
53. No real solution **55.** $-1, 3$
57. $\pm 3\sqrt{3}, \pm 2\sqrt{2}$ **59.** $-1, 0, 3$ **61.** No solution **63.** 27, 729
65. $-\frac{1}{2}$ **67.** 20 **69.** $-3, \frac{1 \pm \sqrt{13}}{2}$ **71.** 2 **73.** −1.41, 1.41
75. −5.20, −2.83, 2.83, 5.20 **77.** $\pm\sqrt{a}, \pm 2\sqrt{a}$ **79.** $\sqrt{a^2 + 36}$
81. 50 **83.** 89 days **85.** 7.52 ft **87.** 4.63 mm
89. 16 mi; No **91.** 49 ft, 168 ft, and 175 ft **93.** 132.6 ft

SECTION 1.8 ■ PAGE 149

1. (a) $<$ **(b)** $\le$ **(c)** $\le$ **(d)** $>$ **2.** $x^2 + 5x - 14 \le 0$; $(x + 7)(x - 2) \le 0$; -7 and 2; $(-\infty, -7)$, $(-7, 2)$, and $(2, \infty)$

Interval	$(-\infty, 7)$	$(-7, 2)$	$(2, \infty)$
Sign of $x + 7$ Sign of $x - 2$	$-$ $-$	$+$ $-$	$+$ $+$
Sign of $(x + 7)(x - 2)$	$+$	$-$	$+$

$[-7, 2]$

3. $[-1, 0] \cup [1, 3]$ **4.** $(1, 4)$ **5.** $\{4\}$ **7.** $\{\sqrt{2}, 2, 4\}$

9. $\{4\}$ **11.** $\{-2, -1, 2, 4\}$ **13.** $\{-2, \sqrt{2}, 2, 4\}$

15. $(-\infty, \frac{7}{2}]$

17. $(4, \infty)$

19. $(-\infty, 2]$

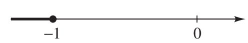
21. $(-\infty, -\frac{1}{2})$

23. $[1, \infty)$

25. $(\frac{16}{3}, \infty)$

27. $(-\infty, -18)$

29. $(-\infty, -1]$

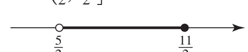


31. $[-3, -1)$

33. $(2, 6)$

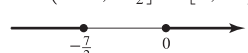
35. $[\frac{9}{2}, 5)$

37. $(\frac{5}{2}, \frac{11}{2}]$

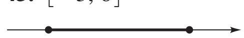


39. $(-2, 3)$

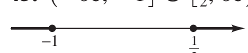
41. $(-\infty, -\frac{7}{2}] \cup [0, \infty)$



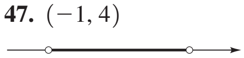
43. $[-3, 6]$



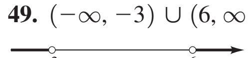
45. $(-\infty, -1] \cup [\frac{1}{2}, \infty)$



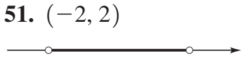
47. $(-1, 4)$



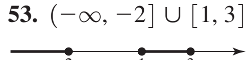
49. $(-\infty, -3) \cup (6, \infty)$



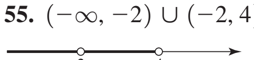
51. $(-2, 2)$



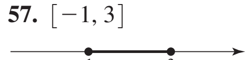
53. $(-\infty, -2] \cup [1, 3]$



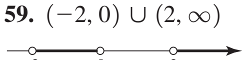
55. $(-\infty, -2) \cup (-2, 4)$



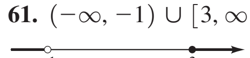
57. $[-1, 3]$



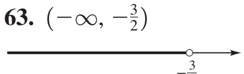
59. $(-2, 0) \cup (2, \infty)$



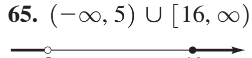
61. $(-\infty, -1) \cup [3, \infty)$



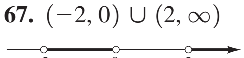
63. $(-\infty, -\frac{3}{2})$



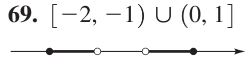
65. $(-\infty, 5) \cup [16, \infty)$



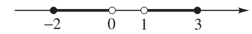
67. $(-2, 0) \cup (2, \infty)$



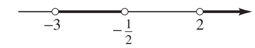
69. $[-2, -1) \cup (0, 1]$



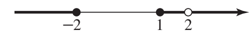
71. $[-2, 0) \cup (1, 3]$



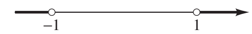
73. $(-3, -\frac{1}{2}) \cup (2, \infty)$



75. $(-\infty, -2] \cup [1, 2) \cup (2, \infty)$



77. $(-\infty, -1) \cup (1, \infty)$



79. $[-2, 5]$ **81.** $(-\infty, 1] \cup [2, 3]$ **83.** $(-1, 0) \cup (1, \infty)$

85. $(-\infty, 0)$ **87.** $-\frac{4}{3} \le x \le \frac{4}{3}$ **89.** $x < -2$ or $x > 7$

91. (a) $x \ge \dfrac{c}{a} + \dfrac{c}{b}$ **(b)** $\dfrac{a - c}{b} \le x < \dfrac{2a - c}{b}$

93. $68 \le F \le 86$ **95.** More than 200 mi

97. Between 12,000 mi and 14,000 mi

99. (a) $-\frac{1}{3}P + \frac{560}{3}$ **(b)** From \$215 to \$290

101. Distances between 20,000 km and 100,000 km

103. From 0 s to 3 s **105.** Between 0 and 60 mi/h

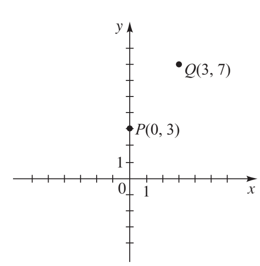
107. Between 20 and 40 ft

SECTION 1.9 ■ PAGE 154

1. $3, -3$ **2.** $[-3, 3]$ **3.** $(-\infty, -3], [3, \infty)$ **4. (a)** < 3 **(b)** > 3 **5.** ± 6 **7.** ± 5 **9.** $1, 5$ **11.** $-4.5, -3.5$ **13.** $-4, \frac{1}{2}$ **15.** $-3, -1$ **17.** $-8, -2$ **19.** $-\frac{25}{2}, \frac{35}{2}$ **21.** $-\frac{3}{2}, -\frac{1}{4}$ **23.** $[-4, 4]$ **25.** $(-\infty, -\frac{7}{2}) \cup (\frac{7}{2}, \infty)$ **27.** $[2, 8]$ **29.** $(-\infty, -2] \cup [0, \infty)$ **31.** $(-\infty, -7] \cup [-3, \infty)$ **33.** $[1.3, 1.7]$ **35.** $(-4, 8)$ **37.** $(-6.001, -5.999)$ **39.** $(-6, 2)$ **41.** $[-\frac{1}{2}, \frac{3}{2}]$ **43.** $(-\infty, -\frac{1}{2}) \cup (\frac{1}{3}, \infty)$ **45.** $[-4, -1] \cup [1, 4]$ **47.** $(-\frac{15}{2}, -7) \cup (-7, -\frac{13}{2})$ **49.** $|x| < 3$ **51.** $|x - 7| \ge 5$ **53.** $|x| \le 2$ **55.** $|x| > 3$ **57. (a)** $|x - 0.020| \le 0.003$ **(b)** $0.017 \le x \le 0.023$

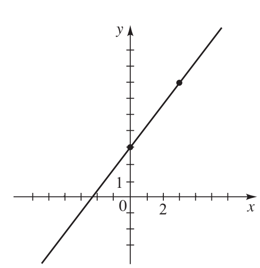
CHAPTER 1 REVIEW ■ PAGE 157

1. (a)

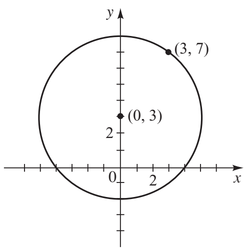


(b) 5 **(c)** $(\frac{3}{2}, 5)$

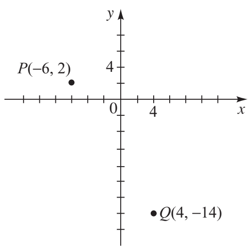
(d) $m = \frac{4}{3}$; point-slope: $y - 7 = \frac{4}{3}(x - 3)$; slope-intercept: $y = \frac{4}{3}x + 3$;



(e) $x^2 + (y - 3)^2 = 25$



3. (a)

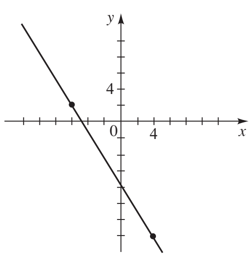


(b) $2\sqrt{89}$ **(c)** $(-1, -6)$

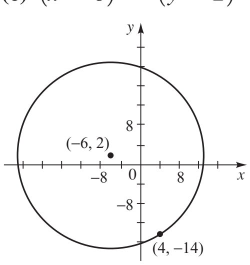
(d) $m = -\frac{8}{5}$;

point-slope: $y + 14 = -\frac{8}{5}(x - 4)$;

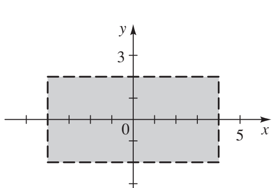
slope-intercept: $y = -\frac{8}{5}x - \frac{38}{5}$;



(e) $(x + 6)^2 + (y - 2)^2 = 356$



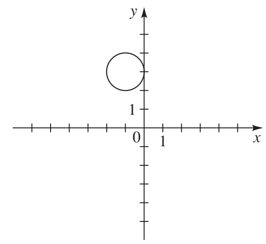
5.



7. B **9.** $(x + 5)^2 + (y + 1)^2 = 26$

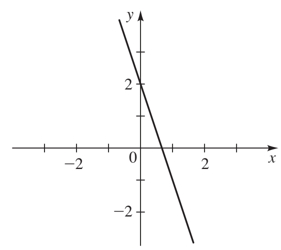
11. **(a)** Circle

(b) Center $(-1, 3)$, radius 1

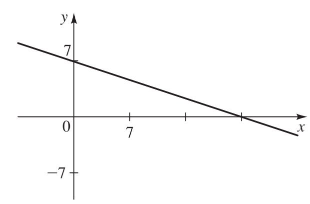


13. **(a)** No graph

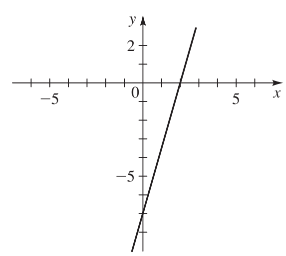
15.



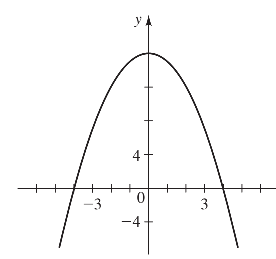
17.



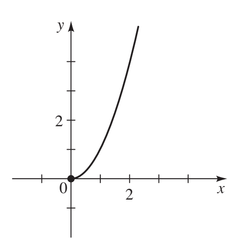
19.



21.



23.



25. **(a)** Symmetry about y-axis

(b) x-intercepts $-3, 3$; y-intercept 9

27. **(a)** Symmetry about y-axis

(b) x-intercept 0; y-intercepts 0, 2

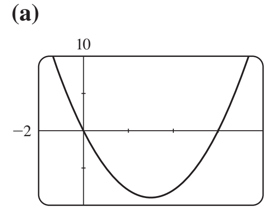
29. **(a)** Symmetry about x- and y-axes and the origin

(b) x-intercepts $-4, 4$; no y-intercept

31. **(a)** Symmetric about origin

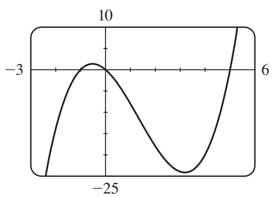
(b) x-intercepts $-1, 1$; y-intercepts $-1, 1$

33. **(a)**



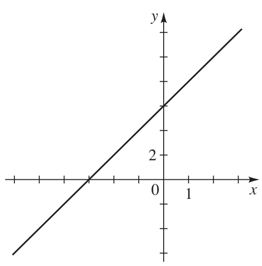
(b) x-intercepts 0, 6; y-intercept 0

35. (a)

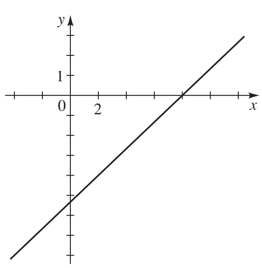


(b) x-intercepts $-1, 0, 5$; y-intercept 0

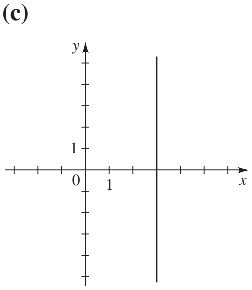
37. (a) $y = 2x + 6$ **(b)** $2x - y + 6 = 0$
(c)



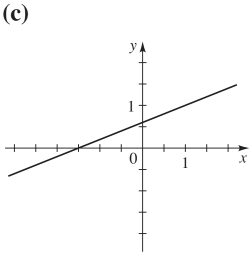
39. (a) $y = \frac{2}{3}x - \frac{16}{3}$ **(b)** $2x - 3y - 16 = 0$
(c)



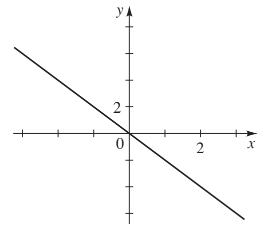
41. (a) $x = 3$ **(b)** $x - 3 = 0$
(c)



43. (a) $y = \frac{2}{5}x + \frac{3}{5}$ **(b)** $2x - 5y + 3 = 0$
(c)



45. (a) $y = -2x$ **(b)** $2x + y = 0$
(c)

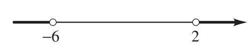


47. Parallel **49. (a)** The slope represents the amount the spring lengthens for a one-pound increase in weight. The S-intercept represents the unstretched length of the spring. **(b)** 4 in.
51. $-1, 6$ **53.** $[-1, 6]$ **55.** $(-\infty, 0] \cup [4, \infty)$
57. $-1, 7$ **59.** $-2.72, -1.15, 1.00, 2.87$ **61.** $[1, 3]$
63. $(-1.85, -0.60) \cup (0.45, 2.00)$
65. $x^2 + y^2 = 169$, $5x - 12y + 169 = 0$ **67.** 2, 7
69. $-1, \frac{1}{2}$ **71.** $0, \pm\frac{5}{2}$ **73.** $\dfrac{-2 \pm \sqrt{7}}{3}$ **75.** $\dfrac{3 \pm \sqrt{6}}{3}$
77. ± 3 **79.** 1 **81.** 3, 11 **83.** $-2, 7$
85. 20 lb raisins, 30 lb nuts
87. $\frac{1}{4}(\sqrt{329} - 3) \approx 3.78$ mi/h
89. \$5475 at 1.5%, \$1525 at 2.5%
91. 12 cm, 16 cm **93.** 23 ft by 46 ft by 8 ft

95. $(-3, \infty)$

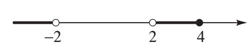
97. $(-3, -1]$

99. $(-\infty, -6) \cup (2, \infty)$



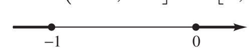
101. $[-4, -1)$

103. $(-\infty, -2) \cup (2, 4]$



105. $[2, 8]$

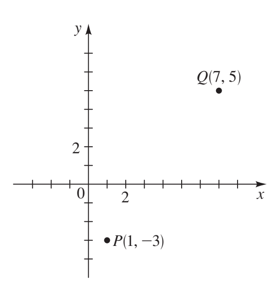
107. $(-\infty, -1] \cup [0, \infty)$



109. (a) $[-3, \frac{8}{3}]$ **(b)** $(0, 1)$

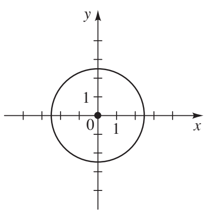
CHAPTER 1 TEST ■ PAGE 160

1. (a)

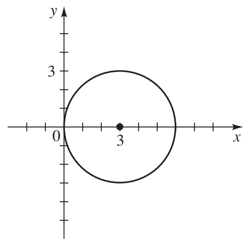


(b) 10 **(c)** $(4, 1)$ **(d)** $\frac{4}{3}$ **(e)** $y = -\frac{3}{4}x + 4$
(f) $(x - 4)^2 + (y - 1)^2 = 25$

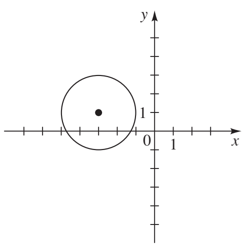
2. (a) $(0, 0); \frac{5}{2}$



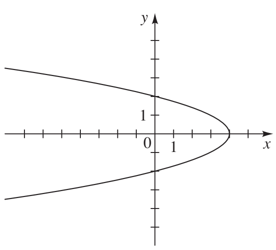
(b) $(3, 0); 3$



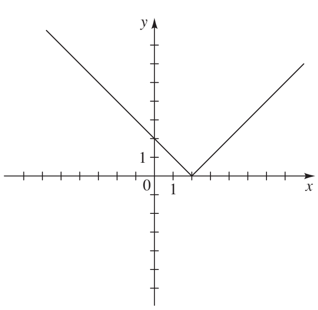
(c) $(-3, 1); 2$



3. (a) symmetry about x-axis; x-intercept 4; y-intercepts $-2, 2$

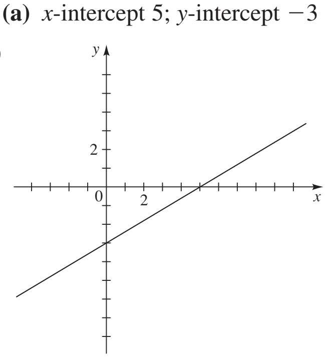


(b) No symmetry; x-intercept 2; y-intercept 2



4. (a) x-intercept 5; y-intercept -3

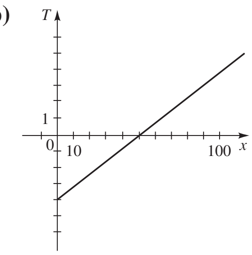
(b)



(c) $y = \frac{3}{5}x - 3$ **(d)** $\frac{3}{5}$ **(e)** $-\frac{5}{3}$

5. (a) $3x + y - 3 = 0$ **(b)** $2x + 3y - 12 = 0$

6. (a) 4°C **(b)**



(c) The slope is the rate of change in temperature, the x-intercept is the depth at which the temperature is 0°C, and the T-intercept is the temperature at ground level.

7. (a) 0, 3 **(b)** $(-\infty, 0) \cup (3, \infty)$

(c) 0, 2 **(d)** $[0, 4]$

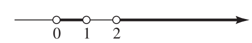
8. (a) $-2.94, -0.11, 3.05$ **(b)** $[-1.07, 3.74]$

9. 150 km **10. (a)** $-3, 4$ **(b)** $-1 \pm \dfrac{\sqrt{10}}{2}$ **(c)** 3

(d) 1, 16 **(e)** $0, \pm 4$ **(f)** $\frac{2}{3}, \frac{22}{3}$ **11.** 50 ft by 120 ft

12. (a) $(-\frac{5}{2}, 3]$

(b) $(0, 1) \cup (2, \infty)$



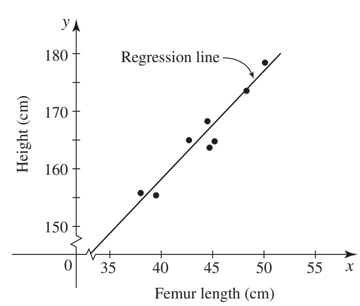
(c) $(1, 5)$

(d) $[-4, -1)$

13. 41°F to 50°F **14.** $0 \le x \le 4$

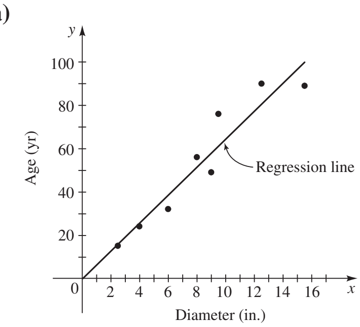
FOCUS ON MODELING ■ PAGE 167

1. (a)



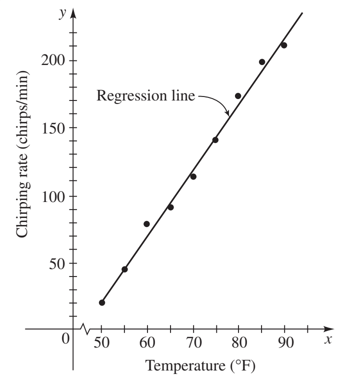
(b) $y = 1.8807x + 82.65$ **(c)** 191.7 cm

3. (a)



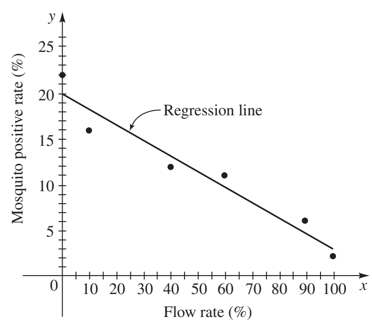
(b) $y = 6.451x - 0.1523$ **(c)** 116 years

5. (a)



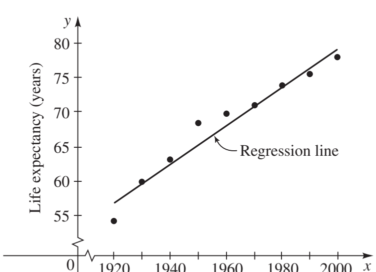
(b) $y = 4.857x - 220.97$ **(c)** 265 chirps/min

7. (a)



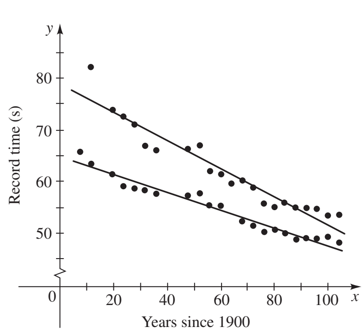
(b) $y = -0.168x + 19.89$ **(c)** 8.13%

9. (a)



(b) $y = 0.2708x - 462.9$ **(c)** 80.3 years

11. (a) Men: $y = -0.1703x + 64.61$, women $y = -0.2603x + 78.27$; x represents years since 1900



(b) 2052

CHAPTER 2

SECTION 2.1 ■ PAGE 181

1. (a) $f(-1) = 0$ **(b)** $f(2) = 9$ **(c)** $f(2) - f(-1) = 9$
2. domain, range **3. (a)** f and g **(b)** $f(5) = 10, g(5) = 0$
4. (a) square, add 3

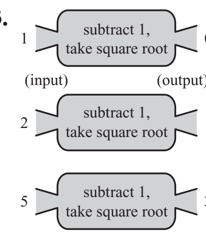
(b)

x	0	2	4	6
$f(x)$	19	7	3	7

5. $f(x) = \dfrac{x-2}{5}$ **7.** $f(x) = 4x - 1$ **9.** Square, then add 2

11. Subtract 4, then divide by 3

13.



15.

x	$f(x)$
−1	8
0	2
1	0
2	2
3	8

17. $3, 3, -6, -\frac{23}{4}$ **19.** $3, -3, 2, 2a + 1, -2a + 1, 2a - 1$

21. $0, 15, 3, a^2 + 2a, x^2 - 2x, \dfrac{1}{a^2} + \dfrac{2}{a}$

23. $-\dfrac{1}{3}$, undefined, $\dfrac{1}{3}, \dfrac{1-a}{1+a}, \dfrac{2-a}{a}, \dfrac{2-x^2}{x^2}$

25. $-4, 10, 3\sqrt{2}, 2x^2 + 7x + 1, 2x^2 - 3x - 4, 2x^6 + 3x^3 - 4$

27. $6, 2, 1, 2, 2|x|, 2(x^2 + 1)$ **29.** $4, 1, 1, 2, 3$

31. $8, -\frac{3}{4}, -1, 0, -1$ **33.** $x^2 + 4x + 5, x^2 + 6$

35. $x^2 + 4, x^2 + 8x + 16$ **37.** 12 **39.** -21

41. $3a + 2, 3(a + h) + 2, 3$

43. $5, 5, 0$ **45.** $\dfrac{a}{a+1}, \dfrac{a+h}{a+h+1}, \dfrac{1}{(a+h+1)(a+1)}$

47. $3 - 5a + 4a^2, 3 - 5a - 5h + 4a^2 + 8ah + 4h^2$, $-5 + 8a + 4h$ **49.** $(-\infty, \infty)$ **51.** $[-1, 5]$

53. $\{x \mid x \neq 3\}$ **55.** $\{x \mid x \neq \pm 1\}$ **57.** $[5, \infty)$

59. $(-\infty, \infty)$ **61.** $[\frac{5}{2}, \infty)$ **63.** $[-2, 3) \cup (3, \infty)$

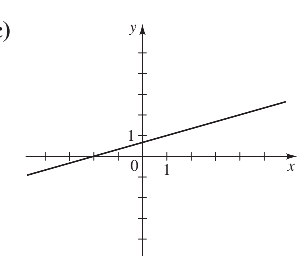
65. $(-\infty, 0] \cup [6, \infty)$ **67.** $(4, \infty)$ **69.** $(\frac{1}{2}, \infty)$

71. (a) $f(x) = \dfrac{x}{3} + \dfrac{2}{3}$

(b)

x	$f(x)$
2	$\frac{4}{3}$
4	2
6	$\frac{8}{3}$
8	$\frac{10}{3}$

(c)

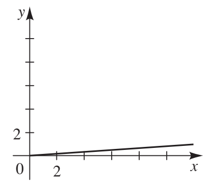


73. (a) $T(x) = 0.08x$

(b)

x	$T(x)$
2	0.16
4	0.32
6	0.48
8	0.64

(c)



75. (a) $C(10) = 1532.1$, $C(100) = 2100$
(b) The cost of producing 10 yd and 100 yd
(c) $C(0) = 1500$
77. (a) 50, 0 **(b)** $V(0)$ is the volume of the full tank, and $V(20)$ is the volume of the empty tank, 20 minutes later.

(c)

x	$V(x)$
0	50
5	28.125
10	12.5
15	3.125
20	0

(d) -50 gal

79. (a) $v(0.1) = 4440$, $v(0.4) = 1665$
(b) Flow is faster near central axis.

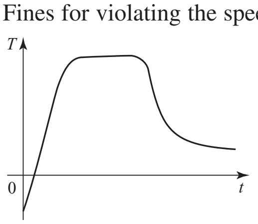
(c)

r	$v(r)$
0	4625
0.1	4440
0.2	3885
0.3	2960
0.4	1665
0.5	0

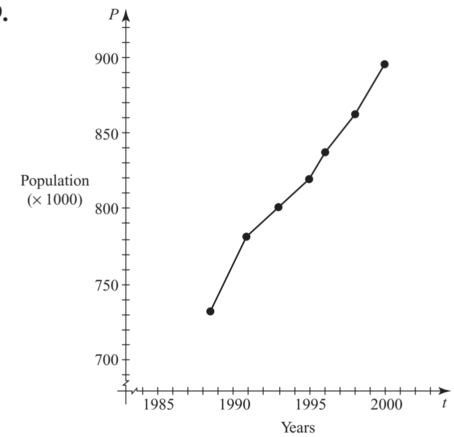
(d) -4440 cm/s
81. (a) 8.66 m, 6.61 m, 4.36 m
(b) It will appear to get shorter.
83. (a) \$90, \$105, \$100, \$105
(b) Total cost of an order, including shipping

85. (a) $F(x) = \begin{cases} 15(40 - x) & \text{if } 0 < x < 40 \\ 0 & \text{if } 40 \le x \le 65 \\ 15(x - 65) & \text{if } x > 65 \end{cases}$

(b) \$150, \$0, \$150
(c) Fines for violating the speed limits
87.



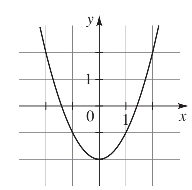
89.



SECTION 2.2 ■ PAGE 191

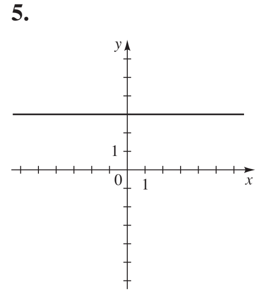
1. $f(x)$, $x^2 - 2$, 7, 7

x	$f(x) = x^2 - 2$
-2	2
-1	-1
0	-2
1	-1
2	2



2. 3 **3.** 3 **4. (a)** IV **(b)** II **(c)** I **(d)** III

5.



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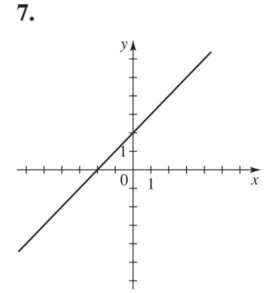

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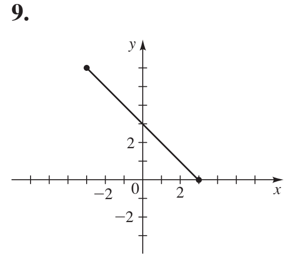

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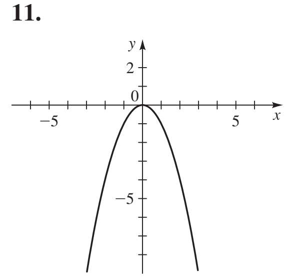

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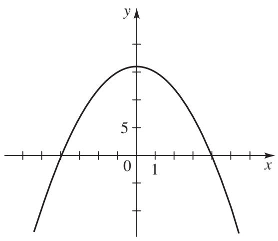

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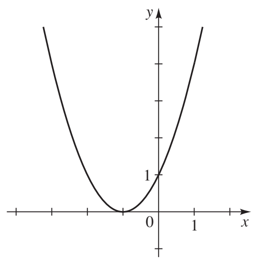

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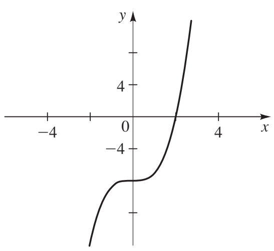

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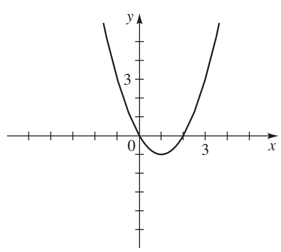
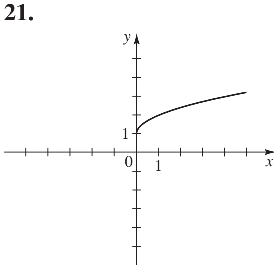


21.



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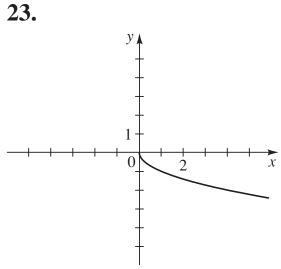
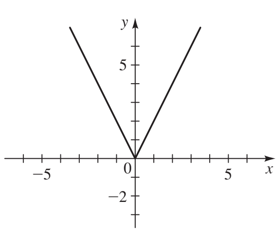
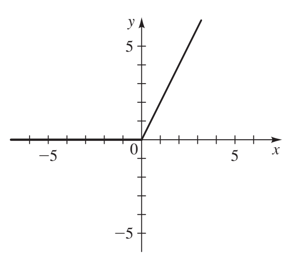


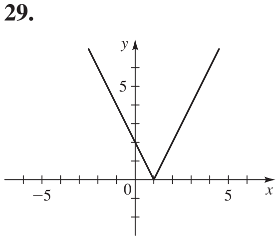
25.



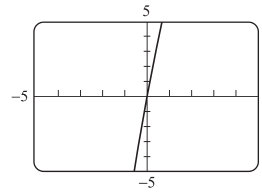
27.



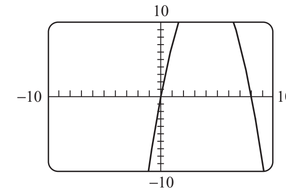
29.



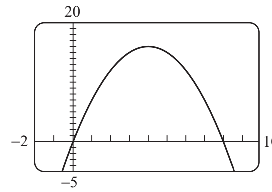
31. (a)



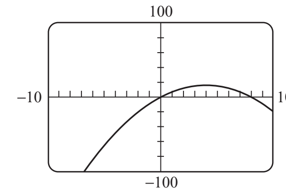
(b)



(c)

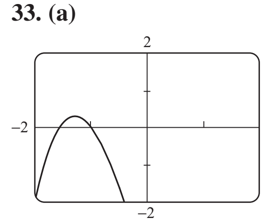


(d)

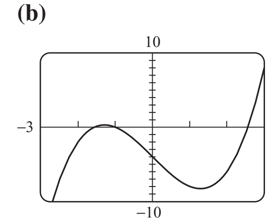


Graph (c) is the most appropriate.

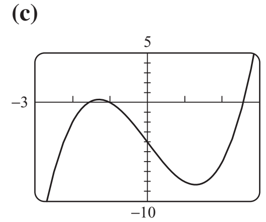
33. (a)



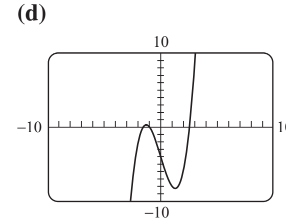
(b)



(c)

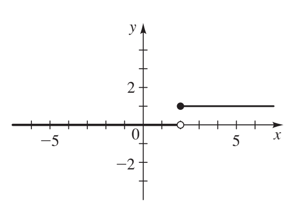


(d)

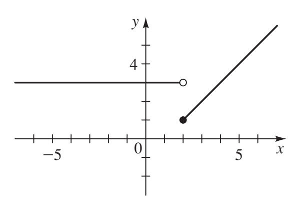


Graph (c) is the most appropriate.

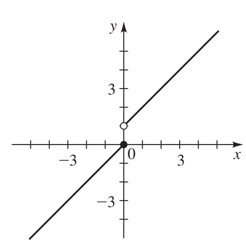
35.



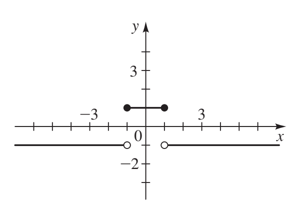
37.



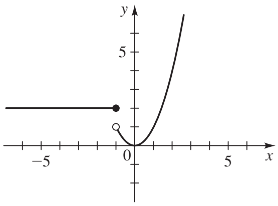
39.



41.



43.



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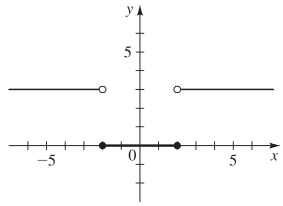
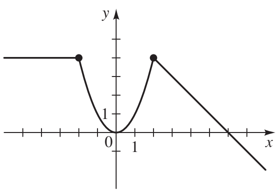
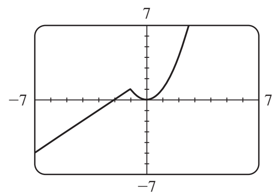


47.

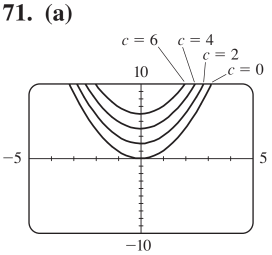


49.

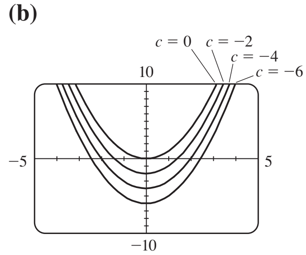


51. $f(x) = \begin{cases} -2 & \text{if } x < -2 \\ x & \text{if } -2 \le x \le 2 \\ 2 & \text{if } x > 2 \end{cases}$

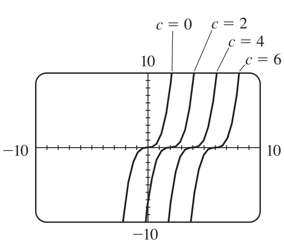
53. **(a)** Yes **(b)** No **(c)** Yes **(d)** No
55. Function, domain $[-3, 2]$, range $[-2, 2]$ **57.** Not a function
59. Yes **61.** No **63.** No **65.** Yes **67.** Yes **69.** Yes
71. **(a)**



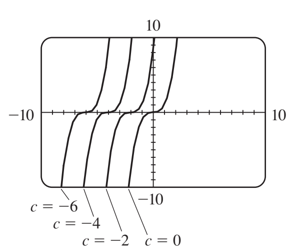
(b)



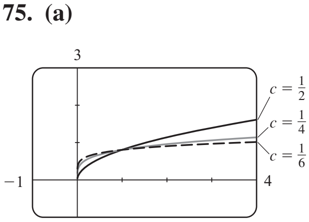
(c) If $c > 0$, then the graph of $f(x) = x^2 + c$ is the same as the graph of $y = x^2$ shifted upward c units. If $c < 0$, then the graph of $f(x) = x^2 + c$ is the same as the graph of $y = x^2$ shifted downward c units.
73. **(a)**



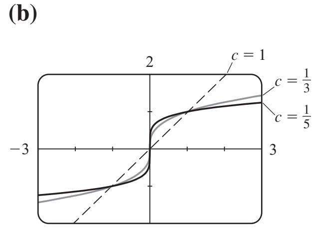
(b)



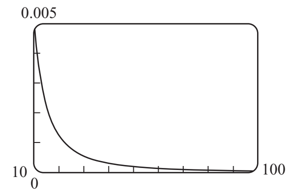
(c) If $c > 0$, then the graph of $f(x) = (x - c)^3$ is the same as the graph of $y = x^3$ shifted to the right c units. If $c < 0$, then the graph of $f(x) = (x - c)^3$ is the same as the graph of $y = x^3$ shifted to the left $|c|$ units.
75. **(a)**



(b)

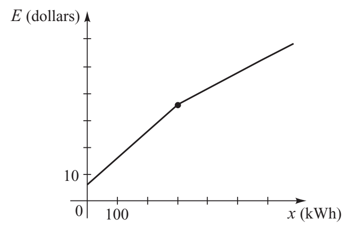


(c) Graphs of even roots are similar to $\sqrt{x}$; graphs of odd roots are similar to $\sqrt[3]{x}$. As c increases, the graph of $y = \sqrt[c]{x}$ becomes steeper near 0 and flatter when $x > 1$.
77. $f(x) = -\frac{7}{6}x - \frac{4}{3}$, $-2 \le x \le 4$
79. $f(x) = \sqrt{9 - x^2}$, $-3 \le x \le 3$
81.

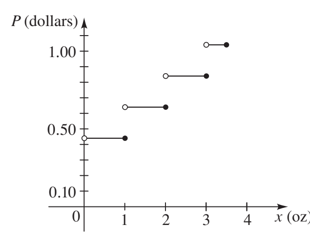


83. **(a)** $E(x) = \begin{cases} 6 + 0.10x & 0 \le x \le 300 \\ 36 + 0.06(x - 300), & x > 300 \end{cases}$

(b)

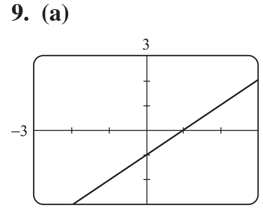


85. $P(x) = \begin{cases} 0.44 & \text{if } 0 < x \le 1 \\ 0.64 & \text{if } 1 < x \le 2 \\ 0.84 & \text{if } 2 < x \le 3 \\ 1.04 & \text{if } 3 < x \le 3.5 \end{cases}$



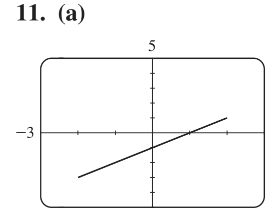
SECTION 2.3 ■ PAGE 200

1. a, 4, 1, $f(3) - f(1) = 3$ **2.** x, y, $[1, 6]$, $[1, 7]$
3. **(a)** increase, $[1, 2]$, $[4, 5]$ **(b)** decrease, $[2, 4]$, $[5, 6]$
4. **(a)** largest, 7, 2 **(b)** smallest, 2, 4 **5.** **(a)** 1, −1, 3, 4
(b) Domain $[-3, 4]$, range $[-1, 4]$ **(c)** −3, 2, 4
(d) $-3 \le x \le 2$ and $x = 4$ **(e)** 1 **7.** **(a)** 3, 2, −2, 1, 0
(b) Domain $[-4, 4]$, range $[-2, 3]$
9. **(a)**



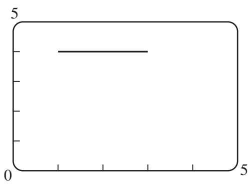
(b) Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

11. **(a)**



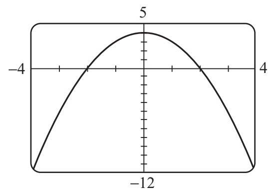
(b) Domain $[-2, 2]$, range $[-3, 1]$

13. **(a)**



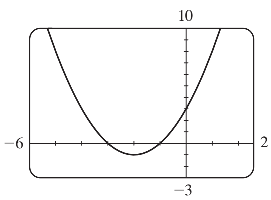
(b) Domain $[1, 3]$, range $\{4\}$

15. **(a)**



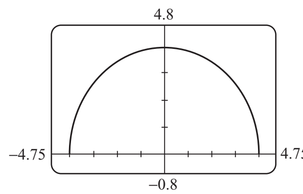
(b) Domain $(-\infty, \infty)$, range $(-\infty, 4]$

17. **(a)**



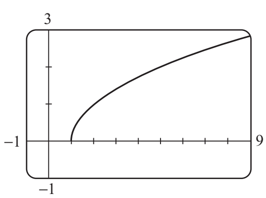
(b) Domain $(-\infty, \infty)$, range $[-1, \infty)$

19. **(a)**



(b) Domain $[-4, 4]$, range $[0, 4]$

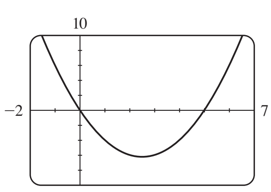
21. **(a)**



(b) Domain $[1, \infty)$, range $[0, \infty)$

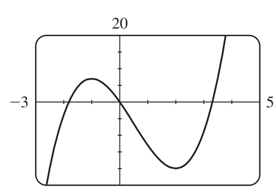
23. **(a)** Domain $[-1, 4]$, range $[-1, 3]$ **(b)** Increasing on $[-1, 1]$ and $[2, 4]$, decreasing on $[1, 2]$
25. **(a)** Domain $[-3, 3]$, range $[-2, 2]$ **(b)** Increasing on $[-2, -1]$ and $[1, 2]$, decreasing on $[-3, -2]$, $[-1, 1]$, and $[2, 3]$

27. **(a)**



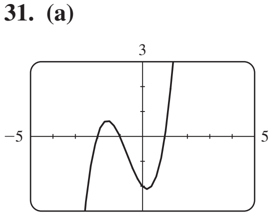
(b) Domain $(-\infty, \infty)$, range $[-6.25, \infty)$
(c) Increasing on $[2.5, \infty)$; decreasing on $(-\infty, 2.5]$

29. **(a)**



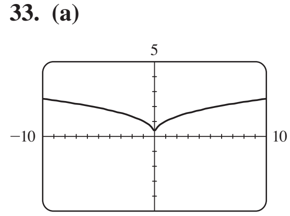
(b) Domain $(-\infty, \infty)$, range $(-\infty, \infty)$
(c) Increasing on $(-\infty, -1]$, $[2, \infty)$; decreasing on $[-1, 2]$

31. **(a)**



(b) Domain $(-\infty, \infty)$, range $(-\infty, \infty)$
(c) Increasing on $(-\infty, -1.55]$, $[0.22, \infty)$; decreasing on $[-1.55, 0.22]$

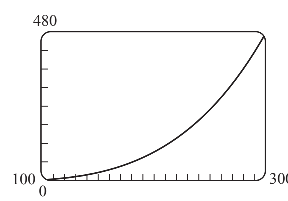
33. **(a)**



(b) Domain $(-\infty, \infty)$, range $[0, \infty)$
(c) Increasing on $[0, \infty)$; decreasing on $(-\infty, 0]$

35. **(a)** Local maximum 2 when $x = 0$; local minimum -1 when $x = -2$, local minimum 0 when $x = 2$ **(b)** Increasing on $[-2, 0] \cup [2, \infty)$; decreasing on $(-\infty, -2] \cup [0, 2]$
37. **(a)** Local maximum 0 when $x = 0$; local maximum 1 when $x = 3$, local minimum -2 when $x = -2$, local minimum -1 when $x = 1$ **(b)** Increasing on $[-2, 0] \cup [1, 3]$; decreasing on $(-\infty, -2] \cup [0, 1] \cup [3, \infty)$ **39.** **(a)** Local maximum ≈ 0.38 when $x \approx -0.58$; local minimum ≈ -0.38 when $x \approx 0.58$ **(b)** Increasing on $(-\infty, -0.58] \cup [0.58, \infty)$; decreasing on $[-0.58, 0.58]$ **41.** **(a)** Local maximum ≈ 0 when $x = 0$; local minimum ≈ -13.61 when $x \approx -1.71$, local minimum ≈ -73.32 when $x \approx 3.21$ **(b)** Increasing on $[-1.71, 0] \cup [3.21, \infty)$; decreasing on $(-\infty, -1.71] \cup [0, 3.21]$ **43.** **(a)** Local maximum ≈ 5.66 when $x \approx 4.00$ **(b)** Increasing on $(-\infty, 4.00]$; decreasing on $[4.00, 6.00]$ **45.** **(a)** Local maximum ≈ 0.38 when $x \approx -1.73$; local minimum ≈ -0.38 when $x \approx 1.73$ **(b)** Increasing on $(-\infty, -1.73] \cup [1.73, \infty]$; decreasing on $[-1.73, 0) \cup (0, 1.73]$
47. **(a)** 500 MW, 725 MW **(b)** Between 3:00 A.M. and 4:00 A.M. **(c)** Just before noon **(d)** -100 MW **49.** **(a)** Increasing on $[0, 30] \cup [32, 68]$; decreasing on $[30, 32]$ **(b)** He went on a crash diet and lost weight, only to regain it again later. **(c)** 100 lb
51. **(a)** Increasing on $[0, 150] \cup [300, \infty)$; decreasing on $[150, 300]$ **(b)** Local maximum when $x = 150$; local minimum when $x = 300$ **(c)** -50 ft **53.** Runner A won the race. All runners finished. Runner B fell but got up again to finish second.

55. **(a)**



(b) Increases
57. 20 mi/h **59.** $r \approx 0.67$ cm

SECTION 2.4 ■ PAGE 208

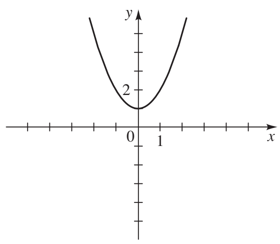
1. $\dfrac{100 \text{ miles}}{2 \text{ hours}} = 50$ mi/h **2.** $\dfrac{f(b) - f(a)}{b - a}$ **3.** $\dfrac{25 - 1}{5 - 1} = 6$
4. **(a)** secant **(b)** 3 **5.** **(a)** 2 **(b)** $\frac{2}{3}$ **7.** **(a)** -4 **(b)** $-\frac{4}{5}$
9. **(a)** 3 **(b)** 3 **11.** **(a)** -5 **(b)** -1 **13.** **(a)** 25 **(b)** 5
15. **(a)** 600 **(b)** 60 **17.** **(a)** $12h + 3h^2$ **(b)** $12 + 3h$
19. **(a)** $\dfrac{1 - a}{a}$ **(b)** $-\dfrac{1}{a}$ **21.** **(a)** $\dfrac{-2h}{a(a + h)}$ **(b)** $\dfrac{-2}{a(a + h)}$
23. **(a)** $\frac{1}{2}$ **25.** -0.25 ft/day **27.** **(a)** 245 persons/yr **(b)** -328.5 persons/yr **(c)** 1997–2001 **(d)** 2001–2006
29. **(a)** 7.2 units/yr **(b)** 8 units/yr **(c)** -55 units/yr **(d)** 2000–2001, 2001–2002 **31.** First 20 minutes: -4.05°F/min, next 20 minutes: -1.5°F/min; first interval

SECTION 2.5 ■ PAGE 219

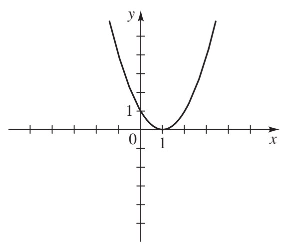
1. **(a)** up **(b)** left **2.** **(a)** down **(b)** right **3.** **(a)** x-axis **(b)** y-axis **4.** **(a)** II **(b)** I **(c)** III **(d)** IV **5.** **(a)** Shift upward 3 units **(b)** Shift to the left 3 units **7.** **(a)** Reflect in the x-axis **(b)** Reflect in the y-axis **9.** **(a)** Shift to the right 5 units, then upward 2 units **(b)** Shift to the left 1 unit, then downward 1 unit **11.** **(a)** Reflect in the x-axis, then shift upward 5 units **(b)** Stretch vertically by a factor of 3, then shift downward 5 units

13. **(a)** Shift to the left 1 unit, stretch vertically by a factor of 2, then shift downward 3 units **(b)** Shift to the right 1 unit, stretch vertically by a factor of 2, then shift upward 3 units
15. **(a)** Shrink horizontally by a factor of $\frac{1}{4}$ **(b)** Stretch horizontally by a factor of 4 **17.** **(a)** Shift to the left 2 units **(b)** Shift upward 2 units **19.** **(a)** Shift to the left 2 units, then shift downward 2 units **(b)** Shift to the right 2 units, then shift upward 2 units

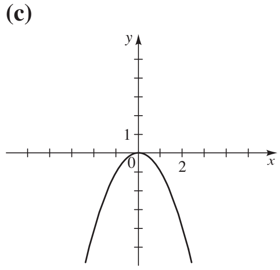
21. **(a)**



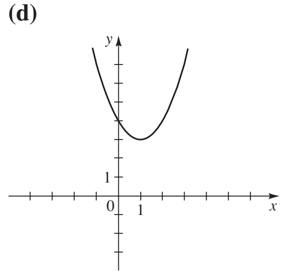
(b)



(c)

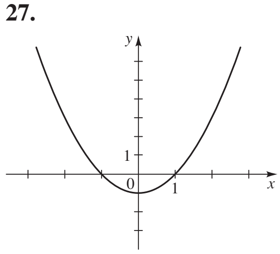


(d)

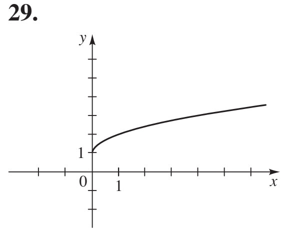


23. II **25.** I

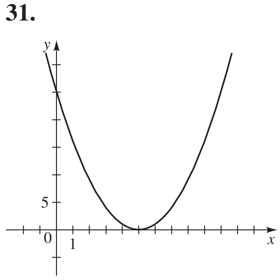
27.



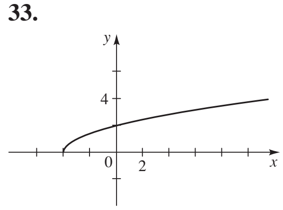
29.



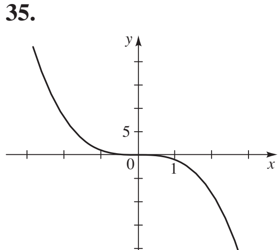
31.



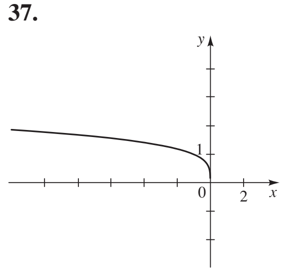
33.



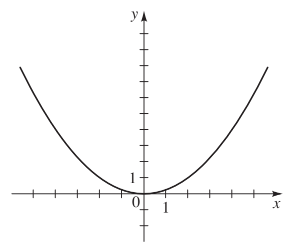
35.



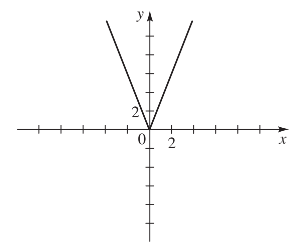
37.



39.



41.



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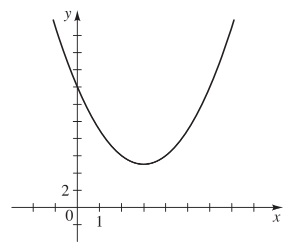
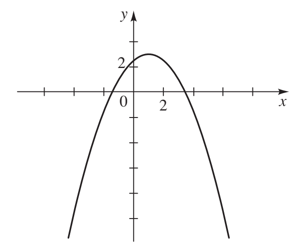
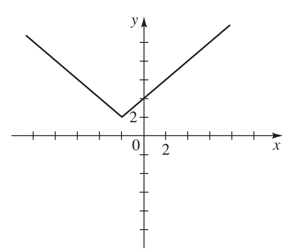


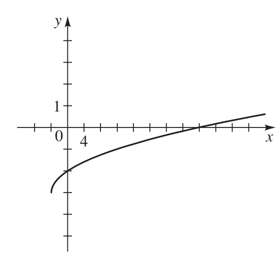
45.



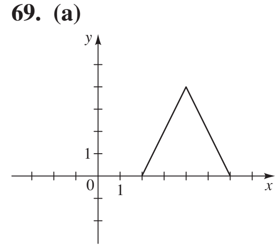
47.



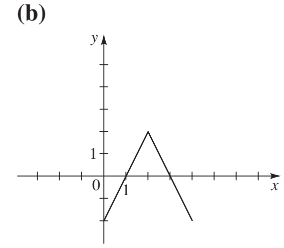
49.



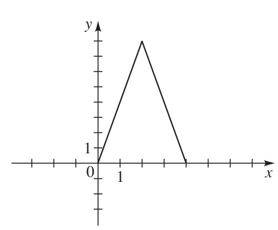
51. $f(x) = x^2 + 3$ **53.** $f(x) = \sqrt{x + 2}$
55. $f(x) = |x - 3| + 1$ **57.** $f(x) = \sqrt[4]{-x} + 1$
59. $f(x) = 2(x - 3)^2 - 2$ **61.** $g(x) = (x - 2)^2$
63. $g(x) = |x + 1| + 2$ **65.** $g(x) = -\sqrt{x + 2}$
67. **(a)** 3 **(b)** 1 **(c)** 2 **(d)** 4
69. **(a)**



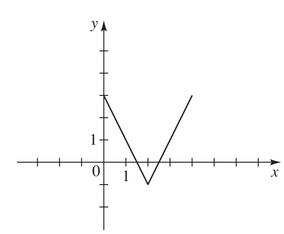
(b)



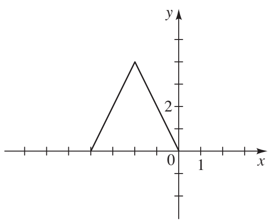
(c)



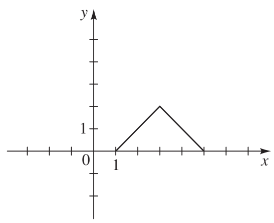
(d)



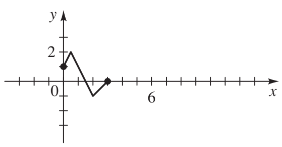
(e)



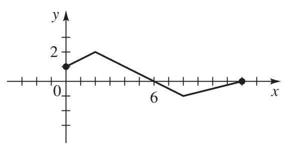
(f)



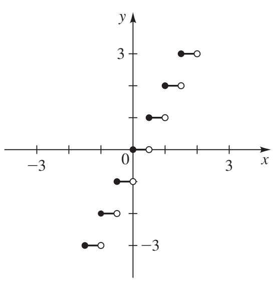
71. (a)



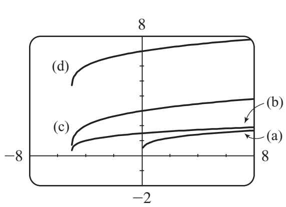
(b)



73.

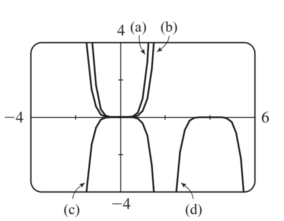


75.



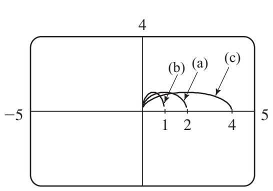
For part (b) shift the graph in (a) to the left 5 units; for part (c) shift the graph in (a) to the left 5 units and stretch vertically by a factor of 2; for part (d) shift the graph in (a) to the left 5 units, stretch vertically by a factor of 2, and then shift upward 4 units.

77.



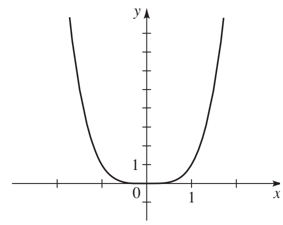
For part (b) shrink the graph in (a) vertically by a factor of $\frac{1}{3}$; for part (c) shrink the graph in (a) vertically by a factor of $\frac{1}{3}$ and reflect in the x-axis; for part (d) shift the graph in (a) to the right 4 units, shrink vertically by a factor of $\frac{1}{3}$, and then reflect in the x-axis.

79.



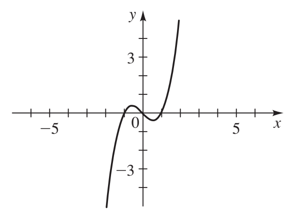
The graph in part (b) is shrunk horizontally by a factor of $\frac{1}{2}$ and the graph in part (c) is stretched by a factor of 2.

81. Even



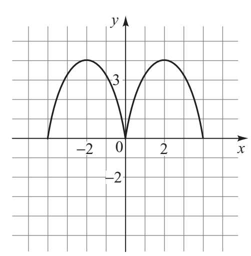
83. Neither

85. Odd

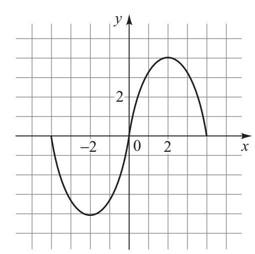


87. Neither

89. (a)

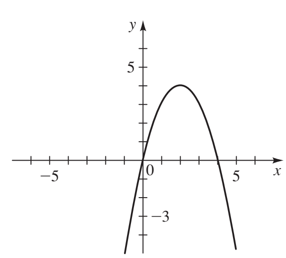


(b)

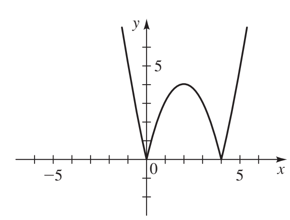


91. To obtain the graph of g, reflect in the x-axis the part of the graph of f that is below the x-axis.

93. (a)



(b)



95. (a) Shift upward 4 units, shrink vertically by a factor of 0.01 **(b)** Shift to the left 10 units; $g(t) = 4 + 0.01(t + 10)^2$

SECTION 2.6 ■ PAGE 228

1. $8, -2, 15, \frac{3}{5}$ **2.** $f(g(x))$, 12 **3.** Multiply by 2, then add 1; Add 1, then multiply by 2 **4.** $x + 1, 2x, 2x + 1, 2(x + 1)$

5. $(f + g)(x) = 3x, (-\infty, \infty)$; $(f - g)(x) = -x, (-\infty, \infty)$; $(fg)(x) = 2x^2, (-\infty, \infty)$; $\left(\dfrac{f}{g}\right)(x) = \dfrac{1}{2}, (-\infty, 0) \cup (0, \infty)$

7. $(f + g)(x) = x + x^2, (-\infty, \infty)$; $(f - g)(x) = x - x^2, (-\infty, \infty)$; $(fg)(x) = x^3, (-\infty, \infty)$; $\left(\dfrac{f}{g}\right)(x) = \dfrac{1}{x}, (-\infty, 0) \cup (0, \infty)$

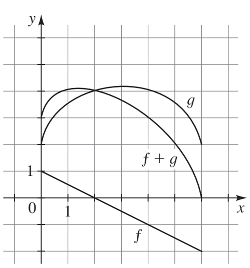
9. $(f+g)(x) = x^2 + x - 3, (-\infty, \infty)$;
$(f-g)(x) = -x^2 + x - 3, (-\infty, \infty)$;
$(fg)(x) = x^3 - 3x^2, (-\infty, \infty)$;
$\left(\frac{f}{g}\right)(x) = \frac{x-3}{x^2}, (-\infty, 0) \cup (0, \infty)$

11. $(f+g)(x) = \sqrt{4-x^2} + \sqrt{1+x}, [-1, 2]$;
$(f-g)(x) = \sqrt{4-x^2} - \sqrt{1+x}, [-1, 2]$;
$(fg)(x) = \sqrt{-x^3 - x^2 + 4x + 4}, [-1, 2]$;
$\left(\frac{f}{g}\right)(x) = \sqrt{\frac{4-x^2}{1+x}}, (-1, 2]$

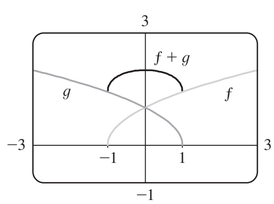
13. $(f+g)(x) = \frac{6x+8}{x^2+4x}, x \neq -4, x \neq 0$;
$(f-g)(x) = \frac{-2x+8}{x^2+4x}, x \neq -4, x \neq 0$;
$(fg)(x) = \frac{8}{x^2+4x}, x \neq -4, x \neq 0$;
$\left(\frac{f}{g}\right)(x) = \frac{x+4}{2x}, x \neq -4, x \neq 0$

15. $[0, 1]$ **17.** $(3, \infty)$

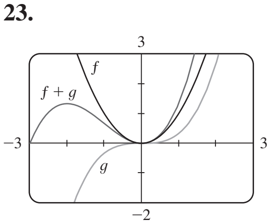
19.



21.



23.



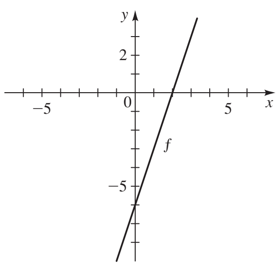
25. **(a)** 1 **(b)** -23 **27.** **(a)** -11 **(b)** -119
29. **(a)** $-3x^2 + 1$ **(b)** $-9x^2 + 30x - 23$
31. 4 **33.** 5 **35.** 4
37. $(f \circ g)(x) = 8x + 1, (-\infty, \infty)$;
$(g \circ f)(x) = 8x + 11, (-\infty, \infty)$; $(f \circ f)(x) = 4x + 9, (-\infty, \infty)$;
$(g \circ g)(x) = 16x - 5, (-\infty, \infty)$
39. $(f \circ g)(x) = (x+1)^2, (-\infty, \infty)$;
$(g \circ f)(x) = x^2 + 1, (-\infty, \infty)$; $(f \circ f)(x) = x^4, (-\infty, \infty)$;
$(g \circ g)(x) = x + 2, (-\infty, \infty)$
41. $(f \circ g)(x) = \frac{1}{2x+4}, x \neq -2$; $(g \circ f)(x) = \frac{2}{x} + 4, x \neq 0$;
$(f \circ f)(x) = x, x \neq 0$, $(g \circ g)(x) = 4x + 12, (-\infty, \infty)$

43. $(f \circ g)(x) = |2x+3|, (-\infty, \infty)$;
$(g \circ f)(x) = 2|x| + 3, (-\infty, \infty)$;
$(f \circ f)(x) = |x|, (-\infty, \infty)$;
$(g \circ g)(x) = 4x + 9, (-\infty, \infty)$
45. $(f \circ g)(x) = \frac{2x-1}{2x}, x \neq 0$;
$(g \circ f)(x) = \frac{2x}{x+1} - 1, x \neq -1$;
$(f \circ f)(x) = \frac{x}{2x+1}, x \neq -1, x \neq -\frac{1}{2}$;
$(g \circ g)(x) = 4x - 3, (-\infty, \infty)$
47. $(f \circ g)(x) = \frac{1}{x+1}, x \neq -1, x \neq 0$;
$(g \circ f)(x) = \frac{x+1}{x}, x \neq -1, x \neq 0$;
$(f \circ f)(x) = \frac{x}{2x+1}, x \neq -1, x \neq -\frac{1}{2}$;
$(g \circ g)(x) = x, x \neq 0$
49. $(f \circ g \circ h)(x) = \sqrt{x-1} - 1$
51. $(f \circ g \circ h)(x) = (\sqrt{x} - 5)^4 + 1$
53. $g(x) = x - 9, f(x) = x^5$
55. $g(x) = x^2, f(x) = x/(x+4)$
57. $g(x) = 1 - x^3, f(x) = |x|$
59. $h(x) = x^2, g(x) = x + 1, f(x) = 1/x$
61. $h(x) = \sqrt[3]{x}, g(x) = 4 + x, f(x) = x^9$
63. $R(x) = 0.15x - 0.000002x^2$
65. **(a)** $g(t) = 60t$ **(b)** $f(r) = \pi r^2$
(c) $(f \circ g)(t) = 3600\pi t^2$
67. $A(t) = 16\pi t^2$ **69.** **(a)** $f(x) = 0.9x$
(b) $g(x) = x - 100$ **(c)** $(f \circ g)(x) = 0.9x - 90$,
$(g \circ f)(x) = 0.9x - 100$, $f \circ g$: first rebate, then discount,
$g \circ f$: first discount, then rebate, $g \circ f$ is the better deal

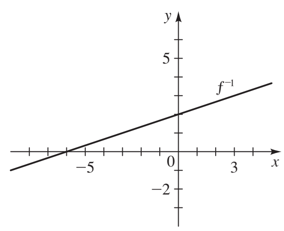
SECTION 2.7 ■ PAGE 237

1. different, Horizontal Line **2.** **(a)** one-to-one, $g(x) = x^3$
(b) $g^{-1}(x) = x^{1/3}$ **3.** **(a)** Take the cube root, subtract 5, then divide the result by 3. **(b)** $f(x) = (3x+5)^3, f^{-1}(x) = \frac{x^{1/3} - 5}{3}$
4. Yes, 4, 5 **5.** (4, 3) **6.** **(a)** False **(b)** True **7.** No **9.** Yes
11. No **13.** Yes **15.** Yes **17.** No **19.** No **21.** No
23. **(a)** 2 **(b)** 3 **25.** 1 **27.** **(a)** 6 **(b)** 2 **(c)** 0
41. $f^{-1}(x) = \frac{1}{2}(x-1)$ **43.** $f^{-1}(x) = \frac{1}{4}(x-7)$
45. $f^{-1}(x) = \sqrt[3]{\frac{1}{4}(5-x)}$ **47.** $f^{-1}(x) = (1/x) - 2$
49. $f^{-1}(x) = \frac{4x}{1-x}$ **51.** $f^{-1}(x) = \frac{7x+5}{x-2}$
53. $f^{-1}(x) = (5x-1)/(2x+3)$
55. $f^{-1}(x) = \frac{1}{5}(x^2 - 2), x \geq 0$
57. $f^{-1}(x) = \sqrt{4-x}, x \leq 4$ **59.** $f^{-1}(x) = (x-4)^3$
61. $f^{-1}(x) = x^2 - 2x, x \geq 1$ **63.** $f^{-1}(x) = \sqrt[4]{x}; x \geq 0$

65. (a)

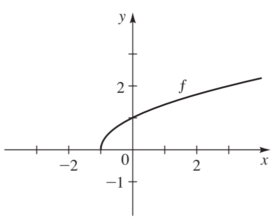


(b)

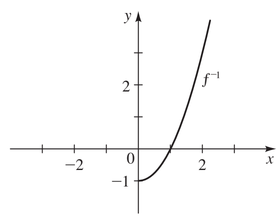


(c) $f^{-1}(x) = \frac{1}{3}(x + 6)$

67. (a)

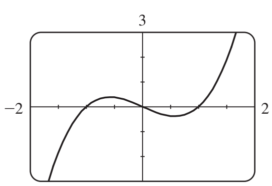


(b)

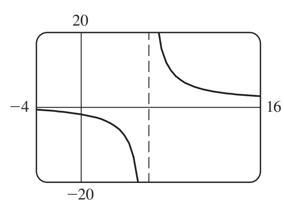


(c) $f^{-1}(x) = x^2 - 1, x \geq 0$

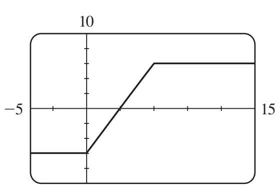
69. Not one-to-one



71. One-to-one

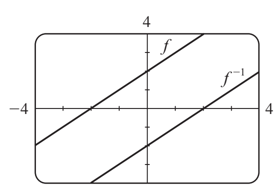


73. Not one-to-one



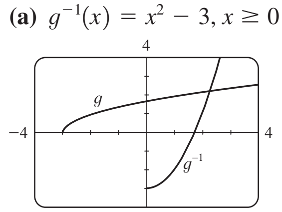
75. (a) $f^{-1}(x) = x - 2$

(b)



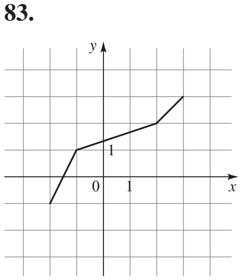
77. (a) $g^{-1}(x) = x^2 - 3, x \geq 0$

(b)



79. $x \geq 0, f^{-1}(x) = \sqrt{4 - x}$ **81.** $x \geq -2, h^{-1}(x) = \sqrt{x} - 2$

83.



85. (a) $f(x) = 500 + 80x$ **(b)** $f^{-1}(x) = \frac{1}{80}(x - 500)$, the number of hours worked as a function of the fee **(c)** 9; if he charges \$1220, he worked 9 h **87. (a)** $v^{-1}(t) = \sqrt{0.25 - \dfrac{t}{18{,}500}}$ **(b)** 0.498; at a distance 0.498 from the central axis the velocity is 30 cm/s **89. (a)** $F^{-1}(x) = \frac{5}{9}(x - 32)$; the Celsius temperature when the Fahrenheit temperature is x **(b)** $F^{-1}(86) = 30$; when the temperature is 86°F, it is 30°C

91. (a) $f(x) = \begin{cases} 0.1x & \text{if } 0 \leq x \leq 20{,}000 \\ 2000 + 0.2(x - 20{,}000) & \text{if } x > 20{,}000 \end{cases}$

(b) $f^{-1}(x) = \begin{cases} 10x & \text{if } 0 \leq x \leq 2000 \\ 10{,}000 + 5x & \text{if } x > 2000 \end{cases}$

If you pay x euros (€) in taxes, your income is $f^{-1}(x)$.
(c) $f^{-1}(10{,}000) =$ € 60,000 **93.** $f^{-1}(x) = \frac{1}{2}(x - 7)$. A pizza costing x dollars has $f^{-1}(x)$ toppings.

CHAPTER 2 REVIEW ■ PAGE 242

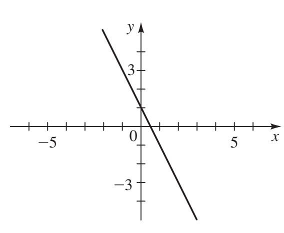
1. $f(x) = x^2 - 5$ **3.** Add 10, then multiply the result by 3.

5.

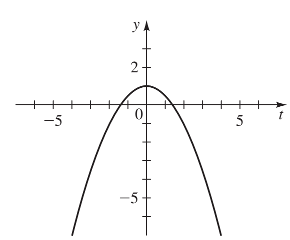
x	$g(x)$
−1	5
0	0
1	−3
2	−4
3	−3

7. (a) $C(1000) = 34{,}000$, $C(10{,}000) = 205{,}000$ **(b)** The costs of printing 1000 and 10,000 copies of the book **(c)** $C(0) = 5000$; fixed costs **(d)** \$171,000
9. $6, 2, 18, a^2 - 4a + 6, a^2 + 4a + 6, x^2 - 2x + 3, 4x^2 - 8x + 6$
11. −6 **13. (a)** Not a function **(b)** Function **(c)** Function, one-to-one **(d)** Not a function **15.** Domain $[-3, \infty)$, range $[0, \infty)$ **17.** $(-\infty, \infty)$
19. $[-4, \infty)$ **21.** $\{x \mid x \neq -2, -1, 0\}$ **23.** $(-\infty, -1] \cup [1, 4]$

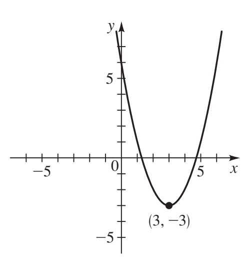
25.



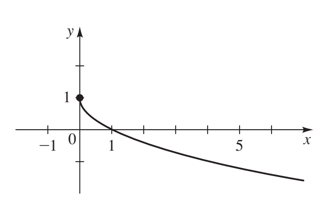
27.



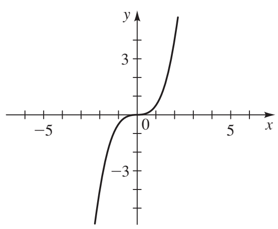
29.



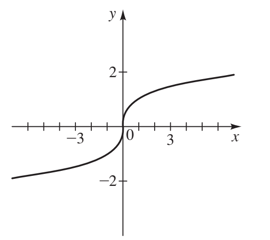
31.



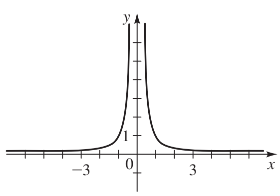
33.



35.



37.



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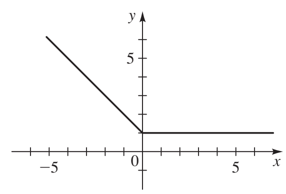
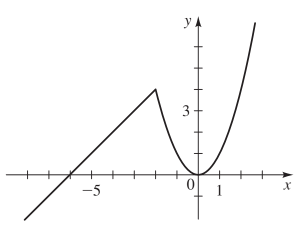
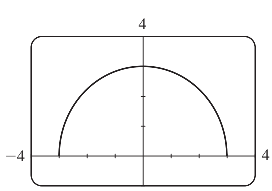


41.



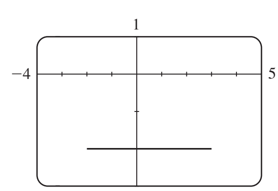
43. No **45.** Yes **47.** (iii)

49. (a)



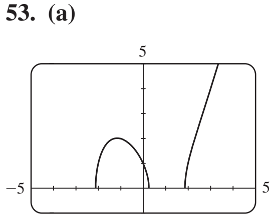
(b) Domain $[-3, 3]$, range $[0, 3]$

51. (a)



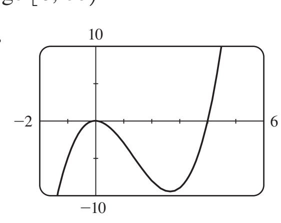
(b) Domain $[-2, 3]$, , range $\{-2\}$

53. (a)



(b) Domain $[-2.11, 0.25] \cup [1.86, \infty)$, range $[0, \infty)$

55.

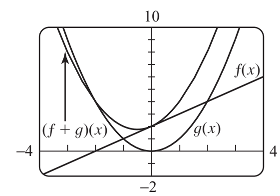


Increasing on $(-\infty, 0]$, $[2.67, \infty)$; decreasing on $[0, 2.67]$

57. 10, 5 **59.** $-\dfrac{h}{3(3+h)}, \dfrac{-1}{3(3+h)}$

61. (a) $P(10) = 5010$, $P(20) = 7040$; the populations in 1995 and 2005 **(b)** 203 people/yr; average annual population increase **63. (a)** $\frac{1}{2}, \frac{1}{2}$ **(b)** Yes, because it is a linear function **65. (a)** Shift upward 8 units **(b)** Shift to the left 8 units **(c)** Stretch vertically by a factor of 2, then shift upward 1 unit **(d)** Shift to the right 2 units and downward 2 units **(e)** Reflect in y-axis **(f)** Reflect in y-axis, then in x-axis **(g)** Reflect in x-axis **(h)** Reflect in line $y = x$ **67. (a)** Neither **(b)** Odd **(c)** Even **(d)** Neither **69.** $g(-1) = -7$ **71.** 68 ft **73.** Local maximum ≈ 3.79 when $x \approx 0.46$; local minimum ≈ 2.81 when $x \approx -0.46$

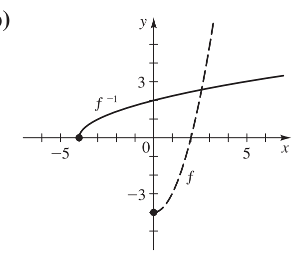
75.



77. (a) $(f+g)(x) = x^2 - 6x + 6$ **(b)** $(f-g)(x) = x^2 - 2$ **(c)** $(fg)(x) = -3x^3 + 13x^2 - 18x + 8$ **(d)** $(f/g)(x) = (x^2 - 3x + 2)/(4 - 3x)$ **(e)** $(f \circ g)(x) = 9x^2 - 15x + 6$ **(f)** $(g \circ f)(x) = -3x^2 + 9x - 2$ **79.** $(f \circ g)(x) = -3x^2 + 6x - 1, (-\infty, \infty)$; $(g \circ f)(x) = -9x^2 + 12x - 3, (-\infty, \infty)$; $(f \circ f)(x) = 9x - 4$, $(-\infty, \infty)$; $(g \circ g)(x) = -x^4 + 4x^3 - 6x^2 + 4x, (-\infty, \infty)$ **81.** $(f \circ g \circ h)(x) = 1 + \sqrt{x}$ **83.** Yes **85.** No **87.** No **89.** $f^{-1}(x) = \dfrac{x+2}{3}$ **91.** $f^{-1}(x) = \sqrt[3]{x} - 1$

93. Yes, 1, 3

95. (a), (b)



(c) $f^{-1}(x) = \sqrt{x+4}$

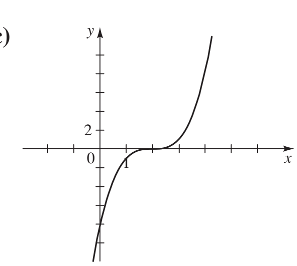
CHAPTER 2 TEST ■ PAGE 245

1. (a) and (b) are graphs of functions, (a) is one-to-one
2. (a) $2/3, \sqrt{6}/5, \sqrt{a}/(a-1)$ **(b)** $[-1, 0) \cup (0, \infty)$
3. (a) $f(x) = (x-2)^3$

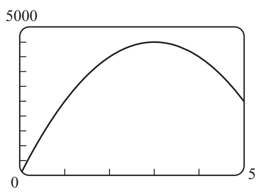
(b)

x	$f(x)$
−1	−27
0	−8
1	−1
2	0
3	1
4	8

(c)

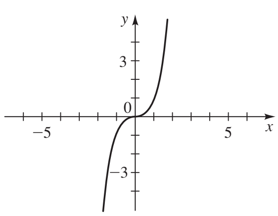


(d) By the Horizontal Line Test; take the cube root, then add 2 **(e)** $f^{-1}(x) = x^{1/3} + 2$ **4. (a)** Local minimum $f(-1) = -4$, local maxima $f(-4) = -1$ and $f(3) = 4$ **(b)** Increasing on $(-\infty, -4]$ and $[-1, 3]$, decreasing on $[-4, -1]$ and $[3, \infty)$
5. (a) $R(2) = 4000$, $R(4) = 4000$; total sales revenue with prices of \$2 and \$4
(b)

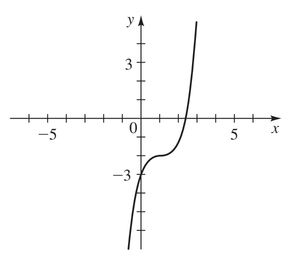


Revenue increases until price reaches \$3, then decreases

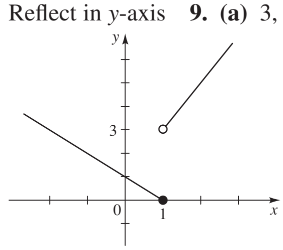
(c) \$4500; \$3 **6.** 15, 5
7. (a)



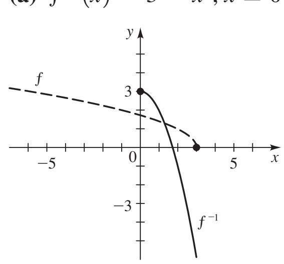
(b)



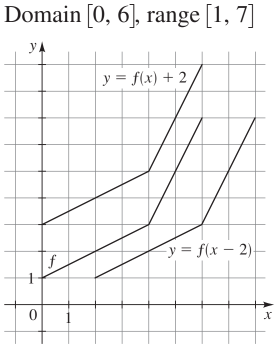
8. (a) Shift to the right 3 units, then shift upward 2 units **(b)** Reflect in y-axis **9. (a)** 3, 0
(b)



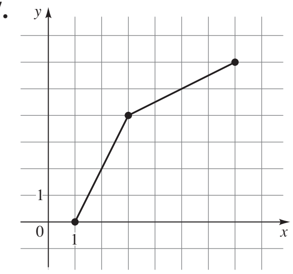
10. (a) $x^2 + 2x - 2$ **(b)** $x^2 + 4$ **(c)** $x^2 - 5x + 7$ **(d)** $x^2 + x - 2$ **(e)** 1 **(f)** 4 **(g)** $x - 9$
11. (a) $f^{-1}(x) = 3 - x^2, x \geq 0$
(b)



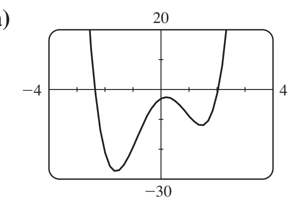
12. Domain $[0, 6]$, range $[1, 7]$ **13.** 1, 3
14.



15. 5, $\frac{5}{4}$ **16.** 0, 4
17.



18. (a)



(b) No

(c) Local minimum ≈ -27.18 when $x \approx -1.61$; local maximum ≈ -2.55 when $x \approx 0.18$; local minimum ≈ -11.93 when $x \approx 1.43$
(d) $[-27.18, \infty)$ **(e)** Increasing on $[-1.61, 0.18] \cup [1.43, \infty)$; decreasing on $(-\infty, -1.61] \cup [0.18, 1.43]$

FOCUS ON MODELING ■ PAGE 252

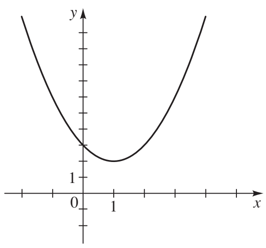
1. $A(w) = 3w^2, w > 0$ **3.** $V(w) = \frac{1}{2}w^3, w > 0$
5. $A(x) = 10x - x^2, 0 < x < 10$
7. $A(x) = (\sqrt{3}/4)x^2, x > 0$
9. $r(A) = \sqrt{A/\pi}, A > 0$
11. $S(x) = 2x^2 + (240/x), x > 0$
13. $D(t) = 25t, t \geq 0$ **15.** $A(b) = b\sqrt{4 - b}, 0 < b < 4$
17. $A(h) = 2h\sqrt{100 - h^2}, 0 < h < 10$
19. (b) $p(x) = x(19 - x)$ **(c)** 9.5, 9.5
21. (b) $A(x) = x(2400 - 2x)$ **(c)** 600 ft by 1200 ft
23. (a) $f(w) = 8w + (7200/w)$ **(b)** Width along road is 30 ft, length is 40 ft **(c)** 15 ft to 60 ft
25. (a) $A(x) = 15x - \left(\dfrac{\pi + 4}{8}\right)x^2$
(b) Width $\approx$ 8.40 ft, height of rectangular part $\approx$ 4.20 ft
27. (a) $A(x) = x^2 + (48/x)$
(b) Height $\approx$ 1.44 ft, width $\approx$ 2.88 ft
29. (a) $A(x) = 2x + \dfrac{200}{x}$ **(b)** 10 m by 10 m
31. (b) To point C, 5.1 mi from B

CHAPTER 3

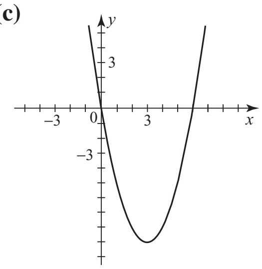
SECTION 3.1 ■ PAGE 263

1. square **2. (a)** (h, k) **(b)** upward, minimum **(c)** downward, maximum **3.** upward, (3, 5), 5, minimum
4. downward, (3, 5), 5, maximum
5. (a) (3, 4) **(b)** maximum 4 **(c)** $\mathbb{R}$, $(-\infty, 4]$
7. (a) $(1, -3)$ **(b)** minimum -3 **(c)** $\mathbb{R}$, $[-3, \infty)$

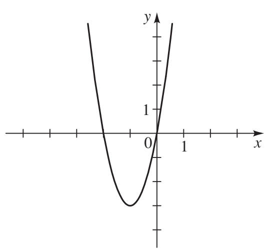
9. **(a)** $f(x) = (x-1)^2 + 2$
(b) Vertex $(1, 2)$
no x-intercepts
y-intercept 3
(c)



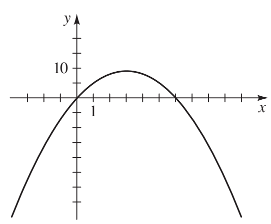
11. **(a)** $f(x) = (x-3)^2 - 9$
(b) Vertex $(3, -9)$
x-intercepts 0, 6
y-intercept 0
(c)



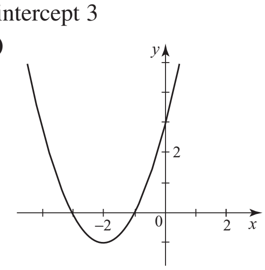
13. **(a)** $f(x) = 3(x+1)^2 - 3$
(b) Vertex $(-1, -3)$
x-intercepts -2, 0
y-intercept 0
(c)



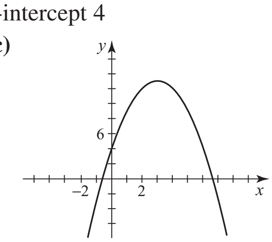
15. **(a)** $f(x) = -(x-3)^2 + 9$
(b) Vertex $(3, 9)$
x-intercepts 0, 6
y-intercept 0
(c)



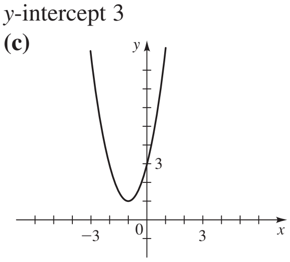
17. **(a)** $f(x) = (x+2)^2 - 1$
(b) Vertex $(-2, -1)$
x-intercepts -1, -3
y-intercept 3
(c)



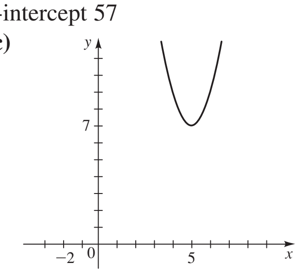
19. **(a)** $f(x) = -(x-3)^2 + 13$
(b) Vertex $(3, 13)$
x-intercepts $3 \pm \sqrt{13}$
y-intercept 4
(c)



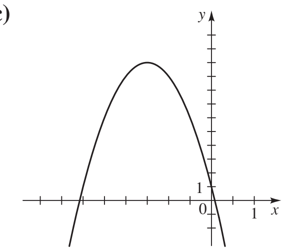
21. **(a)** $f(x) = 2(x+1)^2 + 1$
(b) Vertex $(-1, 1)$
no x-intercept
y-intercept 3
(c)



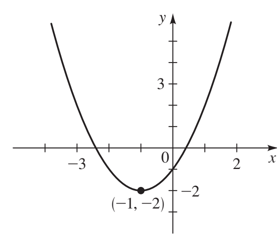
23. **(a)** $f(x) = 2(x-5)^2 + 7$
(b) Vertex $(5, 7)$
no x-intercept
y-intercept 57
(c)



25. **(a)** $f(x) = -4\left(x + \frac{3}{2}\right)^2 + 10$
(b) Vertex $\left(-\frac{3}{2}, 10\right)$; x-intercepts $-\frac{3}{2} - \frac{\sqrt{10}}{2}$, $-\frac{3}{2} + \frac{\sqrt{10}}{2}$;
y-intercept 1
(c)

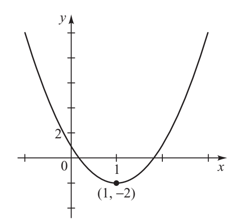


27. **(a)** $f(x) = (x+1)^2 - 2$
(b)



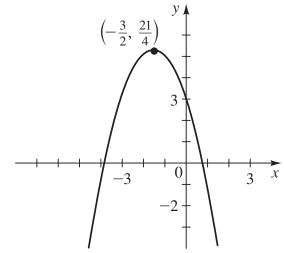
(c) Minimum $f(-1) = -2$

29. **(a)** $f(x) = 3(x-1)^2 - 2$
(b)



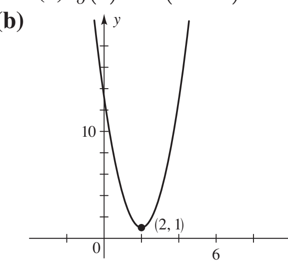
(c) Minimum $f(1) = -2$

31. **(a)** $f(x) = -\left(x + \frac{3}{2}\right)^2 + \frac{21}{4}$
(b)



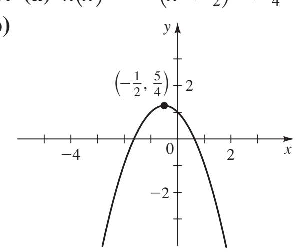
(c) Maximum $f\left(-\frac{3}{2}\right) = \frac{21}{4}$

33. **(a)** $g(x) = 3(x-2)^2 + 1$
(b)



(c) Minimum $g(2) = 1$

35. **(a)** $h(x) = -\left(x + \frac{1}{2}\right)^2 + \frac{5}{4}$
(b)

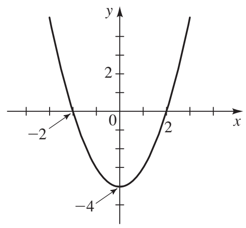


(c) Maximum $h\left(-\frac{1}{2}\right) = \frac{5}{4}$
37. Minimum $f\left(-\frac{1}{2}\right) = \frac{3}{4}$ **39.** Maximum $f(-3.5) = 185.75$
41. Minimum $f(0.6) = 15.64$ **43.** Minimum $h(-2) = -8$
45. Maximum $f(-1) = \frac{7}{2}$ **47.** $f(x) = 2x^2 - 4x$
49. $(-\infty, \infty)$, $(-\infty, 1]$ **51.** $(-\infty, \infty)$, $\left[-\frac{23}{2}, \infty\right)$
53. **(a)** -4.01 **(b)** -4.011025 **55.** 25 ft
57. \$4000, 100 units **59.** 30 times
61. 50 trees per acre **63.** 600 ft by 1200 ft
65. Width 8.40 ft, height of rectangular part 4.20 ft
67. **(a)** $f(x) = x(1200 - x)$ **(b)** 600 ft by 600 ft
69. **(a)** $R(x) = x(57{,}000 - 3000x)$ **(b)** \$9.50 **(c)** \$19.00

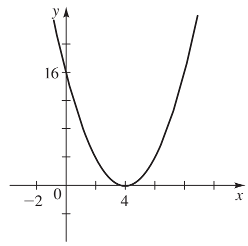
SECTION 3.2 ■ PAGE 277

1. II **2.** **(a)** $\infty, -\infty$ **(b)** $-\infty, -\infty$ **3.** **(a)** 0 **(b)** factor **(c)** x **4.** (a)

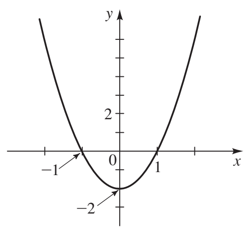
5. **(a)**



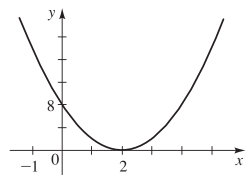
(b)



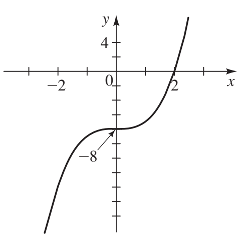
(c)



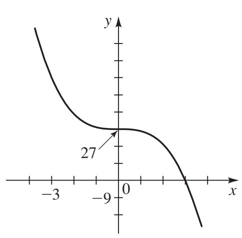
(d)



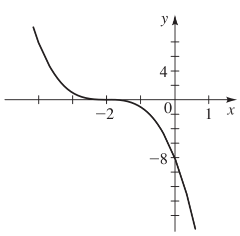
7. **(a)**



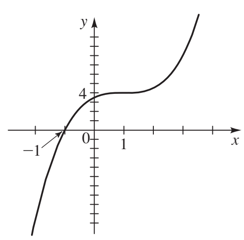
(b)



(c)



(d)

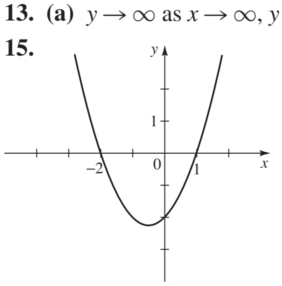


9. **(a)** $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$ **(b)** III

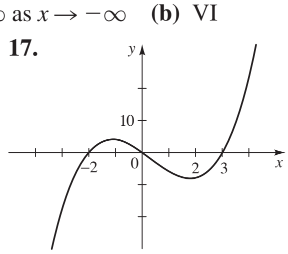
11. **(a)** $y \to -\infty$ as $x \to \infty$, $y \to \infty$ as $x \to -\infty$ **(b)** V

13. **(a)** $y \to \infty$ as $x \to \infty$, $y \to \infty$ as $x \to -\infty$ **(b)** VI

15.



17.



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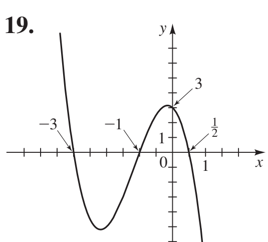

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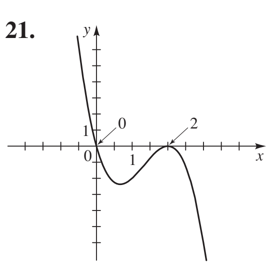
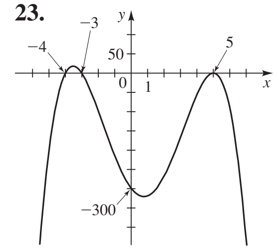
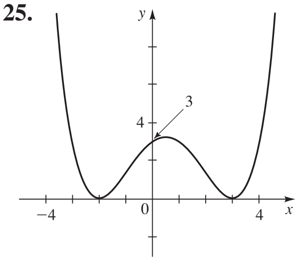


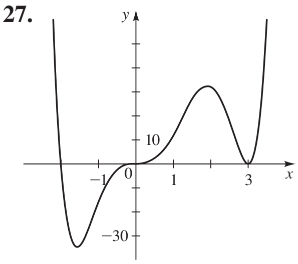
23.



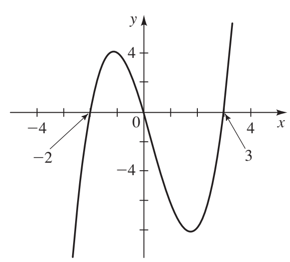
25.



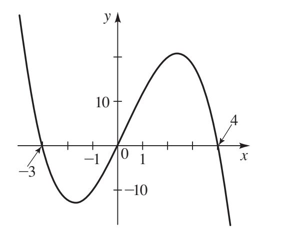
27.



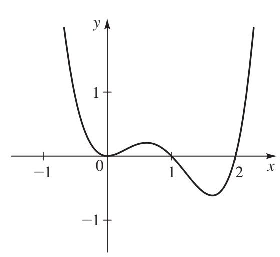
29. $P(x) = x(x + 2)(x - 3)$



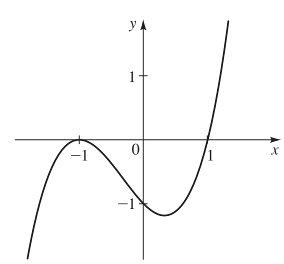
31. $P(x) = -x(x + 3)(x - 4)$



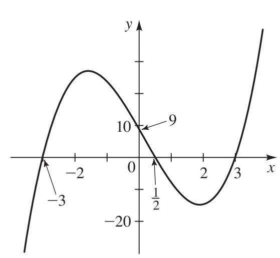
33. $P(x) = x^2(x - 1)(x - 2)$



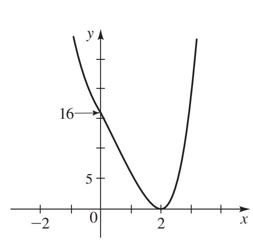
35. $P(x) = (x + 1)^2(x - 1)$



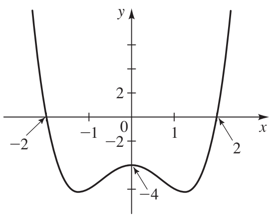
37. $P(x) = (2x - 1)(x + 3)(x - 3)$



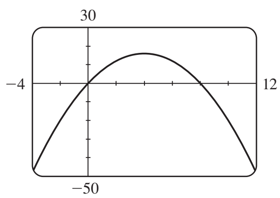
39. $P(x) = (x - 2)^2(x^2 + 2x + 4)$



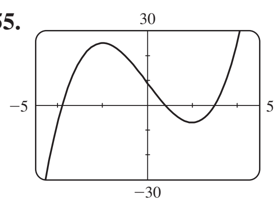
41. $P(x) = (x^2 + 1)(x + 2)(x - 2)$



43. $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$
45. $y \to \infty$ as $x \to \pm\infty$
47. $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$
49. **(a)** x-intercepts 0, 4; y-intercept 0 **(b)** Maximum $(2, 4)$
51. **(a)** x-intercepts -2, 1; y-intercept -1
(b) Minimum $(-1, -2)$, maximum $(1, 0)$
53. local maximum $(4, 16)$

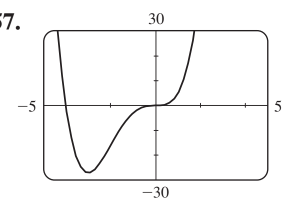


55.



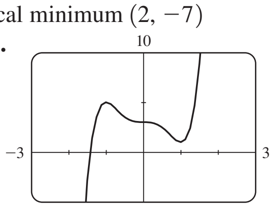
local maximum $(-2, 25)$,
local minimum $(2, -7)$

57.

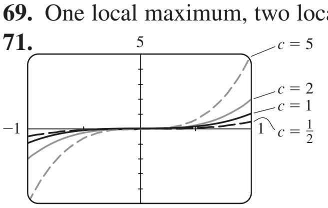


local minimum $(-3, -27)$

59.

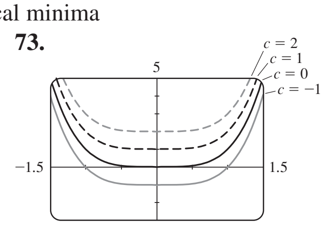


local maximum $(-1, 5)$,
local minimum $(1, 1)$
61. One local maximum, no local minimum
63. One local maximum, one local minimum
65. One local maximum, two local minima
67. No local extrema
69. One local maximum, two local minima
71.



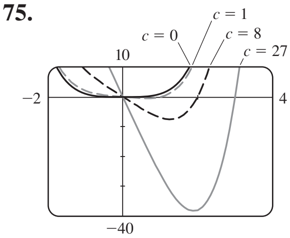
Increasing the value of c stretches the graph vertically.

73.



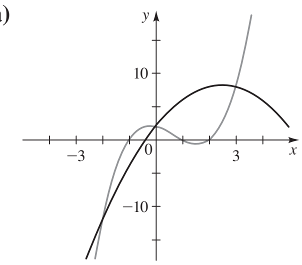
Increasing the value of c moves the graph up.

75.

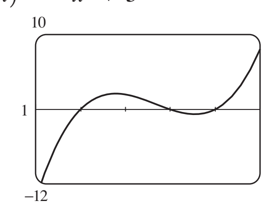


Increasing the value of c causes a deeper dip in the graph in the fourth quadrant and moves the positive x-intercept to the right.

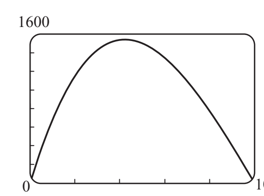
77. **(a)**



(b) Three **(c)** $(0, 2), (3, 8), (-2, -12)$
79. **(d)** $P(x) = P_O(x) + P_E(x)$, where
$P_O(x) = x^5 + 6x^3 - 2x$ and $P_E(x) = -x^2 + 5$
81. **(a)** Two local extrema



83. **(a)** 26 blenders **(b)** No; \$3276.22
85. **(a)** $V(x) = 4x^3 - 120x^2 + 800x$ **(b)** $0 < x < 10$
(c) Maximum volume $\approx 1539.6 \text{ cm}^3$



SECTION 3.3 ■ PAGE 285

1. quotient, remainder **2.** **(a)** factor **(b)** k

3. $x + 1 + \dfrac{-11}{x + 3}$ **5.** $2x - \dfrac{1}{2} + \dfrac{-\frac{15}{2}}{2x - 1}$

7. $2x^2 - x + 1 + \dfrac{4x - 4}{x^2 + 4}$ **9.** $(x + 3)(3x - 4) + 8$

11. $(2x - 3)(x^2 - 1) - 3$

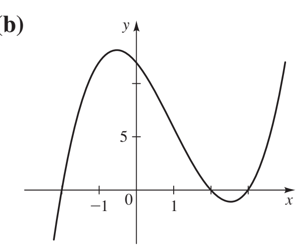
13. $(x^2 + 3)(x^2 - x - 3) + (7x + 11)$

In answers 15–37 the first polynomial given is the quotient, and the second is the remainder.
15. $x - 2, -16$ **17.** $2x^2 - 1, -2$ **19.** $x + 2, 8x - 1$
21. $3x + 1, 7x - 5$ **23.** $x^4 + 1, 0$ **25.** $x - 2, -2$
27. $3x + 23, 138$ **29.** $x^2 + 2, -3$ **31.** $x^2 - 3x + 1, -1$
33. $x^4 + x^3 + 4x^2 + 4x + 4, -2$ **35.** $2x^2 + 4x, 1$
37. $x^2 + 3x + 9, 0$ **39.** -3 **41.** 12 **43.** -7 **45.** -483
47. 2159 **49.** $\frac{7}{3}$ **51.** -8.279 **57.** $-1 \pm \sqrt{6}$
59. $x^3 - 3x^2 - x + 3$ **61.** $x^4 - 8x^3 + 14x^2 + 8x - 15$

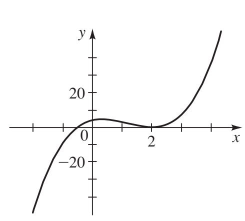
63. $-\frac{3}{2}x^3 + 3x^2 + \frac{15}{2}x - 9$ **65.** $(x + 1)(x - 1)(x - 2)$
67. $(x + 2)^2(x - 1)^2$

SECTION 3.4 ■ PAGE 294

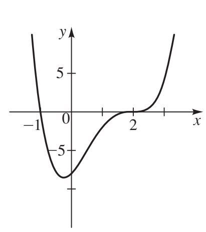
1. $a_0, a_n, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm 2, \pm\frac{2}{3}, \pm 5, \pm\frac{5}{2}, \pm\frac{5}{3}, \pm\frac{5}{6}, \pm 10, \pm\frac{10}{3}$
2. 1, 3, 5; 0 **3.** True **4.** False **5.** $\pm 1, \pm 3$ **7.** $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{2}$ **9.** $\pm 1, \pm 7, \pm\frac{1}{2}, \pm\frac{7}{2}, \pm\frac{1}{4}, \pm\frac{7}{4}$ **11. (a)** $\pm 1, \pm\frac{1}{5}$
(b) $-1, 1, \frac{1}{5}$ **13. (a)** $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$ **(b)** $-\frac{1}{2}, 1, 3$
15. $-1, 2, 3; P(x) = (x + 1)(x - 2)(x - 3)$
17. $-2, 1; P(x) = (x + 2)^2(x - 1)$
19. $-3, 2; P(x) = (x + 3)^2(x - 2)$ **21.** $2; P(x) = (x - 2)^3$
23. $-1, 2, 3; P(x) = (x + 1)(x - 2)(x - 3)$
25. $-3, -1, 1; P(x) = (x + 3)(x + 1)(x - 1)$
27. $\pm 1, \pm 2; P(x) = (x - 2)(x + 2)(x - 1)(x + 1)$
29. $-4, -2, -1, 1; P(x) = (x + 4)(x + 2)(x - 1)(x + 1)$
31. $\pm 2, \pm\frac{3}{2}; P(x) = (x - 2)(x + 2)(2x - 3)(2x + 3)$
33. $\pm 2, \frac{1}{3}, 3; P(x) = (x - 2)(x + 2)(x - 3)(3x - 1)$
35. $-1, \pm\frac{1}{2}; P(x) = (x + 1)(2x - 1)(2x + 1)$
37. $-\frac{3}{2}, \frac{1}{2}, 1; P(x) = (x - 1)(2x + 3)(2x - 1)$
39. $-\frac{5}{2}, -1, \frac{3}{2}; P(x) = (x + 1)(2x + 5)(2x - 3)$
41. $-\frac{1}{2}, \frac{2}{5}, \frac{1}{2}; P(x) = (2x - 1)(5x - 2)(2x + 1)$
43. $-1, \frac{1}{2}, 2; P(x) = (x + 1)(x - 2)^2(2x - 1)$
45. $-3, -2, 1, 3; P(x) = (x + 3)(x + 2)^2(x - 1)(x - 3)$
47. $-1, -\frac{1}{3}, 2, 5; P(x) = (x + 1)^2(x - 2)(x - 5)(3x + 1)$
49. $-2, -1 \pm \sqrt{2}$ **51.** $-1, 4, \dfrac{3 \pm \sqrt{13}}{2}$
53. $3, \dfrac{1 \pm \sqrt{5}}{2}$ **55.** $\frac{1}{2}, \dfrac{1 \pm \sqrt{3}}{2}$ **57.** $-1, -\frac{1}{2}, -3 \pm \sqrt{10}$
59. (a) $-2, 2, 3$ **(b)**



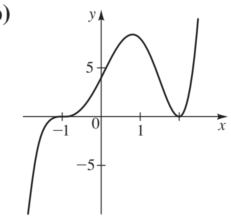
61. (a) $-\frac{1}{2}, 2$ **(b)**



63. (a) $-1, 2$ **(b)**



65. (a) $-1, 2$ **(b)**



67. 1 positive, 2 or 0 negative; 3 or 1 real **69.** 1 positive, 1 negative; 2 real **71.** 2 or 0 positive, 0 negative; 3 or 1 real (since 0 is a zero but is neither positive nor negative) **81.** $3, -2$
83. $3, -1$ **85.** $-2, \frac{1}{2}, \pm 1$ **87.** $\pm\frac{1}{2}, \pm\sqrt{5}$ **89.** $-2, 1, 3, 4$
95. $-2, 2, 3$ **97.** $-\frac{3}{2}, -1, 1, 4$ **99.** $-1.28, 1.53$ **101.** -1.50
105. 11.3 ft **107. (a)** It began to snow again. **(b)** No
(c) Just before midnight on Saturday night **109.** 2.76 m
111. 88 in. (or 3.21 in.)

SECTION 3.5 ■ PAGE 302

1. -1 **2.** 3, 4 **3. (a)** $3 - 4i$ **(b)** $9 + 16 = 25$ **4.** $3 - 4i$
5. Real part 5, imaginary part -7 **7.** Real part $-\frac{2}{3}$, imaginary part $-\frac{5}{3}$ **9.** Real part 3, imaginary part 0 **11.** Real part 0, imaginary part $-\frac{2}{3}$ **13.** Real part $\sqrt{3}$, imaginary part 2 **15.** $3 + 7i$
17. $5 - i$ **19.** $3 + 5i$ **21.** $2 - 2i$ **23.** $-19 + 4i$
25. $-4 + 8i$ **27.** $-3 - 15i$ **29.** $30 + 10i$ **31.** $-33 - 56i$
33. $27 - 8i$ **35.** $-i$ **37.** $\frac{8}{5} + \frac{1}{5}i$ **39.** $-5 + 12i$ **41.** $-4 + 2i$
43. $2 - \frac{4}{3}i$ **45.** $-i$ **47.** $-i$ **49.** $243i$ **51.** 1 **53.** $5i$
55. -6 **57.** $(3 + \sqrt{5}) + (3 - \sqrt{5})i$ **59.** 2 **61.** $-i\sqrt{2}$
63. $\pm 7i$ **65.** $2 \pm i$ **67.** $-1 \pm 2i$ **69.** $-\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$
71. $\frac{1}{2} \pm \frac{1}{2}i$ **73.** $-\dfrac{3}{2} \pm \dfrac{\sqrt{3}}{2}i$ **75.** $\dfrac{-6 \pm \sqrt{6}i}{6}$ **77.** $1 \pm 3i$

SECTION 3.6 ■ PAGE 310

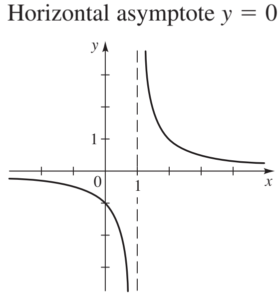
1. $5, -2, 3, 1$ **2. (a)** $x - a$ **(b)** $(x - a)^m$ **3.** n **4.** $a - bi$
5. (a) $0, \pm 2i$ **(b)** $x^2(x - 2i)(x + 2i)$
7. (a) $0, 1 \pm i$ **(b)** $x(x - 1 - i)(x - 1 + i)$
9. (a) $\pm i$ **(b)** $(x - i)^2(x + i)^2$
11. (a) $\pm 2, \pm 2i$ **(b)** $(x - 2)(x + 2)(x - 2i)(x + 2i)$
13. (a) $-2, 1 \pm i\sqrt{3}$
(b) $(x + 2)(x - 1 - i\sqrt{3})(x - 1 + i\sqrt{3})$
15. (a) $\pm 1, \frac{1}{2} \pm \frac{1}{2}i\sqrt{3}, -\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$
(b) $(x - 1)(x + 1)(x - \frac{1}{2} - \frac{1}{2}i\sqrt{3})(x - \frac{1}{2} + \frac{1}{2}i\sqrt{3}) \times (x + \frac{1}{2} - \frac{1}{2}i\sqrt{3})(x + \frac{1}{2} + \frac{1}{2}i\sqrt{3})$

In answers 17–33 the factored form is given first, then the zeros are listed with the multiplicity of each in parentheses.
17. $(x - 5i)(x + 5i); \pm 5i\ (1)$
19. $[x - (-1 + i)][x - (-1 - i)]; -1 + i\ (1), -1 - i\ (1)$
21. $x(x - 2i)(x + 2i); 0\ (1), 2i\ (1), -2i\ (1)$
23. $(x - 1)(x + 1)(x - i)(x + i); 1\ (1), -1\ (1), i\ (1), -i\ (1)$
25. $16(x - \frac{3}{2})(x + \frac{3}{2})(x - \frac{3}{2}i)(x + \frac{3}{2}i); \frac{3}{2}\ (1), -\frac{3}{2}\ (1), \frac{3}{2}i\ (1), -\frac{3}{2}i\ (1)$
27. $(x + 1)(x - 3i)(x + 3i); -1\ (1), 3i\ (1), -3i\ (1)$
29. $(x - i)^2(x + i)^2; i\ (2), -i\ (2)$
31. $(x - 1)(x + 1)(x - 2i)(x + 2i); 1\ (1), -1\ (1), 2i\ (1), -2i\ (1)$

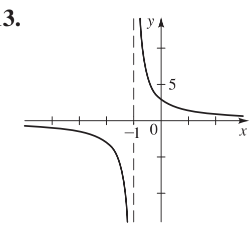
33. $x(x - i\sqrt{3})^2(x + i\sqrt{3})^2$; 0 (1), $i\sqrt{3}$ (2), $-i\sqrt{3}$ (2)
35. $P(x) = x^2 - 2x + 2$ **37.** $Q(x) = x^3 - 3x^2 + 4x - 12$
39. $P(x) = x^3 - 2x^2 + x - 2$
41. $R(x) = x^4 - 4x^3 + 10x^2 - 12x + 5$
43. $T(x) = 6x^4 - 12x^3 + 18x^2 - 12x + 12$ **45.** $-2, \pm 2i$
47. $1, \dfrac{1 \pm i\sqrt{3}}{2}$ **49.** $2, \dfrac{1 \pm i\sqrt{3}}{2}$ **51.** $-\frac{3}{2}, -1 \pm i\sqrt{2}$
53. $-2, 1, \pm 3i$ **55.** $1, \pm 2i, \pm i\sqrt{3}$
57. 3 (multiplicity 2), $\pm 2i$ **59.** $-\frac{1}{2}$ (multiplicity 2), $\pm i$
61. 1 (multiplicity 3), $\pm 3i$
63. **(a)** $(x - 5)(x^2 + 4)$ **(b)** $(x - 5)(x - 2i)(x + 2i)$
65. **(a)** $(x - 1)(x + 1)(x^2 + 9)$
(b) $(x - 1)(x + 1)(x - 3i)(x + 3i)$
67. **(a)** $(x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4)$
(b) $(x - 2)(x + 2)[x - (1 + i\sqrt{3})][x - (1 - i\sqrt{3})] \times [x + (1 + i\sqrt{3})][x + (1 - i\sqrt{3})]$
69. **(a)** 4 real **(b)** 2 real, 2 imaginary **(c)** 4 imaginary

SECTION 3.7 ■ PAGE 323

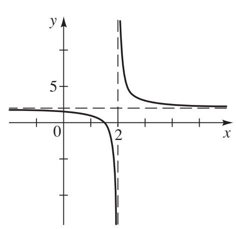
1. $-\infty, \infty$ **2.** 2 **3.** $-1, 2$ **4.** $\frac{1}{3}$ **5.** $-2, 3$ **6.** 1
7. **(a)** $-3, -19, -199, -1999$; $5, 21, 201, 2001$;
$1.2500, 1.0417, 1.0204, 1.0020$; $0.8333, 0.9615, 0.9804, 0.9980$
(b) $r(x) \to -\infty$ as $x \to 2^-$; $r(x) \to \infty$ as $x \to 2^+$
(c) Horizontal asymptote $y = 1$
9. **(a)** $-22, -430, -40{,}300, -4{,}003{,}000$; $-10, -370, -39{,}700, -3{,}997{,}000$; $0.3125, 0.0608, 0.0302, 0.0030$; $-0.2778, -0.0592, -0.0298, -0.0030$
(b) $r(x) \to -\infty$ as $x \to 2^-$; $r(x) \to -\infty$ as $x \to 2^+$
(c) Horizontal asymptote $y = 0$

11.


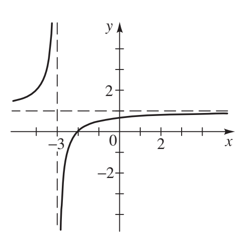
domain $\{x \mid x \neq 1\}$
range $\{y \mid y \neq 0\}$

13.


domain $\{x \mid x \neq -1\}$
range $\{y \mid y \neq 0\}$

15.


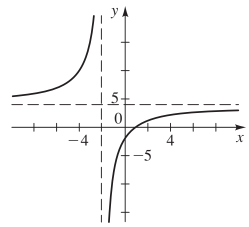
domain $\{x \mid x \neq 2\}$
range $\{y \mid y \neq 2\}$

17.


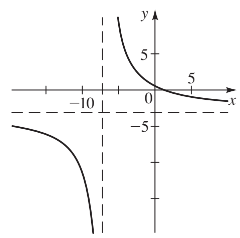
domain $\{x \mid x \neq -3\}$
range $\{y \mid y \neq 1\}$

19. x-intercept 1, y-intercept $-\frac{1}{4}$
21. x-intercepts $-1, 2$; y-intercept $\frac{1}{3}$
23. x-intercepts $-3, 3$; no y-intercept **25.** x-intercept 3, y-intercept 3, vertical $x = 2$; horizontal $y = 2$ **27.** x-intercepts $-1, 1$; y-intercept $\frac{1}{4}$; vertical $x = -2, x = 2$; horizontal $y = 1$
29. Vertical $x = 2$; horizontal $y = 0$ **31.** Horizontal $y = 0$

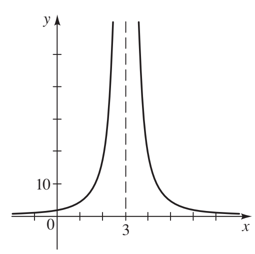
33. Vertical $x = \frac{1}{2}, x = -1$; horizontal $y = 3$
35. Vertical $x = \frac{1}{3}, x = -2$; horizontal $y = \frac{5}{3}$
37. Vertical $x = 0$; horizontal $y = 3$ **39.** Vertical $x = 1$

41.


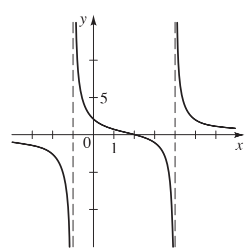
x-intercept 1
y-intercept -2
vertical $x = -2$
horizontal $y = 4$
domain $\{x \mid x \neq -2\}$
range $\{y \mid y \neq 4\}$

43.


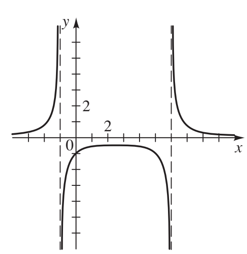
x-intercept $\frac{4}{3}$
y-intercept $\frac{4}{7}$
vertical $x = -7$
horizontal $y = -3$
domain $\{x \mid x \neq -7\}$
range $\{y \mid y \neq -3\}$

45.


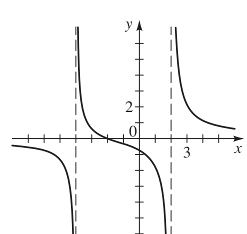
y-intercept 2
vertical $x = 3$
horizontal $y = 0$
domain $\{x \mid x \neq 3\}$
range $\{y \mid y > 0\}$

47.


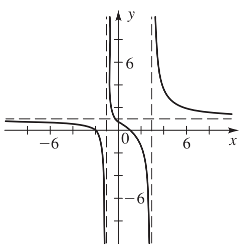
x-intercept 2
y-intercept 2
vertical $x = -1, x = 4$
horizontal $y = 0$
domain $\{x \mid x \neq -1, 4\}$
range $\mathbb{R}$

49.


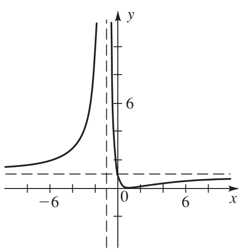
y-intercept -1
vertical $x = -1, x = 6$
horizontal $y = 0$
domain $\{x \mid x \neq -1, 6\}$
range $\{y \mid y \leq -0.5 \text{ or } y > 0\}$

51.


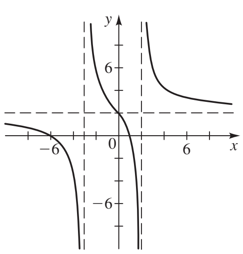
x-intercept -2
y-intercept $-\frac{3}{4}$
vertical $x = -4, x = 2$
horizontal $y = 0$
domain $\{x \mid x \neq -4, 2\}$
range $\mathbb{R}$

53. 

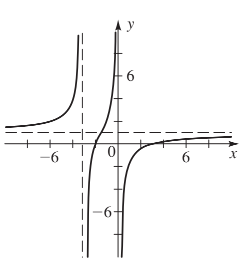
x-intercepts $-2, 1$
y-intercept $\frac{2}{3}$
vertical $x = -1, x = 3$
horizontal $y = 1$
domain $\{x \mid x \neq -1, 3\}$
range $\mathbb{R}$

55. 

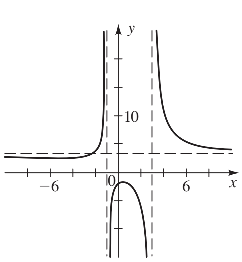
x-intercept 1
y-intercept 1
vertical $x = -1$
horizontal $y = 1$
domain $\{x \mid x \neq -1\}$
range $\{y \mid y \geq 0\}$

57. 

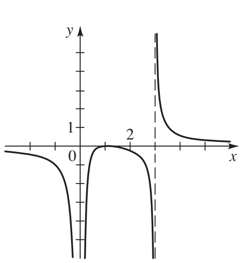
x-intercepts $-6, 1$
y-intercept 2
vertical $x = -3, x = 2$
horizontal $y = 2$
domain $\{x \mid x \neq -3, 2\}$
range $\mathbb{R}$

59. 

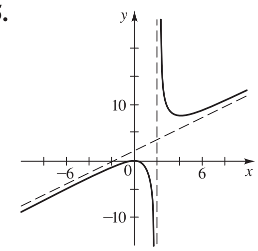
x-intercepts $-2, 3$
vertical $x = -3, x = 0$
horizontal $y = 1$
domain $\{x \mid x \neq -3, 0\}$
range $\mathbb{R}$

61. 

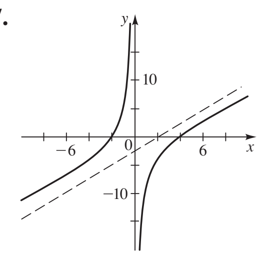
y-intercept -2
vertical $x = -1, x = 3$
horizontal $y = 3$
domain $\{x \mid x \neq -1, 3\}$
range $\{y \mid y \leq -1.5 \text{ or } y \geq 2.4\}$

63. 

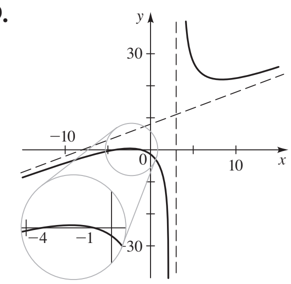
x-intercept 1
vertical $x = 0, x = 3$
horizontal $y = 0$
domain $\{x \mid x \neq 0, 3\}$
range $\mathbb{R}$

65. 

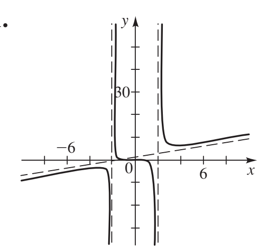
slant $y = x + 2$
vertical $x = 2$

67. 

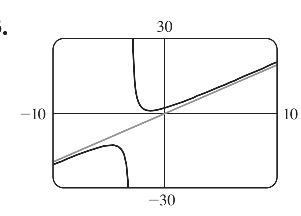
slant $y = x - 2$
vertical $x = 0$

69. 

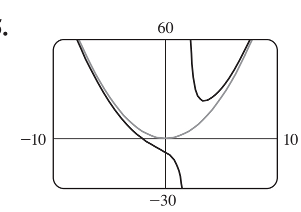
slant $y = x + 8$
vertical $x = 3$

71. 

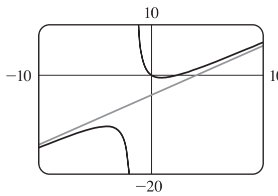
slant $y = x + 1$
vertical $x = 2, x = -2$

73. 

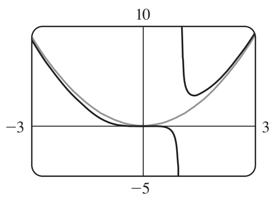
vertical $x = -3$

75. 

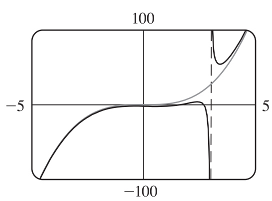
vertical $x = 2$

77. 

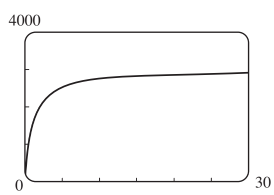
vertical $x = -1.5$
x-intercepts 0, 2.5
y-intercept 0, local maximum $(-3.9, -10.4)$
local minimum $(0.9, -0.6)$
end behavior: $y = x - 4$

79. 

vertical $x = 1$
x-intercept 0
y-intercept 0
local minimum $(1.4, 3.1)$
end behavior: $y = x^2$

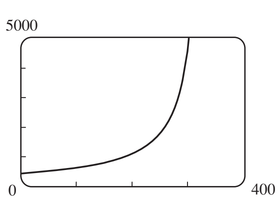
81. 

vertical $x = 3$
x-intercepts 1.6, 2.7
y-intercept -2
local maxima $(-0.4, -1.8)$, $(2.4, 3.8)$,
local minima $(0.6, -2.3)$, $(3.4, 54.3)$
end behavior $y = x^3$

83. **(a)** 

(b) It levels off at 3000.

85. **(a)** 2.50 mg/L **(b)** It decreases to 0. **(c)** 16.61 h

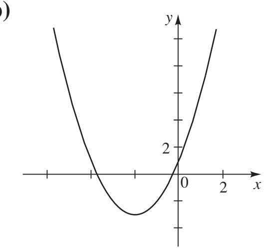
87. 

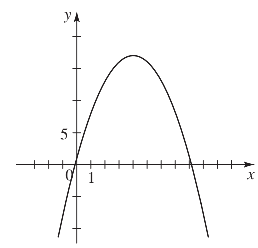
If the speed of the train approaches the speed of sound, then the pitch increases indefinitely (a sonic boom).

SECTION 3.8 ■ PAGE 330

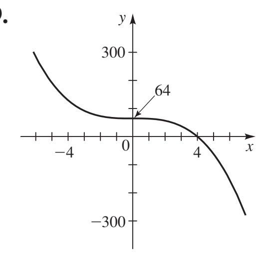
1. directly proportional; proportionality **2.** inversely proportional; proportionality **3.** directly proportional; inversely proportional
4. $\frac{1}{2}xy$ **5.** $T = kx$ **7.** $v = k/z$ **9.** $y = ks/t$ **11.** $z = k\sqrt{y}$
13. $V = klwh$ **15.** $R = k\dfrac{i}{Pt}$ **17.** $y = 7x$ **19.** $R = 12/s$
21. $M = 15x/y$ **23.** $W = 360/r^2$ **25.** $C = 16lwh$
27. $s = 500/\sqrt{t}$ **29.** **(a)** $z = k\dfrac{x^3}{y^2}$ **(b)** $\frac{27}{4}$ **31.** **(a)** $z = kx^3y^5$
(b) 864 **33.** **(a)** $F = kx$ **(b)** 8 **(c)** 32 N **35.** **(a)** $P = ks^3$
(b) 0.012 **(c)** 324 **37.** 40 mi/h (for safety round down, not up)
39. 5.3 mi/h **41.** **(a)** $P = kT/V$ **(b)** 8.3 **(c)** 51.9 kPa
43. **(a)** $L = \dfrac{k}{d^2}$ **(b)** 7000 **(c)** $\frac{1}{4}$ **(d)** 4
45. **(a)** $R = kL/d^2$ **(b)** $0.00291\overline{6}$ **(c)** $R \approx 137\ \Omega$ **(d)** $\frac{3}{4}$
47. **(a)** 160,000 **(b)** 1,930,670,340
49. **(a)** $T = k\sqrt{l}$ **(b)** quadruple the length l
51. **(a)** $f = k/L$ **(b)** Halves it
53. $3.47 \times 10^{-14}\ \mathrm{W/m^2}$

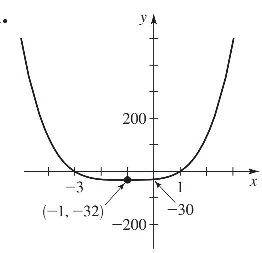
CHAPTER 3 REVIEW ■ PAGE 335

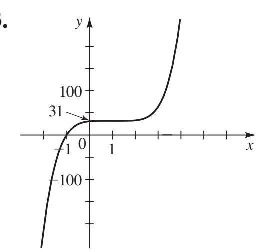
1. **(a)** $f(x) = (x + 2)^2 - 3$
(b) 

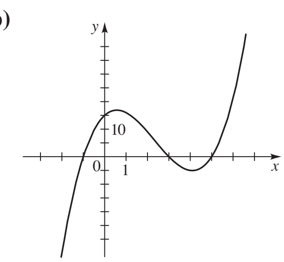
3. **(a)** $g(x) = -(x - 4)^2 + 17$
(b) 

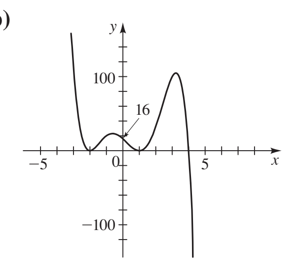
5. Minimum $f(-1) = -7$ **7.** 68 feet

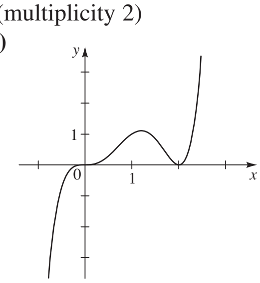
9. 

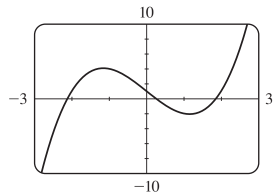
11. 

13. 

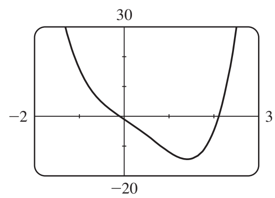
15. **(a)** $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$
(b) 

17. **(a)** $y \to -\infty$ as $x \to \infty$, $y \to \infty$ as $x \to -\infty$
(b) 

19. **(a)** 0 (multiplicity 3), 2 (multiplicity 2)
(b) 

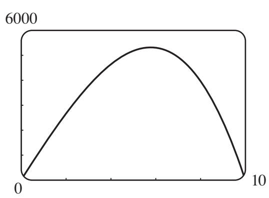
21. 

x-intercepts $-2.1, 0.3, 1.9$
y-intercept 1
local maximum $(-1.2, 4.1)$
local minimum $(1.2, -2.1)$
$y \to \infty$ as $x \to \infty$
$y \to -\infty$ as $x \to -\infty$

23. 

x-intercepts $-0.1, 2.1$
y-intercept -1
local minimum $(1.4, -14.5)$
$y \to \infty$ as $x \to \infty$
$y \to \infty$ as $x \to -\infty$

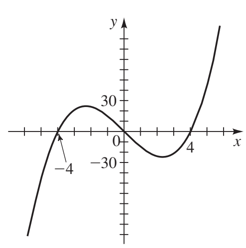
25. (a) $S = 13.8x(100 - x^2)$ **(b)** $0 \le x \le 10$

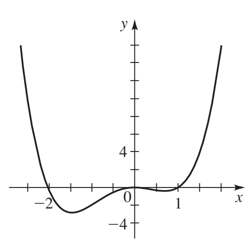
(c) 

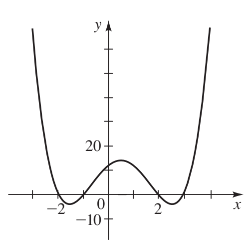
(d) 5.8 in.

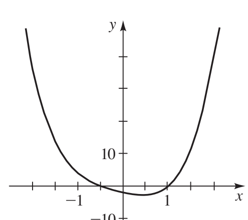
In answers 27–33 the first polynomial given is the quotient, and the second is the remainder.

27. $x - 1, 3$ **29.** $x^2 + 3x + 23, 94$
31. $x^3 - 5x^2 + 17x - 83, 422$ **33.** $2x - 3, 12$ **35.** 3
39. 8 **41. (a)** $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$
(b) 2 or 0 positive, 3 or 1 negative

43. (a) $-4, 0, 4$
(b) 

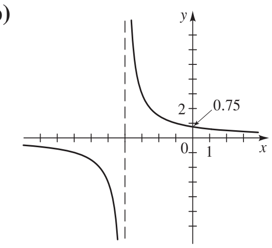
45. (a) -2, 0 (multiplicity 2), 1
(b) 

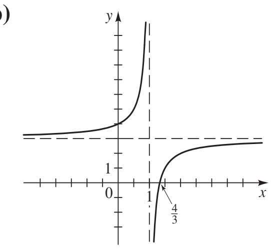
47. (a) $-2, -1, 2, 3$
(b) 

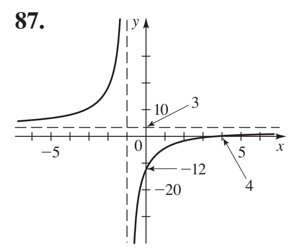
49. (a) $-\frac{1}{2}, 1$
(b) 

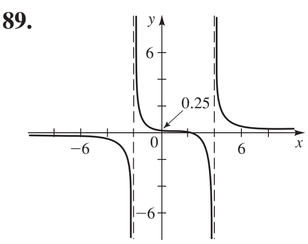
51. $3 + i$ **53.** $8 - i$ **55.** $\frac{6}{5} + \frac{8}{5}i$ **57.** i **59.** 2
61. $4x^3 - 18x^2 + 14x + 12$
63. No; since the complex conjugates of imaginary zeros will also be zeros, the polynomial would have 8 zeros, contradicting the requirement that it have degree 4. **65.** $1, \pm i$ **67.** $-3, 1, 5$
69. $-1 \pm 2i$, -2 (multiplicity 2)

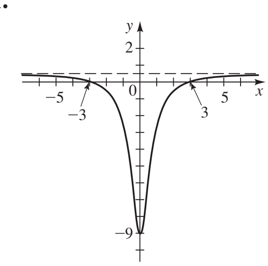
71. ± 2, 1 (multiplicity 3) **73.** $\pm 2, \pm 1 \pm i\sqrt{3}$
75. $1, 3, \dfrac{-1 \pm i\sqrt{7}}{2}$ **77.** $x = -0.5, 3$
79. $x \approx -0.24, 4.24$
81. $2, P(x) = (x - 2)(x^2 + 2x + 2)$

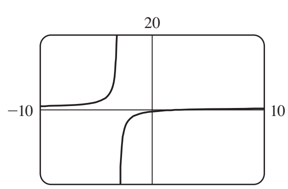
83. (a) Vertical asymptote $x = -4$, horizontal asymptote $y = 0$, no x-intercept, y-intercept $\frac{3}{4}$, domain $\{x \mid x \ne -4\}$ range $\{y \mid y \ne 0\}$
(b) 

85. (a) Vertical asymptote $x = 1$, horizontal asymptote $y = 3$, x-intercept $\frac{4}{3}$, y-intercept -4, domain $\{x \mid x \ne 1\}$ range $\{y \mid y \ne 3\}$
(b) 

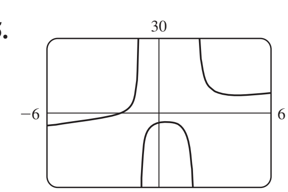
87. 

89. 

91. 

93. 

x-intercept 3
y-intercept -0.5
vertical $x = -3$
horizontal $y = 0.5$
no local extrema

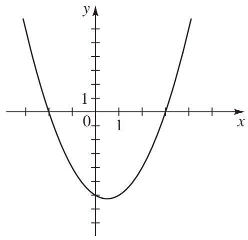
95. 

x-intercept -2
y-intercept -4
vertical $x = -1, x = 2$
slant $y = x + 1$
local maximum $(0.425, -3.599)$
local minimum $(4.216, 7.175)$

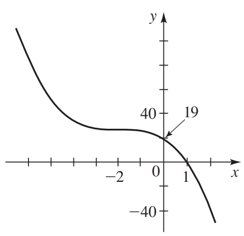
97. $(-2, -28), (1, 26), (2, 68), (5, 770)$
99. $z = 192/y$ **101.** 8 in. **103.** 329.4 ft

CHAPTER 3 TEST ■ PAGE 338

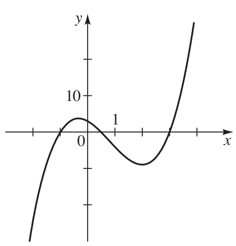
1. $f(x) = \left(x - \frac{1}{2}\right)^2 - \frac{25}{4}$



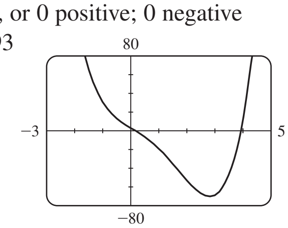
2. Minimum $f\left(-\frac{3}{2}\right) = -\frac{3}{2}$
3. (a) 2500 ft **(b)** 1000 ft
4.



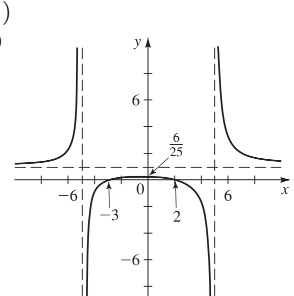
5. (a) $x^3 + 2x^2 + 2, 9$ **(b)** $x^3 + 2x^2 + \frac{1}{2}, \frac{15}{2}$
6. (a) $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$ **(b)** $2(x - 3)\left(x - \frac{1}{2}\right)(x + 1)$
(c) $-1, \frac{1}{2}, 3$ **(d)**



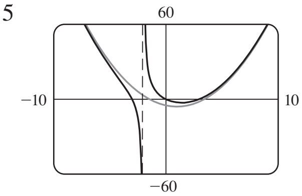
7. (a) $7 + i$ **(b)** $-1 - 5i$ **(c)** $18 + i$ **(d)** $\frac{6}{25} - \frac{17}{25}i$
(e) 1 **(f)** $6 - 2i$
8. $3, -1 \pm i$
9. $(x - 1)^2(x - 2i)(x + 2i)$
10. $x^4 + 2x^3 + 10x^2 + 18x + 9$
11. (a) 4, 2, or 0 positive; 0 negative
(c) 0.17, 3.93



(d) Local minimum $(2.8, -70.3)$
12. (a) r, u **(b)** s **(c)** s **(d)**



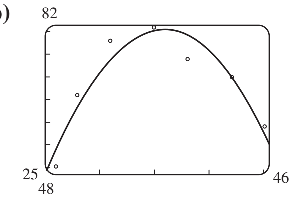
(e) $x^2 - 2x - 5$



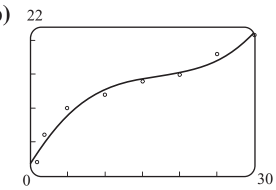
13. (a) $M = kwh^2/L$ **(b)** 400 **(c)** 12,000 lb

FOCUS ON MODELING ■ PAGE 342

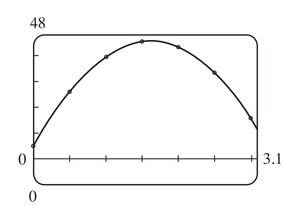
1. (a) $y = -0.275428x^2 + 19.7485x - 273.5523$
(b)



(c) 35.85 lb/in^2
3. (a) $y = 0.00203708x^3 - 0.104521x^2 + 1.966206x + 1.45576$
(b)



(c) 43 vegetables **(d)** 2.0 s
5. (a) Degree 2
(b) $y = -16.0x^2 + 51.8429x + 4.20714$

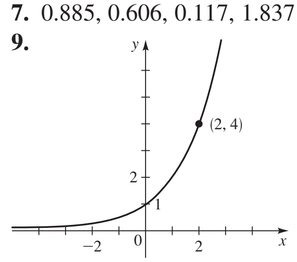


(c) 0.3 s and 2.9 s **(d)** 46.2 ft

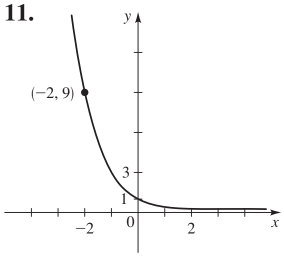
CHAPTER 4

SECTION 4.1 ■ PAGE 351

1. $5; \frac{1}{25}, 1, 25, 15, 625$ **2. (a)** III **(b)** I **(c)** II **(d)** IV **3. (a)** downward **(b)** right **4.** principal, interest rate per year, number of times interest is compounded per year, number of years, amount after t years; \$112.65 **5.** 2.000, 7.103, 0.25, 1.587 **7.** 0.885, 0.606, 0.117, 1.837
9.



11.



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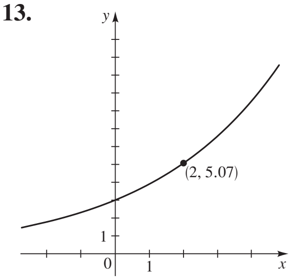

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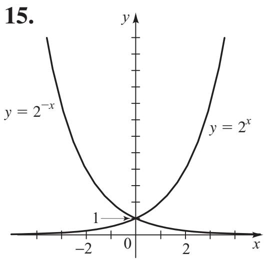
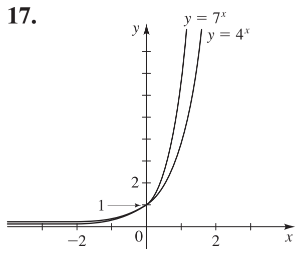
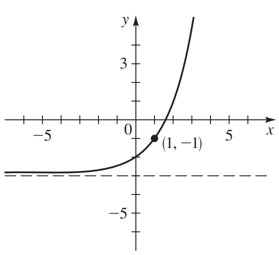


17.

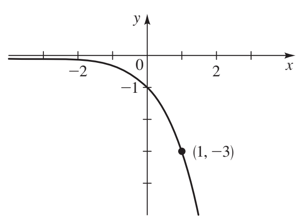


19. $f(x) = 3^x$ **21.** $f(x) = \left(\frac{1}{4}\right)^x$ **23.** II

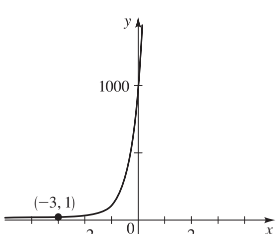
25. $\mathbb{R}, (-3, \infty), y = -3$



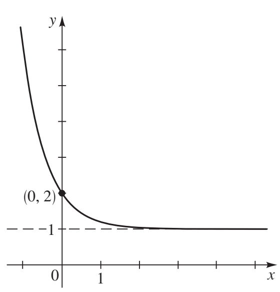
27. $\mathbb{R}, (-\infty, 0), y = 0$



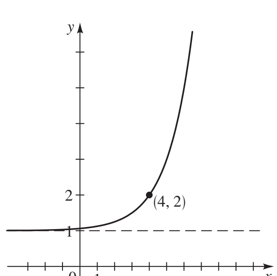
29. $\mathbb{R}, (0, \infty), y = 0$



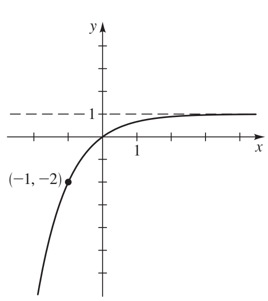
31. $\mathbb{R}, (1, \infty), y = 1$



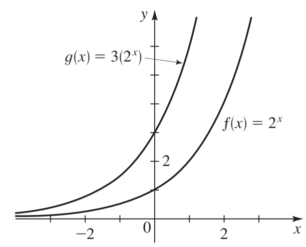
33. $\mathbb{R}, (1, \infty), y = 1$



35. $\mathbb{R}, (-\infty, 1), y = 1$

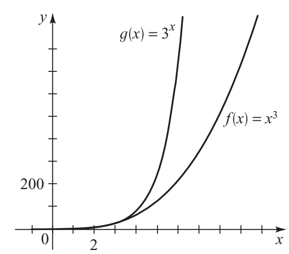


37. **(a)**



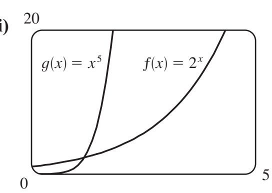
(b) The graph of g is steeper than that of f.

39.

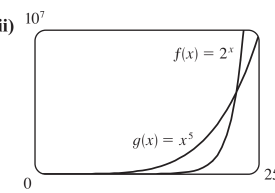


41. **(a)**

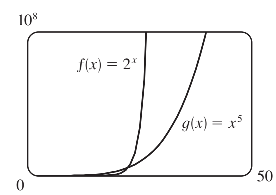
(i)



(ii)



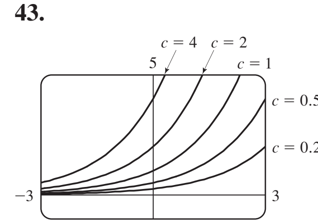
(iii)



The graph of f ultimately increases much more quickly than that of g.

(b) 1.2, 22.4

43.



The larger the value of c, the more rapidly the graph increases.

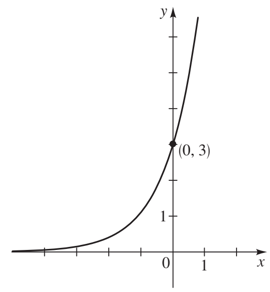
45. **(a)** Increasing on $(-\infty, 0.50]$; decreasing on $[0.50, \infty)$
(b) $(0, 1.78]$
47. **(a)** $1500 \cdot 2^t$ **(b)** 25,165,824,000
49. \$5203.71, \$5415.71, \$5636.36, \$5865.99, \$6104.98, \$6353.71
51. **(a)** \$11,605.41 **(b)** \$13,468.55 **(c)** \$15,630.80
53. **(a)** \$519.02 **(b)** \$538.75 **(c)** \$726.23
55. \$7678.96 **57.** 8.30%

SECTION 4.2 ■ PAGE 356

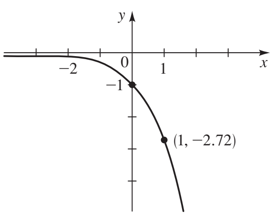
1. natural; 2.71828 **2.** principal, interest rate per year, number of years; amount after t years; \$112.75
3. 20.085, 1.259, 2.718, 0.135

5.

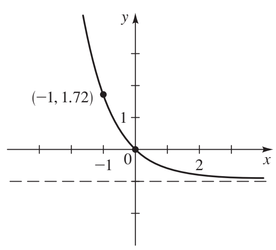
x	y
-2	0.41
-1	1.10
-0.5	1.82
0	3
0.5	4.95
1	8.15
2	22.17



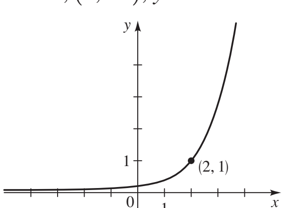
7. $\mathbb{R}, (-\infty, 0), y = 0$



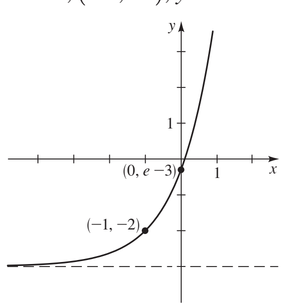
9. $\mathbb{R}, (-1, \infty), y = -1$



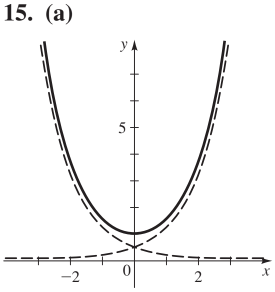
11. $\mathbb{R}, (0, \infty), y = 0$



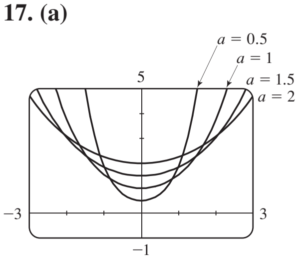
13. $\mathbb{R}, (-3, \infty), y = -3$



15. (a)

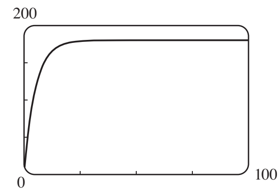


17. (a)

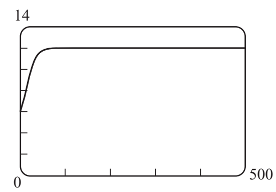


(b) The larger the value of a, the wider the graph.

19. Local minimum $\approx (0.27, 1.75)$
21. (a) 13 kg **(b)** 6.6 kg
23. (a) 0 **(b)** 113.8 ft/s, 155.6 ft/s
(c)

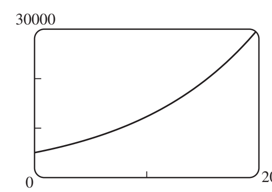


(d) 180 ft/s
25. (a) 100 **(b)** 482, 999, 1168 **(c)** 1200
27. (a) 11.79 billion, 11.97 billion
(b)



(c) 12 billion

29. \$7213.18, \$7432.86, \$7659.22, \$7892.48, \$8132.84, \$8380.52
31. (a) \$2145.02 **(b)** \$2300.55 **(c)** \$3043.92 **33. (a)** \$768.05 **(b)** \$769.22 **(c)** \$769.82 **(d)** \$770.42 **35.** (a) is best.
37. (a) $A(t) = 5000e^{0.09t}$ **(b)**



(c) After 17.88 yr

SECTION 4.3 ■ PAGE 366

1. x

x	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	$10^{1/2}$
$\log x$	3	2	1	0	-1	-2	-3	$\frac{1}{2}$

2. 9; 1, 0, -1, 2, $\frac{1}{2}$
3. (a) $\log_5 125 = 3$ **(b)** $5^2 = 25$
4. (a) III **(b)** II **(c)** I **(d)** IV

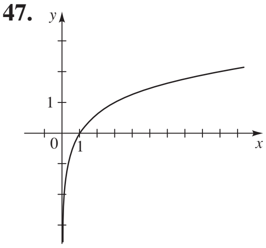
5.

Logarithmic form	Exponential form
$\log_8 8 = 1$	$8^1 = 8$
$\log_8 64 = 2$	$8^2 = 64$
$\log_8 4 = \frac{2}{3}$	$8^{2/3} = 4$
$\log_8 512 = 3$	$8^3 = 512$
$\log_8 \frac{1}{8} = -1$	$8^{-1} = \frac{1}{8}$
$\log_8 \frac{1}{64} = -2$	$8^{-2} = \frac{1}{64}$

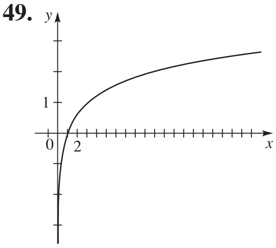
7. (a) $5^2 = 25$ **(b)** $5^0 = 1$ **9. (a)** $8^{1/3} = 2$ **(b)** $2^{-3} = \frac{1}{8}$
11. (a) $3^x = 5$ **(b)** $7^2 = 3y$ **13. (a)** $5 = e^{3y}$ **(b)** $t + 1 = e^{-1}$
15. (a) $\log_5 125 = 3$ **(b)** $\log_{10} 0.0001 = -4$

17. **(a)** $\log_8 \frac{1}{8} = -1$ **(b)** $\log_2 \frac{1}{8} = -3$ **19.** **(a)** $x = \log_5 3$ **(b)** $5 = \log_4 z$ **21.** **(a)** $\ln 2 = x$ **(b)** $\ln y = 3$ **23.** **(a)** 1 **(b)** 0 **(c)** 2 **25.** **(a)** 2 **(b)** 2 **(c)** 10 **27.** **(a)** -3 **(b)** $\frac{1}{2}$ **(c)** -1 **29.** **(a)** 37 **(b)** 8 **(c)** $\sqrt{5}$ **31.** **(a)** $-\frac{2}{3}$ **(b)** 4 **(c)** -1 **33.** **(a)** 32 **(b)** 4 **35.** **(a)** e^3 **(b)** 2 **37.** **(a)** 5 **(b)** 27 **39.** **(a)** 100 **(b)** 25 **41.** **(a)** 2 **(b)** 4
43. **(a)** 0.3010 **(b)** 1.5465 **(c)** -0.1761 **45.** **(a)** 1.6094 **(b)** 3.2308 **(c)** 1.0051

47.

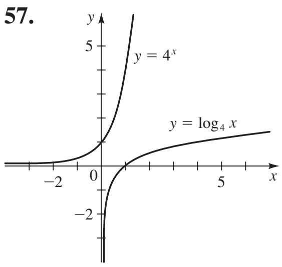


49.

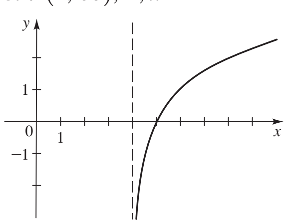


51. $y = \log_5 x$ **53.** $y = \log_9 x$ **55.** I

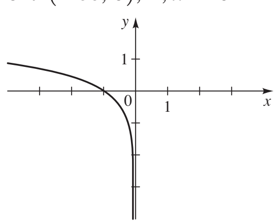
57.



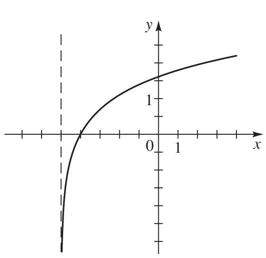
59. $(4, \infty), \mathbb{R}, x = 4$



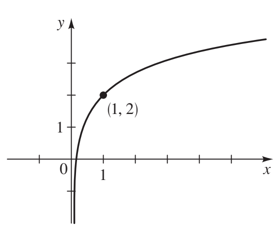
61. $(-\infty, 0), \mathbb{R}, x = 0$



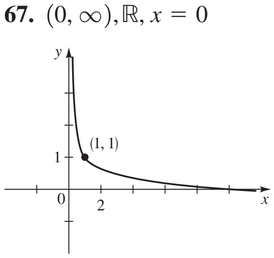
63. $(-5, \infty), \mathbb{R}, x = -5$



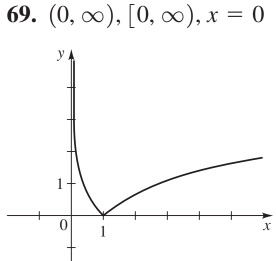
65. $(0, \infty), \mathbb{R}, x = 0$



67. $(0, \infty), \mathbb{R}, x = 0$

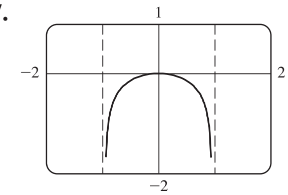


69. $(0, \infty), [0, \infty), x = 0$



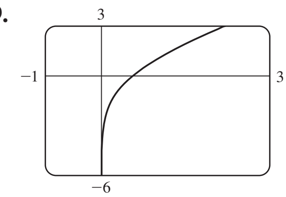
71. $(-3, \infty)$ **73.** $(-\infty, -1) \cup (1, \infty)$ **75.** $(0, 2)$

77.



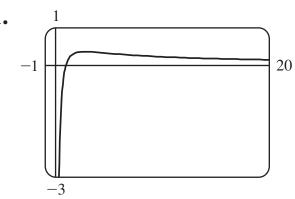
domain $(-1, 1)$
vertical asymptotes $x = 1$, $x = -1$
local maximum $(0, 0)$

79.



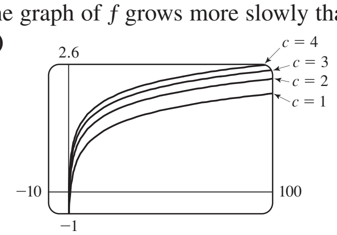
domain $(0, \infty)$
vertical asymptote $x = 0$
no maximum or minimum

81.



domain $(0, \infty)$
vertical asymptote $x = 0$
horizontal asymptote $y = 0$
local maximum
$\approx (2.72, 0.37)$

83. $(f \circ g)(x) = 2^{x+1}, (-\infty, \infty); (g \circ f)(x) = 2^x + 1, (-\infty, \infty)$
85. $(f \circ g)(x) = \log_2 (x - 2), (2, \infty)$; $(g \circ f)(x) = \log_2 x - 2, (0, \infty)$
87. The graph of f grows more slowly than g.
89. **(a)**



(b) The graph of $f(x) = \log(cx)$ is the graph of $f(x) = \log(x)$ shifted upward $\log c$ units.

91. **(a)** $(1, \infty)$ **(b)** $f^{-1}(x) = 10^{2^x}$

93. **(a)** $f^{-1}(x) = \log_2\left(\dfrac{x}{1 - x}\right)$ **(b)** $(0, 1)$

95. 2602 yr **97.** 11.5 yr, 9.9 yr, 8.7 yr **99.** 5.32, 4.32

SECTION 4.4 ■ PAGE 373

1. sum; $\log_5 25 + \log_5 125 = 2 + 3$
2. difference; $\log_5 25 - \log_5 125 = 2 - 3$ **3.** times; $10 \cdot \log_5 25$
4. **(a)** $2 \log x + \log y - \log z$ **(b)** $\log\left(\dfrac{x^2 y}{z}\right)$
5. 10, e; Change of Base; $\log_7 12 = \dfrac{\log 12}{\log 7} = 1.277$ **6.** True
7. 2 **9.** 3 **11.** -0.903 **13.** $\frac{3}{2}$ **15.** 3 **17.** 200 **19.** 4
21. $1 + \log_2 x$ **23.** $\log_2 x + \log_2(x - 1)$ **25.** $10 \log 6$
27. $\frac{1}{2} \ln z$ **29.** $\frac{1}{3} \log_5(x^2 + 1)$ **31.** $10(\log_2 x + \log_2 y)$
33. $\log_2 A + 2 \log_2 B$ **35.** $3 \log x + 4 \log y - 6 \log z$
37. $\log_2 x + \log_2(x^2 + 1) - \frac{1}{2} \log_2(x^2 - 1)$
39. $\ln x + \frac{1}{2}(\ln y - \ln z)$ **41.** $\frac{1}{4} \log(x^2 + y^2)$
43. $\frac{1}{2}[\log(x^2 + 4) - \log(x^2 + 1) - 2 \log(x^3 - 7)]$

45. $3 \ln x + \frac{1}{2} \ln(x - 1) - \ln(3x + 4)$

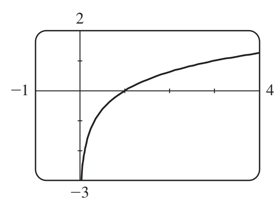
47. $\log_3 160$ **49.** $\ln(5x^2(x^2 + 5)^3)$

51. $\log\left(\dfrac{x^4(x - 1)^2}{\sqrt[3]{x^2 + 1}}\right)$ **53.** $\ln \dfrac{a^2 - b^2}{c^2}$

55. $\log\left(\dfrac{x^2}{x - 3}\right)$ **57.** 2.321928 **59.** 2.523719

61. 0.493008 **63.** 3.482892

65.



71. **(a)** $P = c/W^k$ **(b)** 1866, 64
73. **(a)** $M = -2.5 \log B + 2.5 \log B_0$

SECTION 4.5 ■ PAGE 382

1. **(a)** $e^x = 25$ **(b)** $x = \ln 25$ **(c)** 3.219
2. **(a)** $\log 3(x - 2) = \log x$ **(b)** $3(x - 2) = x$ **(c)** 3
3. 2 **5.** $\frac{3}{2}$ **7.** -3 **9.** $-1, 1$ **11.** **(a)** $2 \log 5$ **(b)** 1.397940

13. **(a)** $-\frac{1}{2} \ln 7$ **(b)** -0.972955 **15.** **(a)** $1 - \dfrac{\log 3}{\log 2}$

(b) -0.584963 **17.** **(a)** $\ln\left(\frac{10}{3}\right)$ **(b)** 1.203973

19. **(a)** $\dfrac{\log 3}{2 \log 1.04}$ **(b)** 14.005511

21. **(a)** $\dfrac{1 - \ln 2}{4}$ **(b)** 0.076713

23. **(a)** $\dfrac{-1 + \ln 200}{2}$ **(b)** 2.149159

25. **(a)** $\dfrac{\log 5}{0.4 \log 8}$ **(b)** 1.934940

27. **(a)** $\dfrac{14 \log 0.1}{\log 3}$ **(b)** -29.342646

29. **(a)** $\frac{1}{5}(\log 5 - \log 4)$ **(b)** 0.019382

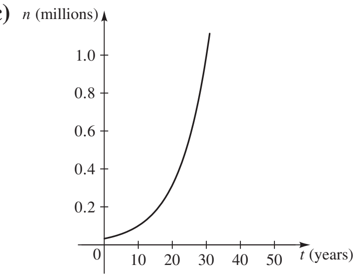
31. **(a)** $\dfrac{\log 4}{\log 5 - \log 4}$ **(b)** 6.212567

33. **(a)** $-\dfrac{\log 2 + 2 \log 3}{3 \log 2 - \log 3}$ **(b)** -2.946865 **35.** **(a)** $-\ln 11.5$

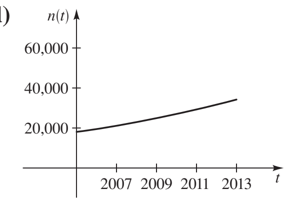
(b) -2.442347 **37.** $\ln 2 \approx 0.6931, 0$ **39.** $\frac{1}{2} \ln 3 \approx 0.5493$
41. ± 1 **43.** $0, \frac{4}{3}$ **45.** 5 **47.** 2, 4 **49.** 5 **51.** $e^{10} \approx 22{,}026$
53. 0.01 **55.** $\frac{95}{3}$ **57.** -7 **59.** 4 **61.** 6 **63.** $\frac{13}{12}$ **65.** $\frac{3}{2}$
67. $1/\sqrt{5} \approx 0.4472$ **69.** 2.21 **71.** 0.00, 1.14
73. -0.57 **75.** 0.36 **77.** $2 < x < 4$ or $7 < x < 9$
79. $\log 2 < x < \log 5$ **81.** $f^{-1}(x) = \dfrac{\ln x}{2 \ln 2}$
83. $f^{-1}(x) = 2^x + 1$ **85.** **(a)** \$6435.09 **(b)** 8.24 yr
87. 6.33 yr **89.** 8.15 yr **91.** 13 days **93.** **(a)** 7337
(b) 1.73 yr **95.** **(a)** $P = P_0 e^{-h/k}$ **(b)** 56.47 kPa
97. **(a)** $t = -\frac{5}{13} \ln(1 - \frac{13}{60} I)$ **(b)** 0.218 s

SECTION 4.6 ■ PAGE 394

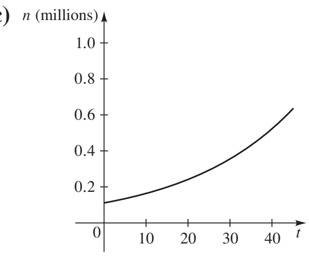
1. **(a)** $n(t) = 10 \cdot 2^{2t/3}$ **(b)** 1.06×10^8 **(c)** 14.9
3. **(a)** 3125 **(b)** 317,480
(c)



5. **(a)** $n(t) = 18{,}000e^{0.08t}$ **(b)** 34,137 **(c)** 4.1
(d)



7. **(a)** 233 million **(b)** 181 million
9. **(a)** $n(t) = 112{,}000 \cdot 2^{t/18}$ **(b)** $n(t) = 112{,}000e^{0.0385t}$
(c)

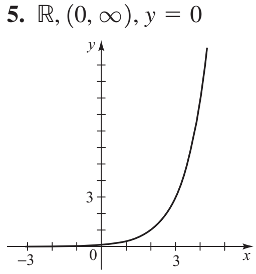


(d) 38.85 years

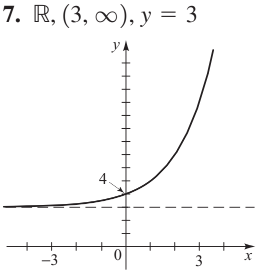
11. **(a)** 20,000 **(b)** $n(t) = 20{,}000e^{0.1096t}$ **(c)** About 48,000
(d) 14.68 years **13.** **(a)** $n(t) = 8600e^{0.1508t}$ **(b)** About 11,600
(c) 4.6 h **15.** **(a)** $n(t) = 29.76e^{0.012936t}$ million **(b)** 53.5 yr
(c) 38.55 million **17.** **(a)** $m(t) = 22 \cdot 2^{-t/1600}$
(b) $m(t) = 22e^{-0.000433t}$ **(c)** 3.9 mg **(d)** 463.4 **19.** 18 yr
21. 149 h **23.** 3560 yr **25.** **(a)** 210°F **(b)** 153°F
(c) 28 min **27.** **(a)** 137°F **(b)** About 2 h **29.** **(a)** 2.3
(b) 3.5 **(c)** 8.3 **31.** **(a)** 10^{-3} M **(b)** 3.2×10^{-7} M
33. $4.8 \le \text{pH} \le 6.4$ **35.** $\log 20 \approx 1.3$
37. Six times as intense **39.** 73 dB **41.** 25

CHAPTER 4 REVIEW ■ PAGE 399

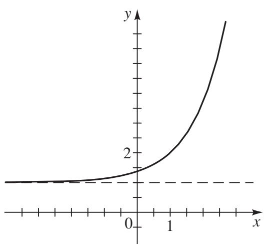
1. 0.089, 9.739, 55.902 **3.** 0.269, 1.472, 12.527
5. $\mathbb{R}, (0, \infty), y = 0$



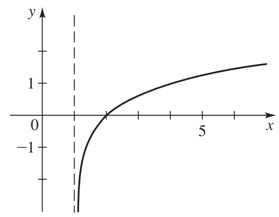
7. $\mathbb{R}, (3, \infty), y = 3$



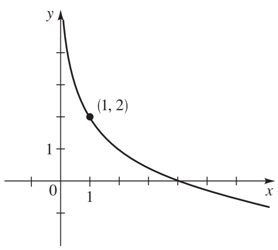
9. $\mathbb{R}, (1, \infty), y = 1$



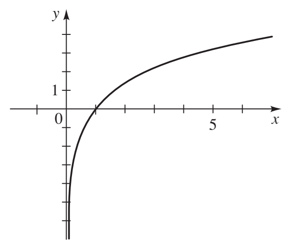
11. $(1, \infty), \mathbb{R}, x = 1$



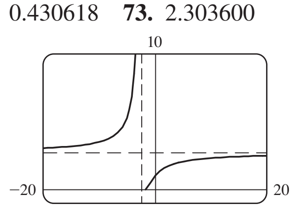
13. $(0, \infty), \mathbb{R}, x = 0$



15. $(0, \infty), \mathbb{R}, x = 0$

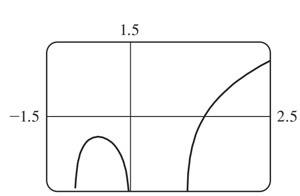


17. $\left(-\infty, \frac{1}{2}\right)$ **19.** $(-\infty, -2) \cup (2, \infty)$
21. $2^{10} = 1024$ **23.** $10^y = x$
25. $\log_2 64 = 6$ **27.** $\log 74 = x$
29. 7 **31.** 45 **33.** 6 **35.** -3
37. $\frac{1}{2}$ **39.** 2 **41.** 92 **43.** $\frac{2}{3}$
45. $\log A + 2 \log B + 3 \log C$
47. $\frac{1}{2}[\ln(x^2 - 1) - \ln(x^2 + 1)]$
49. $2 \log_5 x + \frac{3}{2} \log_5(1 - 5x) - \frac{1}{2} \log_5(x^3 - x)$
51. $\log 96$ **53.** $\log_2\left(\dfrac{(x - y)^{3/2}}{(x^2 + y^2)^2}\right)$
55. $\log\left(\dfrac{x^2 - 4}{\sqrt{x^2 + 4}}\right)$ **57.** 5 **59.** 2.60
61. -1.15 **63.** $-4, 2$
65. 3 **67.** -15 **69.** 9
71. 0.430618 **73.** 2.303600
75.



vertical asymptote $x = -2$
horizontal asymptote $y = 2.72$
no maximum or minimum

77.



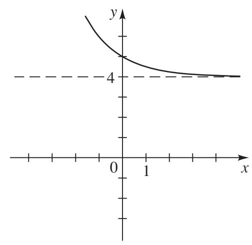
vertical asymptotes $x = -1, x = 0, x = 1$
local maximum $\approx (-0.58, -0.41)$

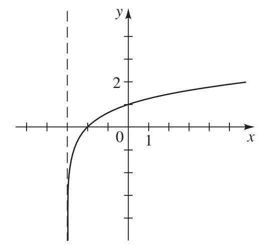
79. 2.42 **81.** $0.16 < x < 3.15$
83. Increasing on $(-\infty, 0]$ and $[1.10, \infty)$, decreasing on $[0, 1.10]$
85. 1.953445 **87.** -0.579352 **89.** $\log_4 258$

91. **(a)** \$16,081.15 **(b)** \$16,178.18 **(c)** \$16,197.64 **(d)** \$16,198.31 **93.** 1.83 yr **95.** 4.341%
97. **(a)** $n(t) = 30e^{0.15t}$ **(b)** 55 **(c)** 19 yr
99. **(a)** 9.97 mg **(b)** 1.39×10^5 yr
101. **(a)** $n(t) = 150e^{-0.0004359t}$ **(b)** 97.0 mg **(c)** 2520 yr
103. **(a)** $n(t) = 1500e^{0.1515t}$ **(b)** 7940
105. 7.9, basic **107.** 8.0

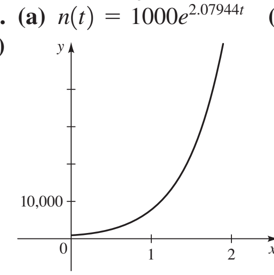
CHAPTER 4 TEST ■ PAGE 402

1. **(a)** $\mathbb{R}, (4, \infty), y = 4$ **(b)** $(-3, \infty), \mathbb{R}, x = -3$





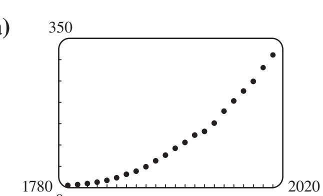
2. **(a)** $\left(\frac{3}{2}, \infty\right)$ **(b)** $(-\infty, -1) \cup (1, \infty)$
3. **(a)** $\log_6 25 = 2x$ **(b)** $e^3 = A$
4. **(a)** 36 **(b)** 3 **(c)** $\frac{3}{2}$ **(d)** 3 **(e)** $\frac{2}{3}$ **(f)** 2
5. **(a)** $\log x + 3 \log y - 2 \log z$ **(b)** $\frac{1}{2} \ln x - \frac{1}{2} \ln y$
(c) $\frac{1}{3}[\log(x + 2) - 4 \log x - \log(x^2 + 4)]$
6. **(a)** $\log(ab^2)$ **(b)** $\ln(x - 5)$ **(c)** $\log_2 \dfrac{3\sqrt{x + 1}}{x^3}$
7. **(a)** 25 **(b)** 1, 2 **(c)** 11.13 **(d)** 5.39
8. **(a)** 500 **(b)** $\frac{2}{3}$ **(c)** 0.774 **(d)** 2 **9.** 1.326
10. **(a)** $n(t) = 1000e^{2.07944t}$ **(b)** 22,627 **(c)** 1.3
(d)



11. **(a)** $A(t) = 12{,}000\left(1 + \dfrac{0.056}{12}\right)^{12t}$ **(b)** \$14,195.06
(c) 9.12 yr **12.** **(a)** $m(t) = 3 \cdot 2^{-t/10}$ **(b)** $m(t) = 3e^{-0.0693t}$
(c) 0.047 g **(d)** after 3.6 min **13.** 1995 times more intense

FOCUS ON MODELING ■ PAGE 409

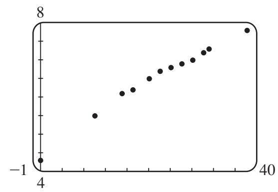
1. **(a)**



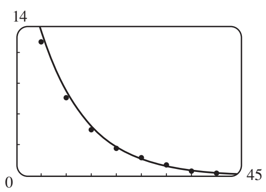
(b) $y = ab^t$, where $a = 3.334926 \times 10^{-15}$, $b = 1.019844$, and y is the population in millions in the year t **(c)** 577.5 million
(d) 353.1 million **(e)** No

3. (a) Yes **(b)** Yes, the scatter plot appears linear.

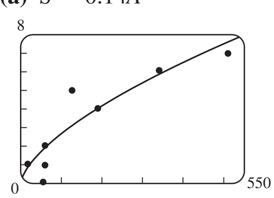
t	$\ln E$
0	4.3
10	5.5
15	6.1
17	6.2
20	6.5
22	6.7
24	6.8
26	6.9
28	7.0
30	7.2
31	7.3
38	7.8



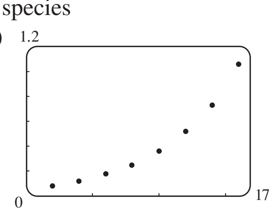
(c) $\ln E = 4.618612 + 0.0881283t$
(d) $101.353256e^{at}$, where $a = 0.0881283$
(e) 5347.50 billion dollars
5. (a) $I_0 = 22.7586444$, $k = 0.1062398$
(b)



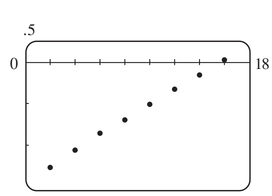
(c) 47.3 ft
7. (a) $S = 0.14A^{0.64}$
(b)

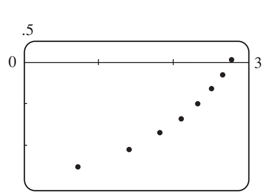


(c) 4 species
9. (a)



(b)



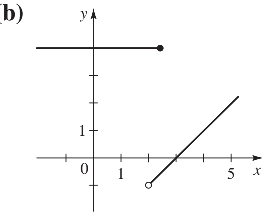


(c) Exponential function
(d) $y = ab^x$ where $a = 0.057697$ and $b = 1.200236$

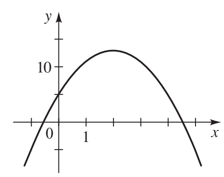
11. (a) $y = \dfrac{c}{1 + ae^{-bx}}$, where $a = 49.10976596$, $b = 0.4981144989$, and $c = 500.855793$
(b) 10.58 days

CUMULATIVE REVIEW TEST FOR CHAPTERS 2, 3, AND 4 ■ PAGE 413

1. (a) $(-\infty, \infty)$ **(b)** $[-4, \infty)$ **(c)** $12, 0, 0, 2, 2\sqrt{3}$, undefined
(d) $x^2 - 4$, $\sqrt{x + 6}$, $-4 + h^2$ **(e)** $\frac{1}{8}$
(f) $f \circ g(x) = x + 4 - 4\sqrt{x + 4}$, $g \circ f(x) = |x - 2|$, $f(g(12)) = 0$, $g(f(12)) = 10$ **(g)** $g^{-1}(x) = x^2 - 4$, $x \ge 0$
2. (a) 4, 4, 4, 0, 1 **(b)**

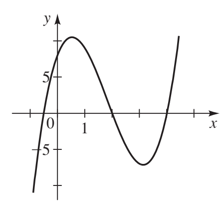


3. (a) $f(x) = -2(x - 2)^2 + 13$ **(b)** Maximum 13
(c)

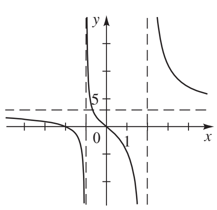


(d) Increasing on $(-\infty, 2]$; decreasing on $[2, \infty)$
(e) Shift upward 5 units
(f) Shift to the left 3 units

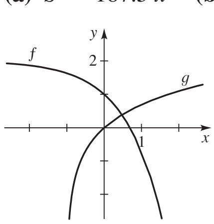
4. f, D; g, C; r, A; s, F; h, B; k, E
5. (a) $\pm1, \pm2, \pm4, \pm8, \pm\frac{1}{2}$ **(b)** $2, 4, -\frac{1}{2}$
(c) $P(x) = 2(x - 2)(x - 4)\left(x + \frac{1}{2}\right)$ **(d)**



6. (a) 1 (multiplicity 2); -1, $1 + i$, $1 - i$ (multiplicity 1)
(b) $Q(x) = (x - 1)^2(x + 1)(x - 1 - i)(x - 1 + i)$
(c) $Q(x) = (x - 1)^2(x + 1)(x^2 - 2x + 2)$
7. x-intercepts $0, -2$; y-intercept 0; horizontal asymptote $y = 3$; vertical asymptotes $x = 2$ and $x = -1$



8. (a) $S = 187.5\,x^2$ **(b)** \$18,750.00 **(c)** 18 years
9.

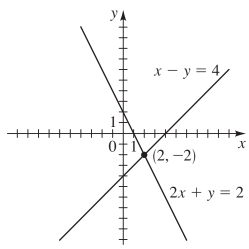


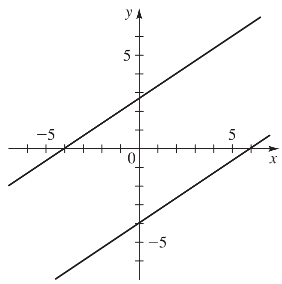
10. (a) -4 **(b)** $5 \log x + \frac{1}{2}\log(x - 1) - \log(2x - 3)$
11. (a) 4 **(b)** $\ln 2$, $\ln 4$ **12. (a)** \$29,396.15
(b) After 6.23 years **(c)** 12.837 years
13. (a) $P(t) = 120e^{0.0565t}$ **(b)** 917 **(c)** After 49.8 months

CHAPTER 5

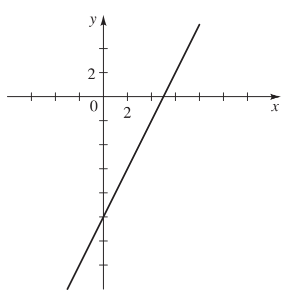
SECTION 5.1 ■ PAGE 424

1. x, y; equation; $(2, 1)$
2. substitution, elimination, graphical **3.** no, infinitely many
4. infinitely many; $1 - t$; $(1, 0), (-3, 4), (5, -4)$
5. $(3, 2)$ **7.** $(3, 1)$ **9.** $(2, 1)$ **11.** $(1, 2)$ **13.** $(-2, 3)$
15. $(2, -2)$ **17.** No solution





19. Infinitely many solutions



21. $(2, 2)$ **23.** $(3, -1)$ **25.** $(2, 1)$ **27.** $(3, 5)$ **29.** $(1, 3)$
31. $(10, -9)$ **33.** $(2, 1)$ **35.** No solution **37.** $\left(x, \frac{1}{3}x - \frac{5}{3}\right)$
39. $\left(x, 3 - \frac{3}{2}x\right)$ **41.** $(-3, -7)$ **43.** $\left(x, 5 - \frac{5}{6}x\right)$ **45.** $(5, 10)$
47. No solution **49.** $(3.87, 2.74)$ **51.** $(61.00, 20.00)$
53. $\left(-\dfrac{1}{a-1}, \dfrac{1}{a-1}\right)$ **55.** $\left(\dfrac{1}{a+b}, \dfrac{1}{a+b}\right)$
57. 22, 12 **59.** 5 dimes, 9 quarters
61. 200 gallons of regular gas, 80 gallons of premium gas
63. Plane's speed 120 mi/h, wind speed 30 mi/h
65. 200 g of A, 40 g of B **67.** 25%, 10%
69. \$14,000 at 5%, \$6,000 at 8%
71. John $2\frac{1}{4}$ h, Mary $2\frac{1}{2}$ h **73.** 25

SECTION 5.2 ■ PAGE 432

1. $x + 3z = 1$ **2.** -3; $4y - 5z = -4$ **3.** Linear **5.** Nonlinear
7. $(5, 1, -2)$ **9.** $(4, 0, 3)$ **11.** $\left(5, 2, -\frac{1}{2}\right)$

13. $\begin{cases} 3x + y + z = 4 \\ -y + z = -1 \\ x - 2y - z = -1 \end{cases}$ **15.** $\begin{cases} 2x + y - 3z = 5 \\ 2x + 3y + z = 13 \\ -8y + 8z = -8 \end{cases}$

17. $(2, 1, -3)$ **19.** $(1, -1, 5)$ **21.** $(1, 2, 1)$ **23.** $(5, 0, 1)$
25. $(0, 1, 2)$ **27.** $\left(\frac{1}{4}, \frac{1}{2}, -\frac{1}{2}\right)$ **29.** No solution **31.** No solution
33. $(3 - t, -3 + 2t, t)$ **35.** $\left(2 - 2t, -\frac{2}{3} + \frac{4}{3}t, t\right)$
37. $(1, -1, 1, 2)$ **39.** \$30,000 in short-term bonds, \$30,000 in intermediate-term bonds, \$40,000 in long-term bonds
41. 250 acres corn, 500 acres wheat, 450 acres soybeans
43. Impossible **45.** 50 Midnight Mango, 60 Tropical Torrent, 30 Pineapple Power **47.** 1500 shares of A, 1200 shares of B, 1000 shares of C

SECTION 5.3 ■ PAGE 439

1. (iii) **2.** (ii) **3.** $\dfrac{A}{x-1} + \dfrac{B}{x+2}$

5. $\dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{x+4}$ **7.** $\dfrac{A}{x-3} + \dfrac{Bx+C}{x^2+4}$

9. $\dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2}$

11. $\dfrac{A}{x} + \dfrac{B}{2x-5} + \dfrac{C}{(2x-5)^2} + \dfrac{D}{(2x-5)^3} + \dfrac{Ex+F}{x^2+2x+5} + \dfrac{Gx+H}{(x^2+2x+5)^2}$

13. $\dfrac{1}{x-1} - \dfrac{1}{x+1}$ **15.** $\dfrac{1}{x-1} - \dfrac{1}{x+4}$

17. $\dfrac{2}{x-3} - \dfrac{2}{x+3}$ **19.** $\dfrac{1}{x-2} - \dfrac{1}{x+2}$

21. $\dfrac{3}{x-4} - \dfrac{2}{x+2}$ **23.** $\dfrac{-\frac{1}{2}}{2x-1} + \dfrac{\frac{3}{2}}{4x-3}$

25. $\dfrac{2}{x-2} + \dfrac{3}{x+2} - \dfrac{1}{2x-1}$ **27.** $\dfrac{2}{x+1} - \dfrac{1}{x} + \dfrac{1}{x^2}$

29. $\dfrac{1}{2x+3} - \dfrac{3}{(2x+3)^2}$ **31.** $\dfrac{2}{x} - \dfrac{1}{x^3} - \dfrac{2}{x+2}$

33. $\dfrac{4}{x+2} - \dfrac{4}{x-1} + \dfrac{2}{(x-1)^2} + \dfrac{1}{(x-1)^3}$

35. $\dfrac{3}{x+2} - \dfrac{1}{(x+2)^2} - \dfrac{1}{(x+3)^2}$ **37.** $\dfrac{x+1}{x^2+3} - \dfrac{1}{x}$

39. $\dfrac{2x-5}{x^2+x+2} + \dfrac{5}{x^2+1}$ **41.** $\dfrac{1}{x^2+1} - \dfrac{x+2}{(x^2+1)^2} + \dfrac{1}{x}$

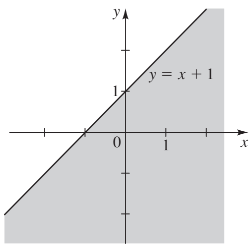
43. $x^2 + \dfrac{3}{x-2} - \dfrac{x+1}{x^2+1}$ **45.** $A = \dfrac{a+b}{2}, B = \dfrac{a-b}{2}$

SECTION 5.4 ■ PAGE 443

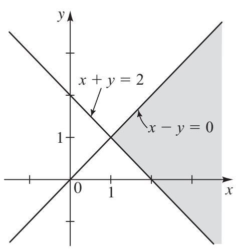
1. $(4, 8), (-2, 2)$ **3.** $(4, 16), (-3, 9)$ **5.** $(2, -2), (-2, 2)$
7. $(-25, 5), (-25, -5)$ **9.** $(-3, 4)\ (3, 4)$
11. $(-2, -1), (-2, 1), (2, -1), (2, 1)$
13. $(-1, \sqrt{2}), (-1, -\sqrt{2}), \left(\frac{1}{2}, \sqrt{\frac{7}{2}}\right), \left(\frac{1}{2}, -\sqrt{\frac{7}{2}}\right)$
15. $(2, 4), \left(-\frac{5}{2}, \frac{7}{4}\right)$ **17.** $(0, 0), (1, -1), (-2, -4)$
19. $(4, 0)$ **21.** $(-2, -2)$
23. $(6, 2), (-2, -6)$ **25.** No solution
27. $(\sqrt{5}, 2), (\sqrt{5}, -2), (-\sqrt{5}, 2), (-\sqrt{5}, -2)$
29. $\left(3, -\frac{1}{2}\right), \left(-3, -\frac{1}{2}\right)$ **31.** $\left(\frac{1}{5}, \frac{1}{3}\right)$ **33.** $(2.00, 20.00), (-8.00, 0)$
35. $(-4.51, 2.17), (4.91, -0.97)$
37. $(1.23, 3.87), (-0.35, -4.21)$
39. $(-2.30, -0.70), (0.48, -1.19)$ **41.** 12 cm by 15 cm
43. 15, 20 **45.** $(400.50, 200.25)$, 447.77 m **47.** $(12, 8)$

SECTION 5.5 ■ PAGE 451

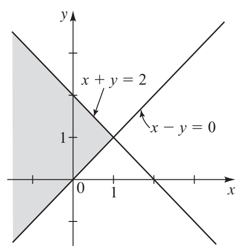
1. equation; $y = x + 1$; test



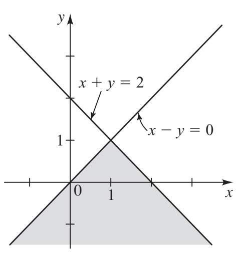
2. (a)



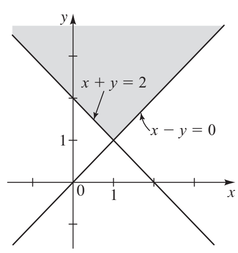
(b)



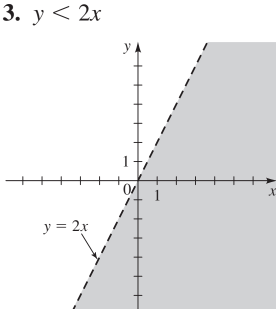
(c)



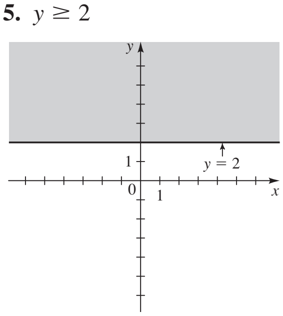
(d)



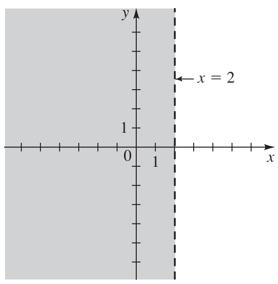
3. $y < 2x$



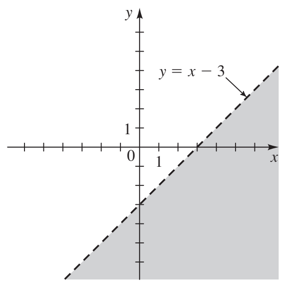
5. $y \geq 2$



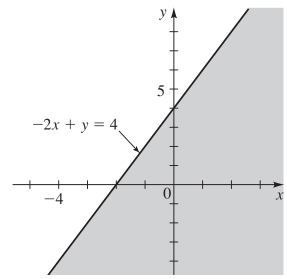
7. $x < 2$



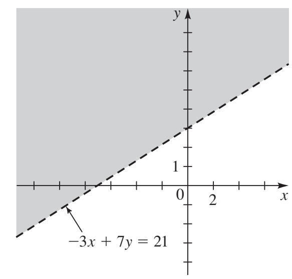
9. $y < x - 3$



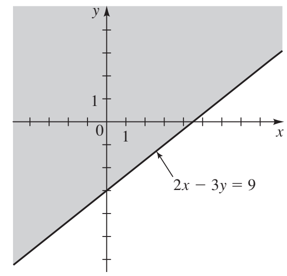
11. $-2x + y \leq 4$



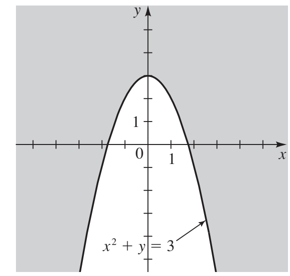
13. $-3x + 7y > 21$



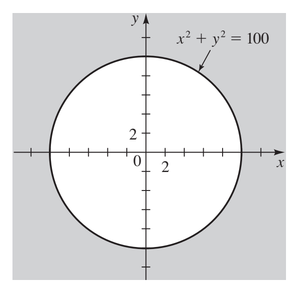
15. $2x - 3y \leq 9$



17. $x^2 + y \geq 3$

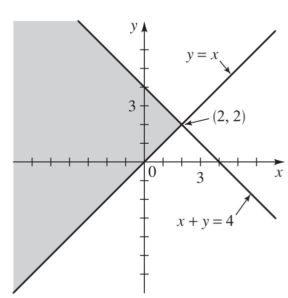


19. $x^2 + y^2 \geq 100$



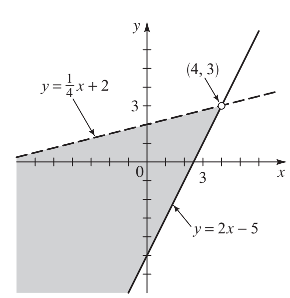
21. $y \leq \frac{1}{2}x - 1$ **23.** $x^2 + y^2 > 4$

25.



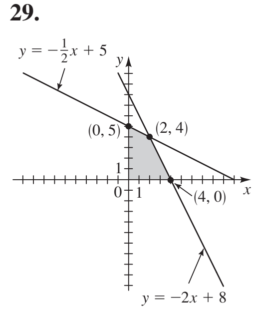
Not bounded

27.



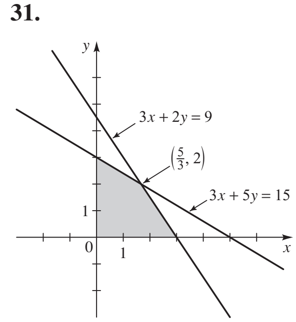
Not bounded

29.



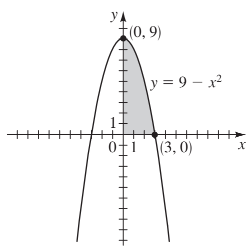
Bounded

31.



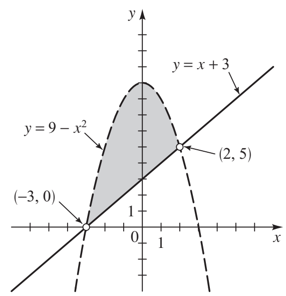
Bounded

33.



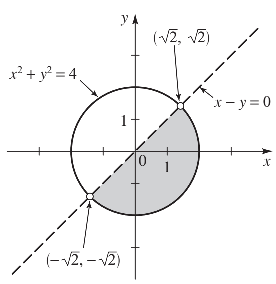
Bounded

35.



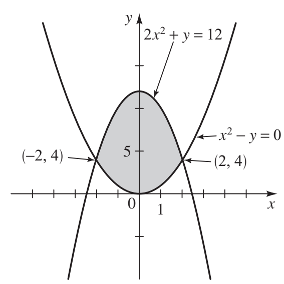
Bounded

37.



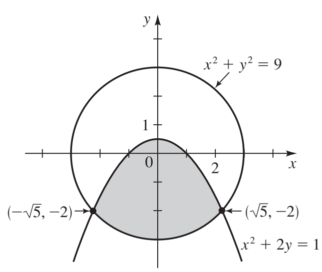
Bounded

39.



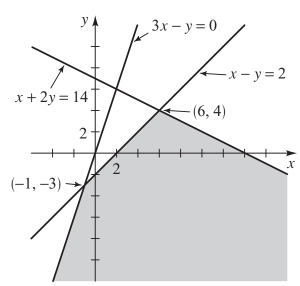
Bounded

41.



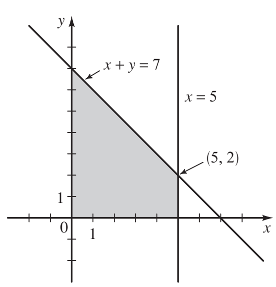
Bounded

43.



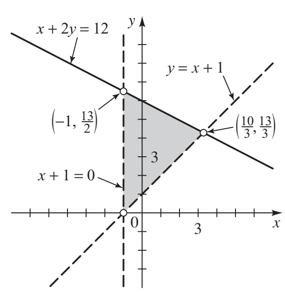
Not bounded

45.



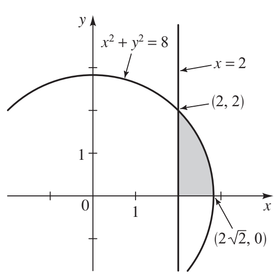
Bounded

47.



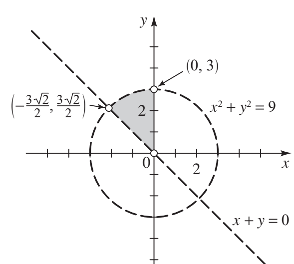
Bounded

49.



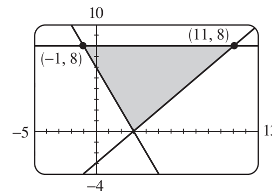
Bounded

51.

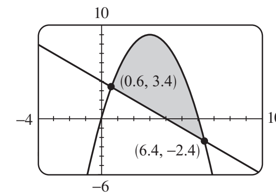


Bounded

53.

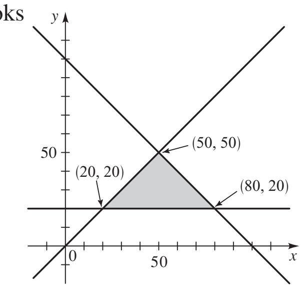


55.



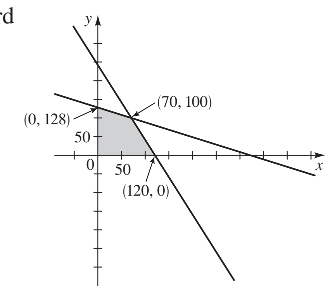
57. $x =$ number of fiction books
$y =$ number of nonfiction books

$$\begin{cases} x + y \le 100 \\ 20 \le y, \quad x \ge y \\ x \ge 0, \quad y \ge 0 \end{cases}$$



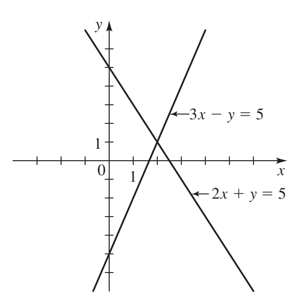
59. $x =$ number of standard packages
$y =$ number of deluxe packages

$$\begin{cases} \frac{1}{4}x + \frac{5}{8}y \le 80 \\ \frac{3}{4}x + \frac{3}{8}y \le 90 \\ x \ge 0, \quad y \ge 0 \end{cases}$$

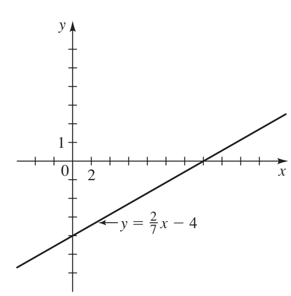


CHAPTER 5 REVIEW ■ PAGE 454

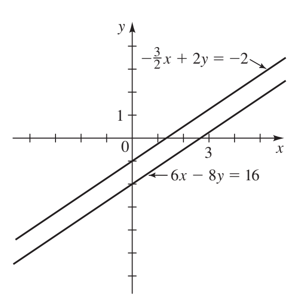
1. (2, 1)



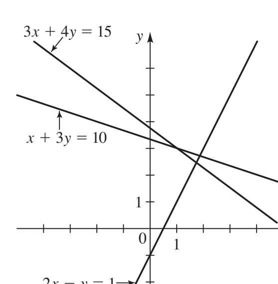
3. $x =$ any number
$y = \frac{2}{7}x - 4$



5. No solution



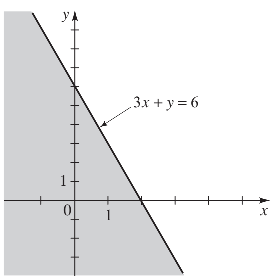
7. No solution



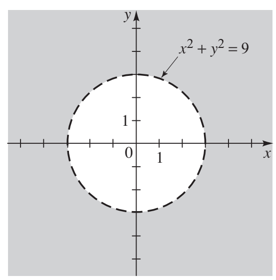
9. $(-3, 3), (2, 8)$ **11.** $\left(\frac{16}{7}, -\frac{14}{3}\right)$ **13.** $(21.41, -15.93)$

15. $(11.94, -1.39), (12.07, 1.44)$ **17.** $(1, 1, 2)$
19. No solution **21.** $x = -4t + 1, y = -t - 1, z = t$
23. $x = 6 - 5t, y = \frac{1}{2}(7 - 3t), z = t$
25. Siobhan is 9 years old; Kieran is 13 years old
27. 12 nickels, 30 dimes, 8 quarters
29. $\dfrac{2}{x-5} + \dfrac{1}{x+3}$ **31.** $\dfrac{-4}{x} + \dfrac{4}{x-1} + \dfrac{-2}{(x-1)^2}$
33. $\dfrac{-1}{x} + \dfrac{x+2}{x^2+1}$ **35.** $\dfrac{3}{x^2+2} - \dfrac{x}{(x^2+2)^2}$ **37.** $(2, 1)$
39. $\left(-\frac{1}{2}, \frac{7}{4}\right), (2, -2)$ **41.** $x + y^2 \le 4$

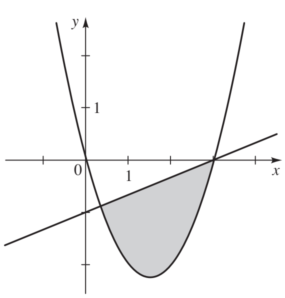
43.



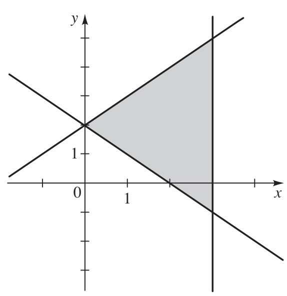
45.



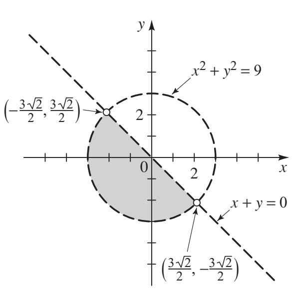
47.



49.

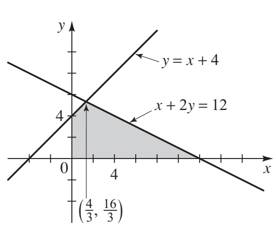


51.



bounded

53.



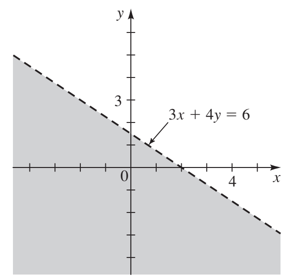
bounded

55. $x = \dfrac{b+c}{2}, y = \dfrac{a+c}{2}, z = \dfrac{a+b}{2}$

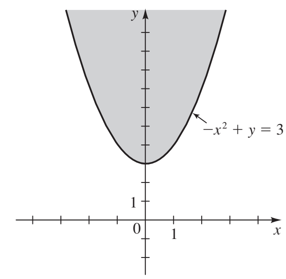
57. 2, 3

CHAPTER 5 TEST ■ PAGE 457

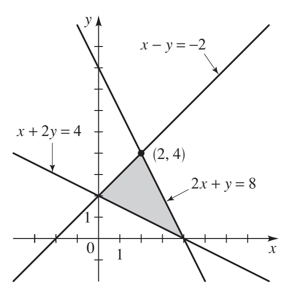
1. **(a)** linear **(b)** $(3, -1)$
2. **(a)** nonlinear **(b)** $\left(\frac{1}{2}, 1\right), (4, -6)$
3. **(a)** Nonlinear **(b)** $\left(-\sqrt{10}, -3\sqrt{10}\right), \left(\sqrt{10}, 3\sqrt{10}\right)$
4. $(-0.55, -0.78), (0.43, -0.29), (2.12, 0.56)$
5. Wind 60 km/h, airplane 300 km/h
6. **(a)** $(2, 1, -1)$ **(b)** Neither
7. **(a)** No solution **(b)** Inconsistent
8. **(a)** $x = \frac{1}{7}(t + 1), y = \frac{1}{7}(9t + 2), z = t$ **(b)** Dependent
9. **(a)** $(10, 0, 1)$ **(b)** neither **10.** Coffee \$1.50, juice \$1.75, donut \$0.75
11. **(a)**



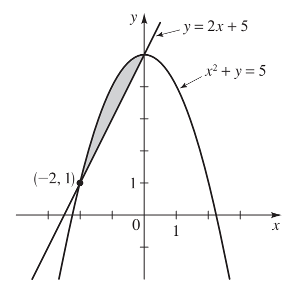
(b)



12.



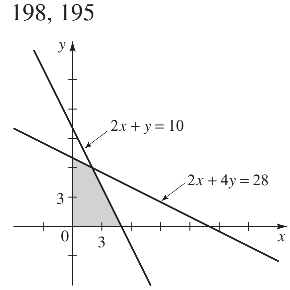
13.



14. $\dfrac{1}{x-1} + \dfrac{1}{(x-1)^2} - \dfrac{1}{x+2}$ **15.** $-\dfrac{1}{x} + \dfrac{x+2}{x^2+3}$

FOCUS ON MODELING ■ PAGE 462

1. 198, 195
3.



maximum 161
minimum 135

5. 3 tables, 34 chairs **7.** 30 grapefruit crates, 30 orange crates
9. 15 Pasadena to Santa Monica, 3 Pasadena to El Toro, 0 Long Beach to Santa Monica, 16 Long Beach to El Toro
11. 90 standard, 40 deluxe **13.** \$7500 in municipal bonds, \$2500 in bank certificates, \$2000 in high-risk bonds
15. 4 games, 32 educational, 0 utility

CHAPTER 6

SECTION 6.1 ■ PAGE 476

1. dependent, inconsistent

2. $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 2 & -3 \\ 0 & 2 & -1 & 3 \end{bmatrix}$

3. (a) x and y **(b)** dependent **(c)** $x = 3 + t, y = 5 - 2t, z = t$
4. (a) $x = 2, y = 1, z = 3$ **(b)** $x = 2 - t, y = 1 - t, z = t$
(c) No solution **5.** 3×2 **7.** 2×1 **9.** 1×3

11. $\begin{bmatrix} 3 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \\ 1 & 0 & -1 & 3 \end{bmatrix}$

13. (a) Yes **(b)** Yes **(c)** $\begin{cases} x = -3 \\ y = 5 \end{cases}$

15. (a) Yes **(b)** No **(c)** $\begin{cases} x + 2y + 8z = 0 \\ y + 3z = 2 \\ 0 = 0 \end{cases}$

17. (a) No **(b)** No **(c)** $\begin{cases} x = 0 \\ 0 = 0 \\ y + 5z = 1 \end{cases}$

19. (a) Yes **(b)** Yes **(c)** $\begin{cases} x + 3y - w = 0 \\ z + 2w = 0 \\ 0 = 1 \\ 0 = 0 \end{cases}$

21. $\begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 4 & 7 & 4 \\ 1 & -2 & -1 & -1 \end{bmatrix}$ **23.** $\begin{bmatrix} 2 & 1 & -3 & 5 \\ 2 & 3 & 1 & 13 \\ 0 & -8 & 8 & -8 \end{bmatrix}$

25. (a) $\begin{cases} x - 2y + 4z = 3 \\ y + 2z = 7 \\ z = 2 \end{cases}$ **(b)** $(1, 3, 2)$

27. (a) $\begin{cases} x + 2y + 3z - w = 7 \\ y - 2z = 5 \\ z + 2w = 5 \\ w = 3 \end{cases}$ **(b)** $(7, 3, -1, 3)$

29. $(1, 1, 2)$ **31.** $(1, 0, 1)$ **33.** $(-1, 0, 1)$ **35.** $(-1, 5, 0)$
37. $(10, 3, -2)$ **39.** No solution **41.** $(2 - 3t, 3 - 5t, t)$
43. No solution **45.** $(-2t + 5, t - 2, t)$
47. $x = -\frac{1}{2}s + t + 6, y = s, z = t$ **49.** $(-2, 1, 3)$
51. No solution **53.** $(-9, 2, 0)$
55. $x = 5 - t, y = -3 + 5t, z = t$ **57.** $(0, -3, 0, -3)$
59. $(-1, 0, 0, 1)$ **61.** $x = \frac{1}{3}s - \frac{2}{3}t, y = \frac{1}{3}s + \frac{1}{3}t, z = s, w = t$
63. $\left(\frac{7}{4} - \frac{7}{4}t, -\frac{7}{4} + \frac{3}{4}t, \frac{9}{4} + \frac{3}{4}t, t\right)$
65. $x = 1.25, y = -0.25, z = 0.75$
67. $x = 1.2, y = 3.4, z = -5.2, w = -1.3$
69. 2 VitaMax, 1 Vitron, 2 VitaPlus **71.** 5-mile run, 2-mile swim, 30-mile cycle **73.** Impossible

SECTION 6.2 ■ PAGE 487

1. dimension **2. (a)** columns, rows **(b)** (ii), (iii) **3.** (i), (ii)

4. $\begin{bmatrix} 4 & 9 & -7 \\ 7 & -7 & 0 \\ 4 & -5 & -5 \end{bmatrix}$ **5.** No **7.** $a = -5, b = 3$ **9.** $\begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 6 \\ 12 & -3 \\ 3 & 0 \end{bmatrix}$ **13.** Impossible **15.** $\begin{bmatrix} 5 & 2 & 1 \\ 7 & 10 & -7 \end{bmatrix}$

17. $\begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 2 \end{bmatrix}$ **19.** No solution **21.** $\begin{bmatrix} 0 & -5 \\ -25 & -20 \\ -10 & 10 \end{bmatrix}$

23. (a) $\begin{bmatrix} 5 & -2 & 5 \\ 1 & 1 & 0 \end{bmatrix}$ **(b)** Impossible

25. (a) $\begin{bmatrix} 10 & -25 \\ 0 & 35 \end{bmatrix}$ **(b)** Impossible

27. (a) Impossible **(b)** $\begin{bmatrix} 14 & -14 \end{bmatrix}$

29. (a) $\begin{bmatrix} -4 & 7 \\ 14 & -7 \end{bmatrix}$ **(b)** $\begin{bmatrix} 6 & -8 \\ 4 & -17 \end{bmatrix}$

31. (a) $\begin{bmatrix} 5 & -3 & 10 \\ 6 & 1 & 0 \\ -5 & 2 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 \\ 8 \\ -1 \end{bmatrix}$

33. (a) $\begin{bmatrix} 4 & -45 \\ 0 & 49 \end{bmatrix}$ **(b)** $\begin{bmatrix} 8 & -335 \\ 0 & 343 \end{bmatrix}$

35. (a) $\begin{bmatrix} 13 \\ -7 \end{bmatrix}$ **(b)** Impossible **37.** $\begin{bmatrix} 1.56 & -5.62 \\ 1.28 & -0.88 \\ -1.09 & 0.97 \end{bmatrix}$

39. $\begin{bmatrix} -0.35 & 0.03 & 0.33 \\ -0.55 & -1.05 & 1.05 \\ -2.41 & -4.31 & 4.46 \end{bmatrix}$ **41.** Impossible

43. $x = 2, y = -1$ **45.** $x = 1, y = -2$

47. $\begin{bmatrix} 2 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

49. $\begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$

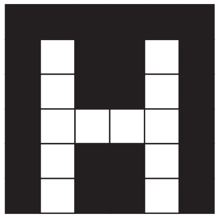
51. Only ACB is defined. $ACB = \begin{bmatrix} -3 & -21 & 27 & -6 \\ -2 & -14 & 18 & -4 \end{bmatrix}$

53. (a) $\begin{bmatrix} 4{,}690 & 1{,}690 & 13{,}210 \end{bmatrix}$ **(b)** Total revenue in Santa Monica, Long Beach, and Anaheim, respectively.
55. (a) $\begin{bmatrix} 105{,}000 & 58{,}000 \end{bmatrix}$ **(b)** The first entry is the total amount (in ounces) of tomato sauce produced, and the second entry is the total amount (in ounces) of tomato paste produced.

57.

(a) $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 & 3 & 0 \\ 1 & 3 & 2 & 3 & 2 & 0 \\ 0 & 3 & 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 & 3 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 1 & 2 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 3 & 2 \\ 2 & 3 & 1 & 1 & 3 & 1 \\ 2 & 3 & 3 & 3 & 3 & 1 \\ 1 & 3 & 1 & 1 & 3 & 2 \\ 2 & 3 & 1 & 2 & 3 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 & 2 & 3 & 2 & 2 \\ 3 & 0 & 3 & 2 & 1 & 2 \\ 2 & 1 & 3 & 3 & 0 & 3 \\ 2 & 0 & 1 & 0 & 1 & 3 \\ 3 & 0 & 3 & 3 & 1 & 2 \\ 2 & 1 & 3 & 2 & 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 & 0 & 3 \\ 3 & 0 & 3 & 3 & 0 & 3 \\ 3 & 0 & 0 & 0 & 0 & 3 \\ 3 & 0 & 3 & 3 & 0 & 3 \\ 3 & 0 & 3 & 3 & 0 & 3 \end{bmatrix}$ 

(e) The letter E

SECTION 6.3 ▪ PAGE 498

1. **(a)** identity **(b)** A, A **(c)** inverse

2. **(a)** $\overset{A}{\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}}\overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{B}{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}$ **(b)** $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

(c) $\overset{A^{-1}}{\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}}\overset{B}{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} = \overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}}$ **(d)** $x = -1, y = 3$ **7.** $\begin{bmatrix} 1 & -2 \\ -\frac{3}{2} & \frac{7}{2} \end{bmatrix}$

9. $\begin{bmatrix} \frac{1}{3} & -\frac{1}{2} \\ 2 & 2 \end{bmatrix}$ **11.** $\begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$ **13.** $\begin{bmatrix} 13 & 5 \\ -5 & -2 \end{bmatrix}$

15. No inverse **17.** $\begin{bmatrix} 1 & 2 \\ -\frac{1}{2} & \frac{2}{3} \end{bmatrix}$ **19.** $\begin{bmatrix} -4 & -4 & 5 \\ 1 & 1 & -1 \\ 5 & 4 & -6 \end{bmatrix}$

21. No inverse **23.** $\begin{bmatrix} -\frac{9}{2} & -1 & 4 \\ 3 & 1 & -3 \\ \frac{7}{2} & 1 & -3 \end{bmatrix}$

25. $\begin{bmatrix} 0 & 0 & -2 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$ **27.** $\begin{bmatrix} \frac{2}{3} & \frac{4}{3} & 3 \\ 1 & 1 & 3 \\ \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$

29. $\begin{bmatrix} -2 & 3 & -1 & -2 \\ 0 & -1 & 0 & \frac{1}{2} \\ -2 & 2 & -1 & -2 \\ -1 & -1 & -1 & 0 \end{bmatrix}$ **31.** $\begin{bmatrix} 1 & -\frac{7}{2} & \frac{1}{6} \\ 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

33. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}$ **35.** $\begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} \\ -\frac{7}{16} & -\frac{23}{16} & -\frac{3}{16} \\ \frac{7}{8} & -\frac{1}{8} & -\frac{5}{8} \end{bmatrix}$

37. $\begin{bmatrix} -7 & -3 & -4 \\ \frac{22}{7} & -\frac{2}{7} & \frac{16}{7} \\ \frac{50}{7} & \frac{26}{7} & \frac{37}{7} \end{bmatrix}$ **39.** $x = 12, y = -8$

41. $x = 126, y = -50$ **43.** $x = -38, y = 9, z = 47$
45. $x = -20, y = 10, z = 16$ **47.** $x = 3, y = 2, z = 1$
49. $x = 3, y = -2, z = 2$ **51.** $x = 8, y = 1, z = 0, w = 3$

53. $\begin{bmatrix} 7 & 2 & 3 \\ 10 & 3 & 5 \end{bmatrix}$ **55.** $\dfrac{1}{2a}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

57. $\begin{bmatrix} 1 & -\dfrac{1}{x} \\ -\dfrac{1}{x} & \dfrac{2}{x^2} \end{bmatrix}$; inverse does not exist for $x = 0$

59. $\frac{1}{2}\begin{bmatrix} 1 & e^{-x} & 0 \\ e^{-x} & -e^{-2x} & 0 \\ 0 & 0 & 1 \end{bmatrix}$; inverse exists for all x

61. **(a)** $\begin{bmatrix} 0 & 1 & -1 \\ -2 & \frac{3}{2} & 0 \\ 1 & -\frac{3}{2} & 1 \end{bmatrix}$ **(b)** 1 oz A, 1 oz B, 2 oz C

(c) 2 oz A, 0 oz B, 1 oz C **(d)** No

63. **(a)** $\begin{cases} 9x + 11y + 8z = 740 \\ 13x + 15y + 16z = 1204 \\ 8x + 7y + 14z = 828 \end{cases}$

(b) $\begin{bmatrix} 9 & 11 & 8 \\ 13 & 15 & 16 \\ 8 & 7 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 740 \\ 1204 \\ 828 \end{bmatrix}$ **(c)** $A^{-1} = \begin{bmatrix} \frac{7}{4} & -\frac{7}{4} & 1 \\ -\frac{27}{28} & \frac{31}{28} & -\frac{5}{7} \\ -\frac{29}{56} & \frac{25}{56} & -\frac{1}{7} \end{bmatrix}$

She earns \$16 on a standard model, \$28 on a deluxe model and \$36 on a super-deluxe model.

SECTION 6.4 ▪ PAGE 509

1. True **2.** True **3.** True **4.** **(a)** $2 \cdot 4 - (-3) \cdot 1 = 11$

(b)
$+1(2 \cdot 4 - (-3) \cdot 1) - 0(3 \cdot 4 - 0 \cdot 1) + 2(3 \cdot (-3) - 0 \cdot 2) = -7$

5. 6 **7.** 0 **9.** -4 **11.** Does not exist **13.** $\frac{1}{8}$ **15.** 20, 20
17. -12, 12 **19.** 0, 0 **21.** 4, has an inverse
23. 5000, has an inverse **25.** 0, does not have an inverse
27. -4, has an inverse **29.** -6, yes **31.** -12, yes **33.** 0, no
35. -18 **37.** 120 **39.** **(a)** -2 **(b)** -2 **(c)** Yes **41.** $(-2, 5)$
43. $(0.6, -0.4)$ **45.** $(4, -1)$ **47.** $(4, 2, -1)$ **49.** $(1, 3, 2)$
51. $(0, -1, 1)$ **53.** $\left(\frac{189}{29}, -\frac{108}{29}, \frac{88}{29}\right)$ **55.** $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, -1\right)$
57. *abcde* **59.** 0, 1, 2 **61.** 1, -1 **63.** 21 **65.** $\frac{63}{2}$

69. **(a)** $\begin{cases} 100a + 10b + c = 25 \\ 225a + 15b + c = 33\frac{3}{4} \\ 1600a + 40b + c = 40 \end{cases}$

(b) $y = -0.05x^2 + 3x$

CHAPTER 6 REVIEW ▪ PAGE 514

1. **(a)** 2×3 **(b)** Yes **(c)** No **(d)** $\begin{cases} x + 2y = -5 \\ y = 3 \end{cases}$

3. **(a)** 3×4 **(b)** Yes **(c)** Yes **(d)** $\begin{cases} x + 8z = 0 \\ y + 5z = -1 \\ 0 = 0 \end{cases}$

5. **(a)** 3×4 **(b)** No **(c)** No **(d)** $\begin{cases} y - 3z = 4 \\ x + y = 7 \\ x + 2y + z = 2 \end{cases}$

7. $(0, 1, 2)$ **9.** No solution **11.** $(1, 0, 1, -2)$
13. $\left(-\frac{4}{3}t + \frac{4}{3}, \frac{5}{3}t - \frac{2}{3}, t\right)$ **15.** $(s + 1, 2s - t + 1, s, t)$
17. No solution **19.** $(1, t + 1, t, 0)$ **21.** Not equal

23. Impossible **25.** $\begin{bmatrix} 4 & 18 \\ 4 & 0 \\ 2 & 2 \end{bmatrix}$ **27.** $\begin{bmatrix} 10 & 0 & -5 \end{bmatrix}$

29. $\begin{bmatrix} -\frac{7}{2} & 10 \\ 1 & -\frac{9}{2} \end{bmatrix}$ **31.** $\begin{bmatrix} 30 & 22 & 2 \\ -9 & 1 & -4 \end{bmatrix}$ **33.** $\begin{bmatrix} -\frac{1}{2} & \frac{11}{2} \\ \frac{15}{4} & -\frac{3}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$

35. $\begin{bmatrix} 27 & 0 & -21 \\ -20 & 5 & 13 \\ -5 & 22 & -7 \end{bmatrix}$ **37.** $\begin{bmatrix} 14 & 26 & -8 \\ -3 & -\frac{7}{3} & \frac{7}{3} \\ 18 & \frac{80}{3} & -\frac{35}{3} \end{bmatrix}$

39. -12 **41.** $\frac{1}{3}$ **43.** -4 **47.** $\frac{1}{3}\begin{bmatrix} -1 & -3 \\ -5 & 2 \end{bmatrix}$ **49.** $\begin{bmatrix} \frac{7}{2} & -2 \\ 0 & 8 \end{bmatrix}$

51. $\begin{bmatrix} 2 & -2 & 6 \\ -4 & 5 & -9 \end{bmatrix}$ **53.** $1, \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$ **55.** 0, no inverse

57. $-1, \begin{bmatrix} 3 & 2 & -3 \\ 2 & 1 & -2 \\ -8 & -6 & 9 \end{bmatrix}$ **59.** $24, \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{4} \\ 0 & 0 & \frac{1}{3} & -\frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

61. $(65, 154)$ **63.** $\left(-\frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$ **65. (a)** Matrix A describes the number of pounds of each vegetable sold on each day; matrix B lists the price per pound of each vegetable. **(b)** $AB = \begin{bmatrix} 68.5 \\ 41.0 \end{bmatrix}$; \$68.50 was the total made on Saturday, and \$41.00 was the total made on Sunday. **67.** $\left(\frac{1}{5}, \frac{9}{5}\right)$ **69.** $\left(-\frac{87}{26}, \frac{21}{26}, \frac{3}{2}\right)$ **71.** 11
73. \$2,500 in bank A, \$40,000 in bank B, \$17,500 in bank C

CHAPTER 6 TEST ■ PAGE 517

1. Row-echelon form **2.** Neither **3.** Reduced row-echelon form
4. Reduced row-echelon form **5.** $\left(\frac{5}{2}, \frac{5}{2}, 0\right)$ **6.** No solution
7. $\left(-\frac{3}{5} + \frac{2}{5}t, \frac{1}{5} + \frac{1}{5}t, t\right)$ **8.** Incompatible dimensions

9. Incompatible dimensions **10.** $\begin{bmatrix} 6 & 10 \\ 3 & -2 \\ -3 & 9 \end{bmatrix}$ **11.** $\begin{bmatrix} 36 & 58 \\ 0 & -3 \\ 18 & 28 \end{bmatrix}$

12. $\begin{bmatrix} 2 & -\frac{3}{2} \\ -1 & 1 \end{bmatrix}$ **13.** B is not square **14.** B is not square

15. -3 **16. (a)** $\begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \end{bmatrix}$ **(b)** $(70, 90)$

17. $|A| = 0, |B| = 2, B^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & \frac{1}{2} & 0 \\ 3 & -6 & 1 \end{bmatrix}$ **18.** $(5, -5, -4)$

19. 1.2 lb almonds, 1.8 lb walnuts

FOCUS ON MODELING ■ PAGE 520

3. (a) Shear to the right **(b)** $T^{-1} = \begin{bmatrix} 1 & -1.5 \\ 0 & 1 \end{bmatrix}$

(c) Shear to the left **(d)** We get back the original square.

5. (a) $D = \begin{bmatrix} 0 & 1 & 1 & 4 & 4 & 1 & 1 & 6 & 6 & 0 & 0 \\ 0 & 0 & 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 & 0 \end{bmatrix}$

(b) $T = \begin{bmatrix} 0.75 & 0 \\ 0 & 1 \end{bmatrix}$

$TD = \begin{bmatrix} 0 & 0.75 & 0.75 & 3 & 3 & 0.75 & 0.75 & 4.5 & 4.5 & 0 & 0 \\ 0 & 0 & 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 & 0 \end{bmatrix}$

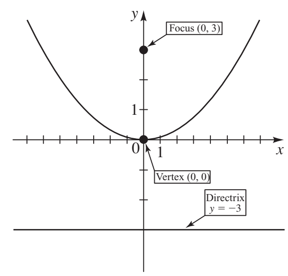
(c) $T = \begin{bmatrix} 1 & 0.25 \\ 0 & 1 \end{bmatrix}$

$TD = \begin{bmatrix} 0 & 1 & 2 & 5 & 5.25 & 2.25 & 2.75 & 7.75 & 8 & 2 & 0 \\ 0 & 0 & 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 & 0 \end{bmatrix}$

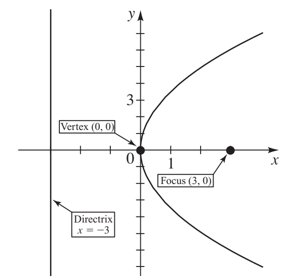
CHAPTER 7

SECTION 7.1 ■ PAGE 530

1. focus, directrix **2.** $F(0, p), y = -p, F(0, 3), y = -3$
3. $F(p, 0), x = -p, F(3, 0), x = -3$
4. (a)



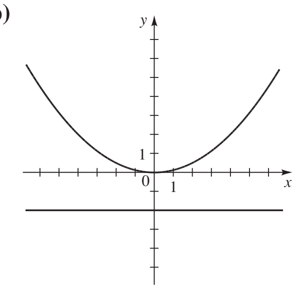
(b)



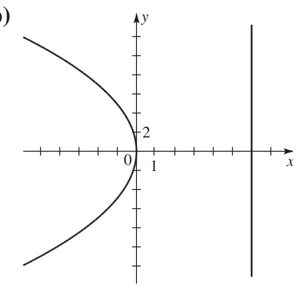
5. III **7.** II **9.** VI

Order of answers for 11–23, part (a): focus; directrix; focal diameter

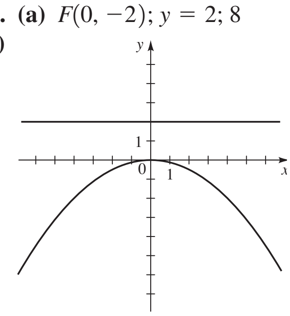
11. (a) $F(0, 2); y = -2; 8$
(b)



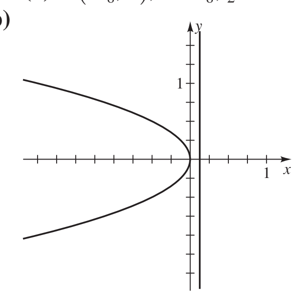
13. (a) $F(-6, 0); x = 6; 24$
(b)



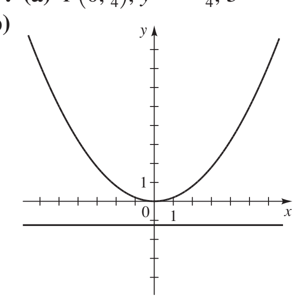
15. (a) $F(0, -2); y = 2; 8$
(b)



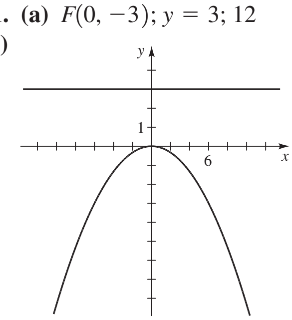
17. (a) $F\left(-\frac{1}{8}, 0\right); x = \frac{1}{8}; \frac{1}{2}$
(b)



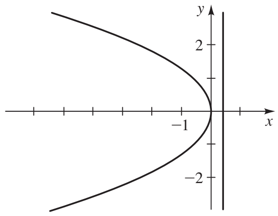
19. (a) $F\left(0, \frac{5}{4}\right); y = -\frac{5}{4}; 5$
(b)



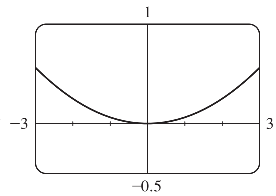
21. (a) $F(0, -3); y = 3; 12$
(b)



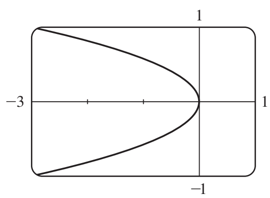
23. (a) $F\left(-\frac{5}{12}, 0\right)$; $x = \frac{5}{12}$; $\frac{5}{3}$
(b)



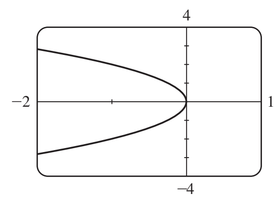
25.



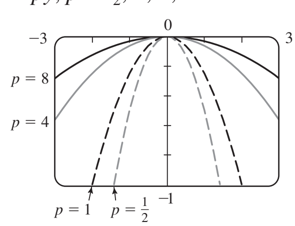
27.



29.



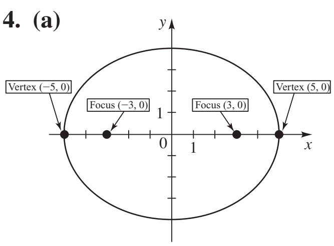
31. $x^2 = 8y$ **33.** $y^2 = -32x$ **35.** $x^2 = -3y$ **37.** $y^2 = -8x$ **39.** $x^2 = 40y$ **41.** $y^2 = -\frac{1}{5}x$ **43.** $y^2 = 4x$ **45.** $x^2 = 20y$ **47.** $x^2 = 8y$ **49.** $y^2 = -16x$ **51.** $y^2 = -3x$ **53.** $x = y^2$ **55.** $x^2 = -4\sqrt{2}y$ **57. (a)** $x^2 = -4py$, $p = \frac{1}{2}$, 1, 4, and 8
(b) The closer the directrix to the vertex, the steeper the parabola.



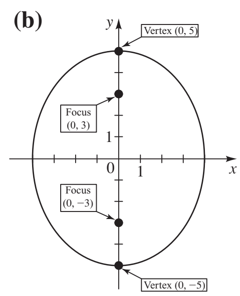
59. (a) $y^2 = 12x$ **(b)** $8\sqrt{15} \approx 31$ cm **61.** $x^2 = 600y$

SECTION 7.2 ■ PAGE 538

1. sum; foci
2. $(a, 0)$, $(-a, 0)$; $c = \sqrt{a^2 - b^2}$; $(5, 0)$, $(-5, 0)$, $(3, 0)$, $(-3, 0)$
3. $(0, a)$, $(0, -a)$; $c = \sqrt{a^2 - b^2}$; $(0, 5)$, $(0, -5)$, $(0, 3)$, $(0, -3)$
4. (a)



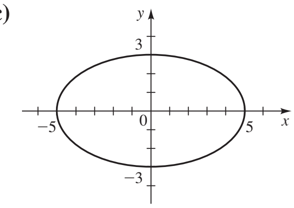
(b)



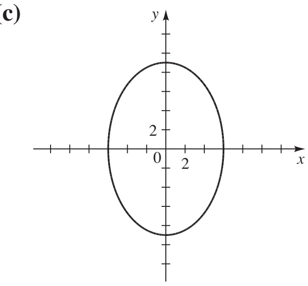
5. II **7.** I

Order of answers for 9–27 part (a): vertices; foci; eccentricity

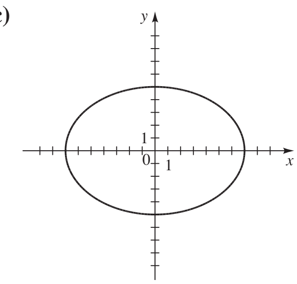
9. (a) $V(\pm 5, 0)$; $F(\pm 4, 0)$; $\frac{4}{5}$
(b) 10, 6
(c)



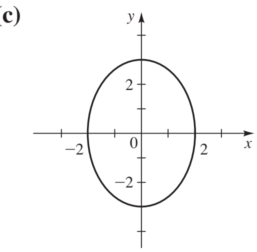
11. (a) $V(0, \pm 9)$; $F(0, \pm 3\sqrt{5})$; $\sqrt{5}/3$
(b) 18, 12
(c)



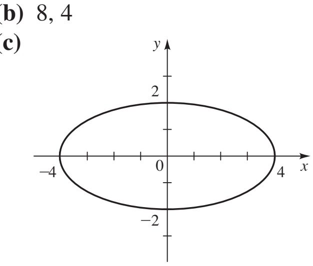
13. (a) $V(\pm 7, 0)$; $F(\pm 2\sqrt{6}, 0)$; $2\sqrt{6}/7$
(b) 14, 10
(c)



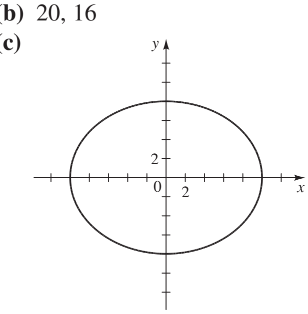
15. (a) $V(0, \pm 3)$; $F(0, \pm\sqrt{5})$; $\sqrt{5}/3$
(b) 6, 4
(c)



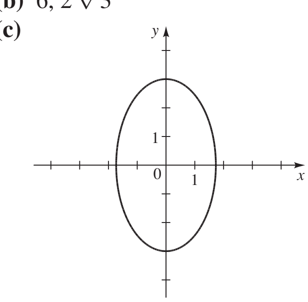
17. (a) $V(\pm 4, 0)$; $F(\pm 2\sqrt{3}, 0)$; $\sqrt{3}/2$
(b) 8, 4
(c)



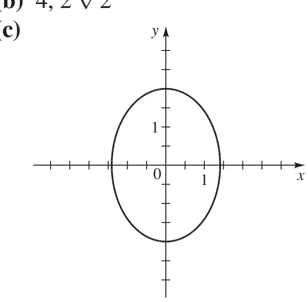
19. (a) $V(\pm 10, 0)$; $F(\pm 6, 0)$; $\frac{3}{5}$
(b) 20, 16
(c)



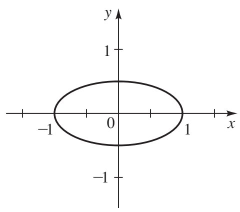
21. (a) $V(0, \pm 3)$; $F(0, \pm\sqrt{6})$; $\sqrt{6}/3$
(b) 6, $2\sqrt{3}$
(c)



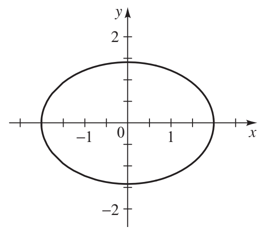
23. (a) $V(0, \pm 2)$; $F(0, \pm\sqrt{2})$; $\sqrt{2}/2$
(b) 4, $2\sqrt{2}$
(c)



25. (a) $V(\pm 1, 0)$; $F(\pm\sqrt{3}/2, 0)$; $\sqrt{3}/2$
(b) 2, 1
(c)

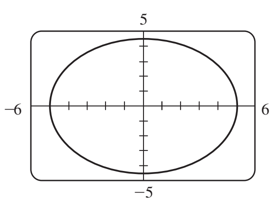


27. (a) $V(\pm 2, 0)$; $F(\pm\sqrt{2}, 0)$; $\sqrt{2}/2$
(b) 4, $2\sqrt{2}$
(c)

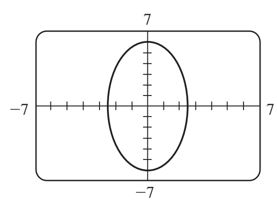


29. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ **31.** $\dfrac{x^2}{4} + \dfrac{y^2}{8} = 1$ **33.** $\dfrac{x^2}{256} + \dfrac{y^2}{48} = 1$

35.



37.

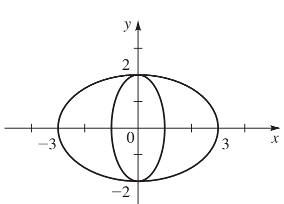


39. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ **41.** $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ **43.** $\dfrac{x^2}{39} + \dfrac{y^2}{49} = 1$

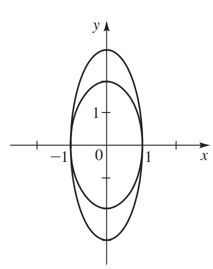
45. $x^2 + \dfrac{y^2}{4} = 1$ **47.** $\dfrac{x^2}{9} + \dfrac{y^2}{13} = 1$ **49.** $\dfrac{x^2}{100} + \dfrac{y^2}{91} = 1$

51. $\dfrac{x^2}{25} + \dfrac{y^2}{5} = 1$ **53.** $\dfrac{x^2}{32} + \dfrac{y^2}{36} = 1$ **55.** $x^2 + \dfrac{y^2}{4} = 1$

57. $(0, \pm 2)$



59. $(\pm 1, 0)$

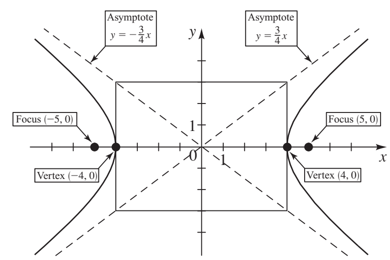


61. (a) $x^2 + y^2 = 4$
65. Perihelion 3.87×10^9 km; aphelion 6.45×10^9 km
67. 8 ft; 6.92 ft apart

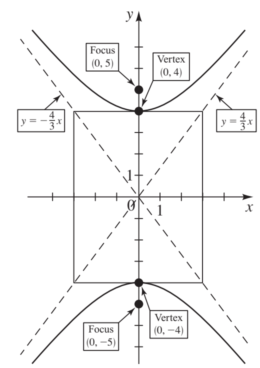
SECTION 7.3 ■ PAGE 547

1. difference; foci
2. horizontal;
$(-a, 0), (a, 0)$; $\sqrt{a^2 + b^2}$; $(-4, 0), (4, 0), (-5, 0), (5, 0)$
3. vertical;
$(0, -a), (0, a)$; $\sqrt{a^2 + b^2}$; $(0, -4), (0, 4), (0, -5), (0, 5)$

4. (a)



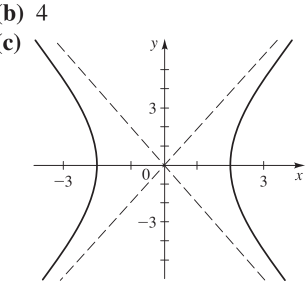
(b)



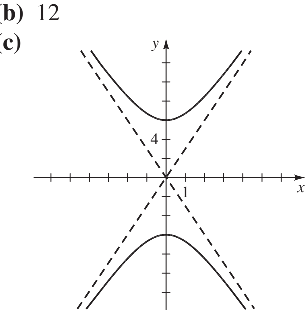
5. III **7.** II

Order of answers for 9–25, part (a): vertices; foci; asymptotes

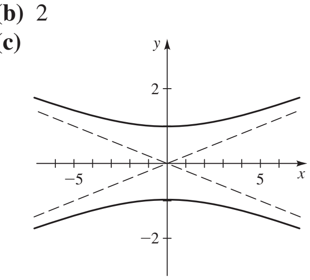
9. (a) $V(\pm 2, 0)$; $F(\pm 2\sqrt{5}, 0)$; $y = \pm 2x$
(b) 4
(c)



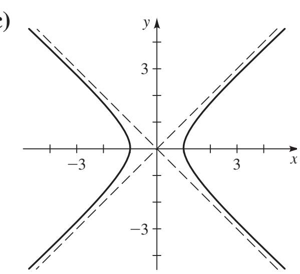
11. (a) $V(0, \pm 6)$; $F(0, \pm 2\sqrt{10})$; $y = \pm 3x$
(b) 12
(c)



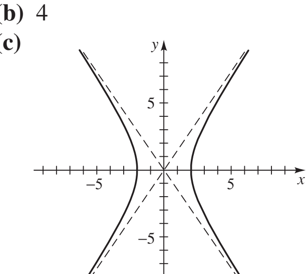
13. (a) $V(0, \pm 1)$; $F(0, \pm\sqrt{26})$; $y = \pm\frac{1}{5}x$
(b) 2
(c)



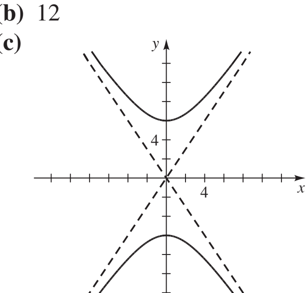
15. (a) $V(\pm 1, 0)$; $F(\pm\sqrt{2}, 0)$; $y = \pm x$
(b) 2
(c)



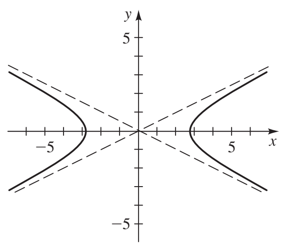
17. (a) $V(\pm 2, 0)$; $F(\pm\sqrt{13}, 0)$; $y = \pm\frac{3}{2}x$
(b) 4
(c)



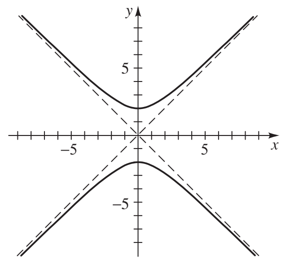
19. (a) $V(0, \pm 6)$; $F(0, \pm 2\sqrt{13})$; $y = \pm\frac{3}{2}x$
(b) 12
(c)



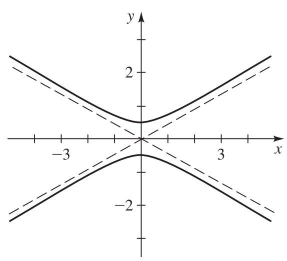
21. (a) $V(\pm 2\sqrt{2}, 0)$; $F(\pm\sqrt{10}, 0)$; $y = \pm\frac{1}{2}x$
(b) $4\sqrt{2}$
(c)



23. (a) $V(0, \pm 2)$; $F(0, \pm 2\sqrt{2})$; $y = \pm x$
(b) 4
(c)

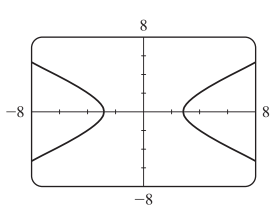


25. (a) $V(0, \pm\frac{1}{2})$; $F(0, \pm\sqrt{5}/2)$; $y = \pm\frac{1}{2}x$ **(b)** 1
(c)

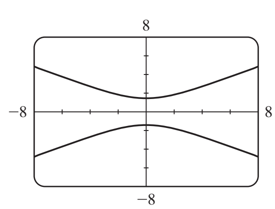


27. $\dfrac{x^2}{4} - \dfrac{y^2}{12} = 1$ **29.** $\dfrac{y^2}{16} - \dfrac{x^2}{16} = 1$ **31.** $\dfrac{y^2}{9} - x^2 = 1$

33.



35.

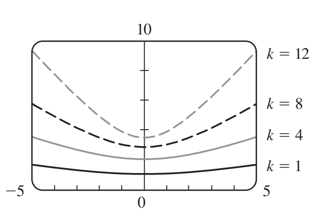


37. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$ **39.** $y^2 - \dfrac{x^2}{3} = 1$ **41.** $x^2 - \dfrac{y^2}{25} = 1$

43. $\dfrac{y^2}{36} - \dfrac{x^2}{20} = 1$ **45.** $\dfrac{x^2}{16} - \dfrac{y^2}{16} = 1$ **47.** $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

49. (b) $x^2 - y^2 = c^2/2$

53. (b)



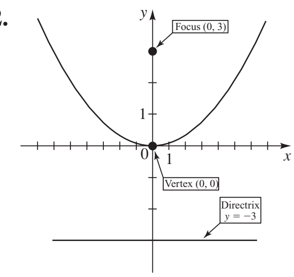
As k increases, the asymptotes get steeper.

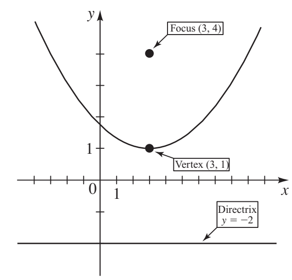
55. $x^2 - y^2 = 2.3 \times 10^{19}$

SECTION 7.4 ■ PAGE 556

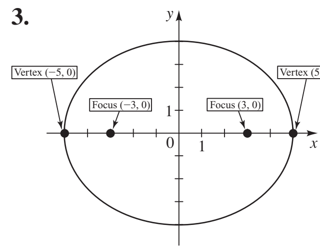
1. (a) right; left **(b)** upward; downward

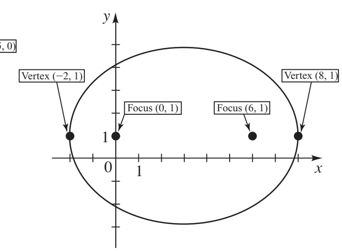
2.



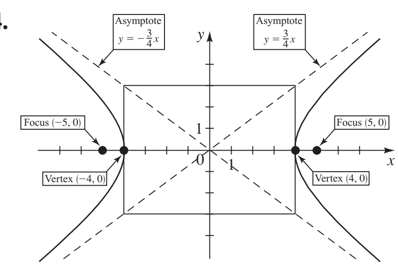


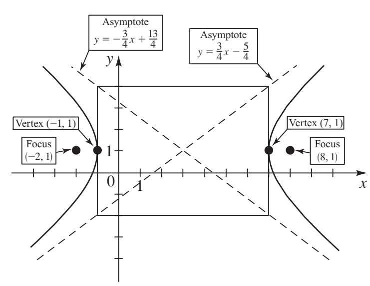
3.



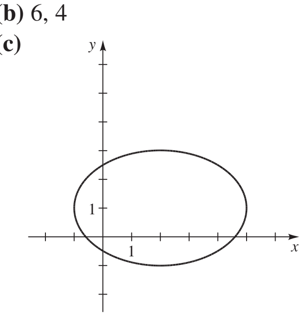


4.

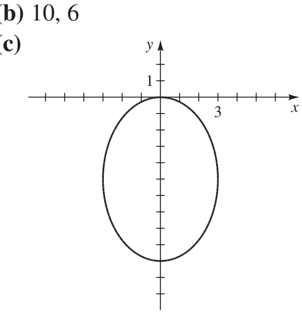




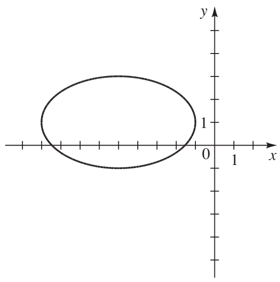
5. (a) $C(2, 1)$; $V_1(-1, 1)$, $V_2(5, 1)$; $F(2 \pm \sqrt{5}, 1)$
(b) 6, 4
(c)



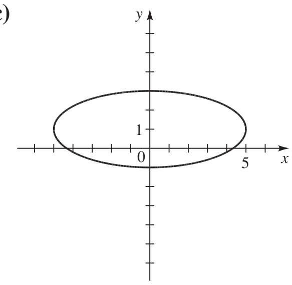
7. (a) $C(0, -5)$; $V_1(0, -10)$, $V_2(0, 0)$; $F_1(0, -9)$, $F_2(0, -1)$
(b) 10, 6
(c)



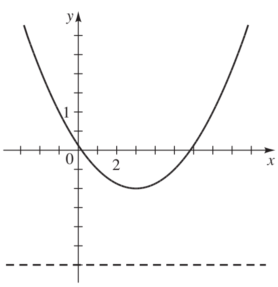
9. (a) $C(-5, 1)$; $V_1(-9, 1)$, $V_2(-1, 1)$; $F(-5 \pm 2\sqrt{3}, 1)$
(b) 8, 4
(c)



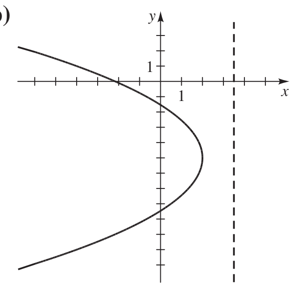
11. (a) $C(0, 1)$; $V(\pm 5, 1)$; $F(\pm\sqrt{21}, 1)$
(b) 10, 4
(c)



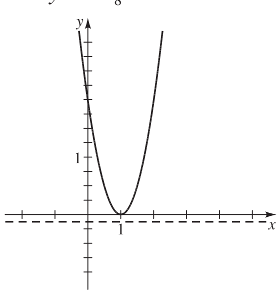
13. (a) $V(3, -1)$; $F(3, 1)$; directrix $y = -3$
(b)



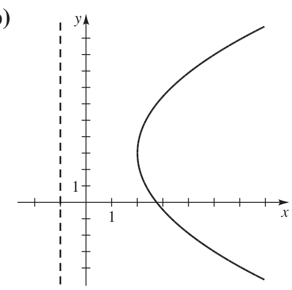
15. (a) $V(2, -5)$; $F\left(\frac{1}{2}, -5\right)$; directrix $x = \frac{7}{2}$
(b)



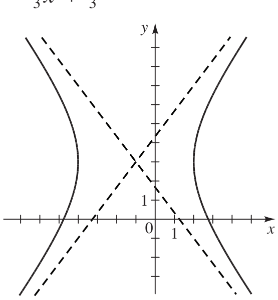
17. (a) $V(1, 0)$; $F\left(1, \frac{1}{8}\right)$; directrix $y = -\frac{1}{8}$
(b)



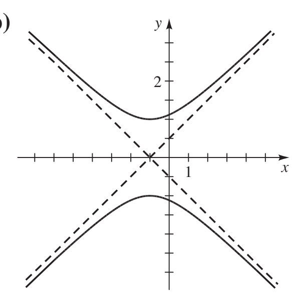
19. (a) $V(2, 3)$; $F(5, 3)$; directrix $x = -1$
(b)



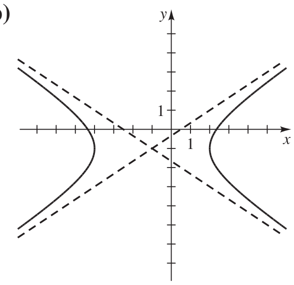
21. (a) $C(-1, 3)$; $V_1(-4, 3)$, $V_2(2, 3)$; $F_1(-6, 3)$, $F_2(4, 3)$; asymptotes $y = \frac{4}{3}x + \frac{13}{3}$ and $y = -\frac{4}{3}x + \frac{5}{3}$
(b)



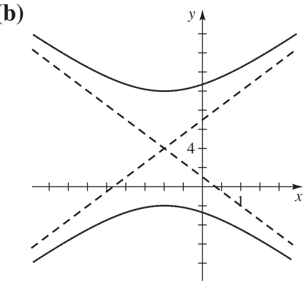
23. (a) $C(-1, 0)$; $V(-1, \pm 1)$; $F(-1, \pm\sqrt{5})$; asymptotes $y = \frac{1}{2}x + \frac{1}{2}$ and $y = -\frac{1}{2}x - \frac{1}{2}$
(b)



25. (a) $C(-1, -1)$; $V_1(-4, -1)$, $V_2(2, -1)$; $F(-1 \pm \sqrt{13}, -1)$; asymptotes $y = \frac{2}{3}x - \frac{1}{3}$ and $y = -\frac{2}{3}x - \frac{5}{3}$
(b)



27. (a) $C(-1, 4)$; $V_1(-1, -2)$, $V_2(-1, 10)$; $F(-1, 4 \pm 2\sqrt{10})$; asymptotes $y = 3x + 7$ and $y = -3x + 1$
(b)



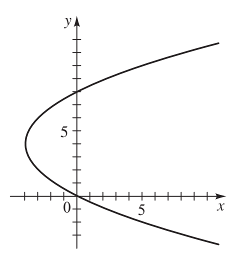
29. $x^2 = -\frac{1}{4}(y - 4)$ **31.** $\dfrac{(x-5)^2}{25} + \dfrac{y^2}{16} = 1$

33. $(y - 1)^2 - x^2 = 1$ **35.** $\dfrac{(x-2)^2}{100} + \dfrac{(y+3)^2}{64} = 1$

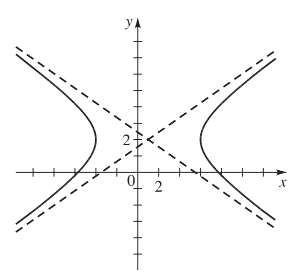
37. $\dfrac{(y-4)^2}{49} - \dfrac{(x+1)^2}{32} = 1$ **39.** $(x + 3)^2 = 12(y - 5)$

41. $\dfrac{y^2}{16} - \dfrac{(x-1)^2}{9} = 1$ **43.** $\dfrac{(x-3)^2}{9} + \dfrac{y^2}{25} = 1$

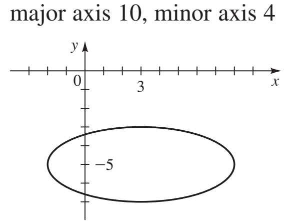
45. Parabola; $V(-4, 4)$; $F(-3, 4)$; directrix $x = -5$



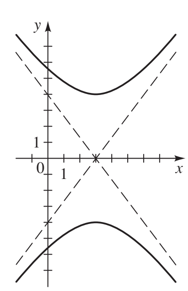
47. Hyperbola; $C(1, 2)$; $F(1 \pm \sqrt{30}, 2)$; $V_1(-4, 2)$, $V_2(6, 2)$; asymptotes $y = \pm\dfrac{\sqrt{5}}{5}(x - 1) + 2$



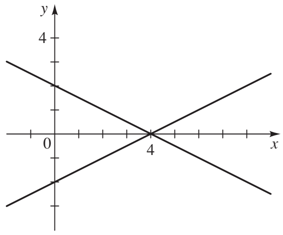
49. Ellipse; $C(3, -5)$; $F\left(3 \pm \sqrt{21}, -5\right)$; $V_1(-2, -5)$, $V_2(8, -5)$; major axis 10, minor axis 4



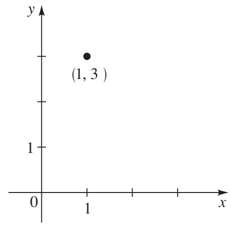
51. Hyperbola; $C(3, 0)$; $F(3, \pm 5)$; $V(3, \pm 4)$; asymptotes $y = \pm\frac{4}{3}(x - 3)$



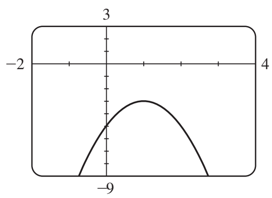
53. Degenerate conic (pair of lines), $y = \pm\frac{1}{2}(x - 4)$



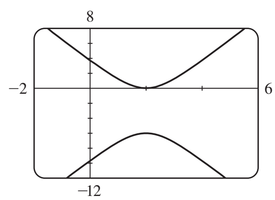
55. Point $(1, 3)$



57.

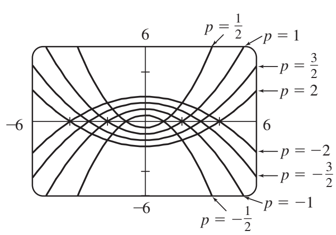


59.



61. (a) $F < 17$ **(b)** $F = 17$ **(c)** $F > 17$

63. (a)

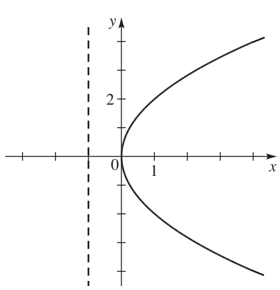


(c) The parabolas become narrower.

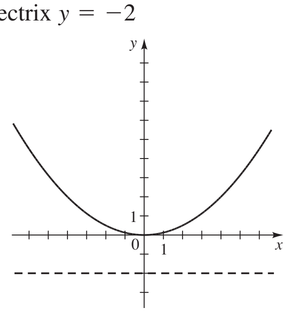
65. $\dfrac{(x + 150)^2}{18{,}062{,}500} + \dfrac{y^2}{18{,}040{,}000} = 1$

CHAPTER 7 REVIEW ■ PAGE 560

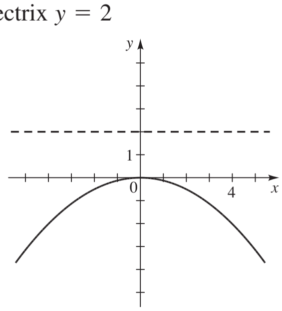
1. (a) $V(0, 0)$; $F(1, 0)$; directrix $x = -1$
(b)



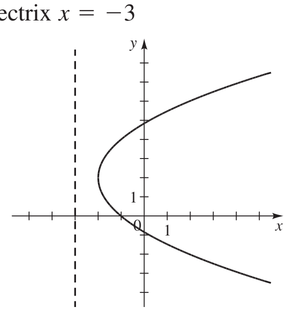
3. (a) $V(0, 0)$; $F(0, 2)$; directrix $y = -2$
(b)



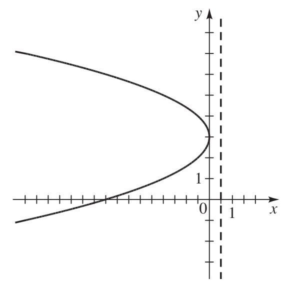
5. (a) $V(0, 0)$; $F(0, -2)$; directrix $y = 2$
(b)



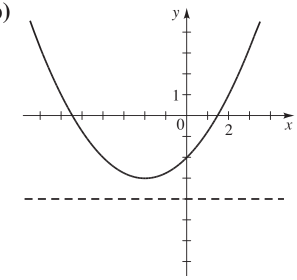
7. (a) $V(-2, 2)$; $F(-1, 2)$; directrix $x = -3$
(b)



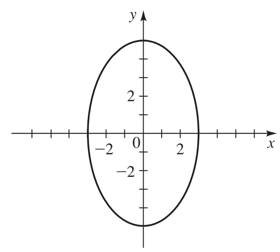
9. (a) $V(0, 3)$; $F\left(-\frac{1}{2}, 3\right)$; directrix $x = \frac{1}{2}$
(b)



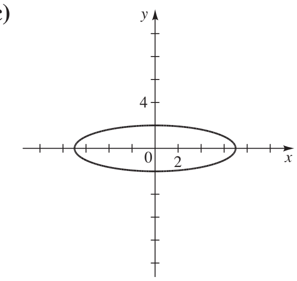
11. (a) $V(-2, -3)$; $F(-2, -2)$; directrix $y = -4$
(b)



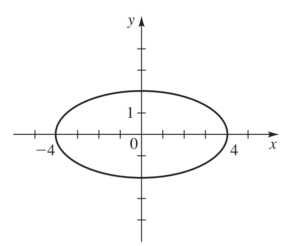
13. (a) $C(0, 0)$; $V(0, \pm 5)$; $F(0, \pm 4)$
(b) 10, 6
(c)



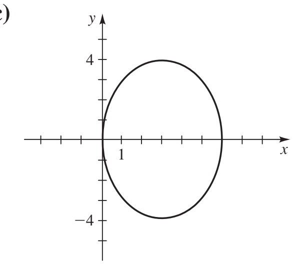
15. (a) $C(0, 0)$; $V(\pm 7, 0)$; $F(\pm 3\sqrt{5}, 0)$
(b) 14, 4
(c)



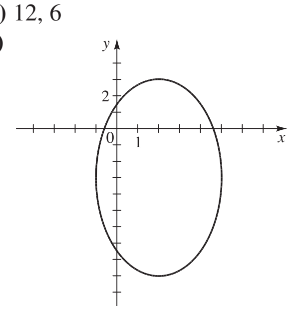
17. (a) $C(0, 0)$; $V(\pm 4, 0)$; $F\left(\pm 2\sqrt{3}, 0\right)$
(b) 8, 4
(c)



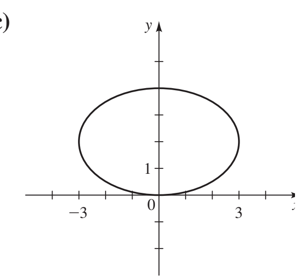
19. (a) $C(3, 0)$; $V(3, \pm 4)$; $F\left(3, \pm\sqrt{7}\right)$
(b) 8, 6
(c)



21. (a) $C(2, -3)$; $V_1(2, -9)$, $V_2(2, 3)$; $F(2, -3 \pm 3\sqrt{3})$
(b) 12, 6
(c)

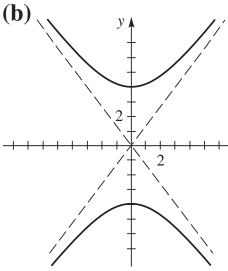


23. (a) $C(0, 2)$; $V(\pm 3, 2)$; $F\left(\pm\sqrt{5}, 2\right)$
(b) 6, 4
(c)



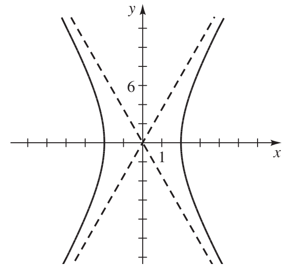
25. (a) $C(0, 0)$; $V(0, \pm 4)$; $F(0, \pm 5)$; asymptotes $y = \pm \frac{4}{3}x$

(b)



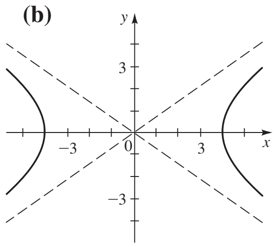
27. (a) $C(0, 0)$; $V(\pm 2, 0)$; $F(\pm\sqrt{53}, 0)$; asymptotes $y = \pm \frac{7}{2}x$

(b)



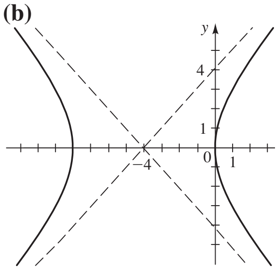
29. (a) $C(0, 0)$; $V(\pm 4, 0)$; $F(\pm 2\sqrt{6}, 0)$; asymptotes $y = \pm \dfrac{1}{\sqrt{2}}x$

(b)



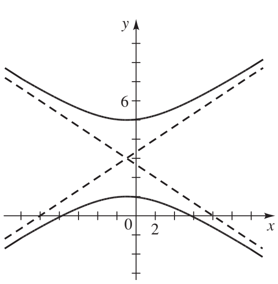
31. (a) $C(-4, 0)$; $V_1(-8, 0)$, $V_2(0, 0)$; $F(-4 \pm 4\sqrt{2}, 0)$; asymptotes $y = \pm(x + 4)$

(b)



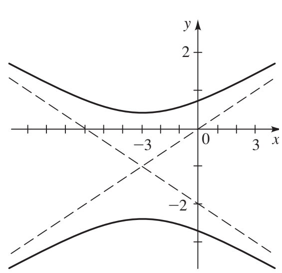
33. (a) $C(-1, 3)$; $V_1(-1, 1)$ $V_2(-1, 5)$; $F(-1, 3 \pm 2\sqrt{10})$; asymptotes $y = \frac{1}{3}x + \frac{10}{3}$ and $y = -\frac{1}{3}x + \frac{8}{3}$

(b)



35. (a) $C(-3, -1)$; $V(-3, -1 \pm \sqrt{2})$; $F(-3, -1 \pm 2\sqrt{5})$; asymptotes $y = \frac{1}{3}x$, $y = -\frac{1}{3}x - 2$

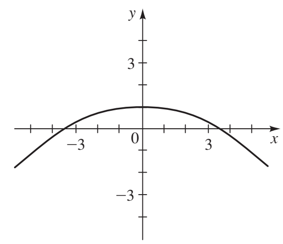
(b)



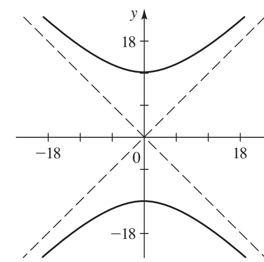
37. $y^2 = 8x$ **39.** $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$

41. $\dfrac{(x - 4)^2}{16} + \dfrac{(y - 2)^2}{4} = 1$

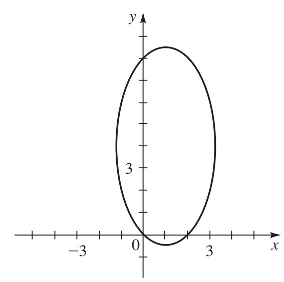
43. Parabola; $V(0, 1)$; $F(0, -2)$; directrix $y = 4$



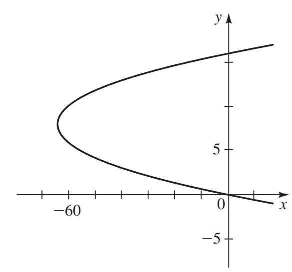
45. Hyperbola; $C(0, 0)$; $F(0, \pm 12\sqrt{2})$; $V(0, \pm 12)$; asymptotes $y = \pm x$



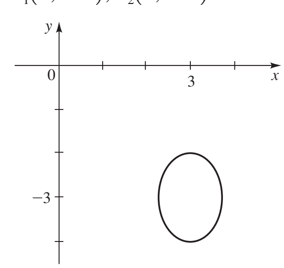
47. Ellipse; $C(1, 4)$; $F(1, 4 \pm \sqrt{15})$; $V(1, 4 \pm 2\sqrt{5})$



49. Parabola; $V(-64, 8)$; $F\left(-\frac{255}{4}, 8\right)$; directrix $x = -\frac{257}{4}$



51. Ellipse; $C(3, -3)$; $F\left(3, -3 \pm \dfrac{\sqrt{2}}{2}\right)$; $V_1(3, -4)$, $V_2(3, -2)$



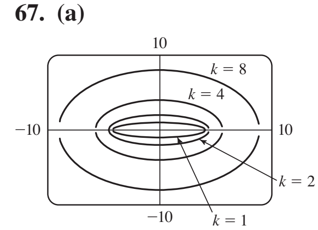
53. Has no graph

55. $x^2 = 4y$ **57.** $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$ **59.** $\dfrac{x^2}{9} + \dfrac{(y - 4)^2}{25} = 1$

61. $\dfrac{(x - 1)^2}{3} + \dfrac{(y - 2)^2}{4} = 1$ **63.** $\dfrac{4(x - 7)^2}{225} + \dfrac{(y - 2)^2}{100} = 1$

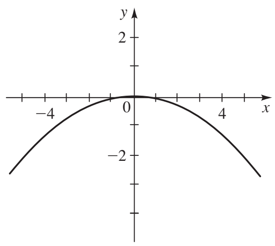
65. (a) 91,419,000 mi **(b)** 94,581,000 mi

67. (a)

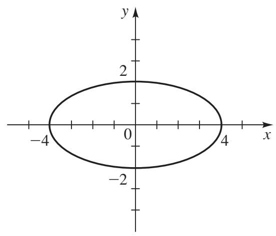


CHAPTER 7 TEST ■ PAGE 562

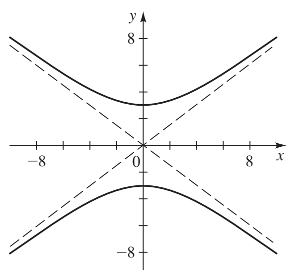
1. $F(0, -3)$, $y = 3$



2. $V(\pm 4, 0)$; $F(\pm 2\sqrt{3}, 0)$; 8, 4



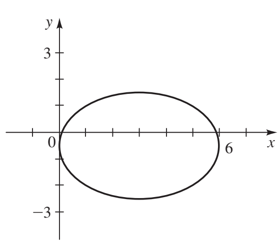
3. $V(0, \pm 3)$; $F(0, \pm 5)$; $y = \pm \frac{3}{4}x$



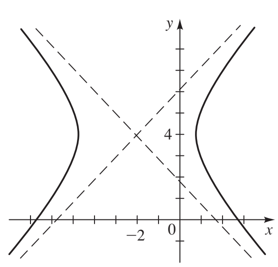
4. $x^2 = 16y$ **5.** $\dfrac{x^2}{16} + \dfrac{y^2}{7} = 1$ **6.** $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$

7. $y^2 = -x$ **8.** $\dfrac{x^2}{16} + \dfrac{(y-3)^2}{9} = 1$ **9.** $(x-2)^2 - \dfrac{y^2}{3} = 1$

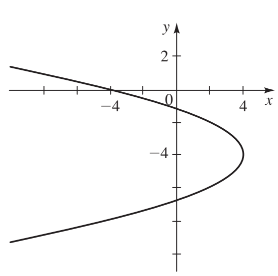
10. Ellipse; $C\left(3, -\frac{1}{2}\right)$; $F\left(3 \pm \sqrt{5}, -\frac{1}{2}\right)$; $V_1\left(0, -\frac{1}{2}\right)$, $V_2\left(6, -\frac{1}{2}\right)$



11. Hyperbola; $C(-2, 4)$, $F(-2 \pm \sqrt{17}, 4)$, $V(-2 \pm 2\sqrt{2}, 4)$, asymptotes $y - 4 = \pm \dfrac{3\sqrt{2}}{4}(x + 2)$



12. Parabola; $V(4, -4)$; $F\left(\frac{7}{2}, -4\right)$; directrix $x = \frac{9}{2}$



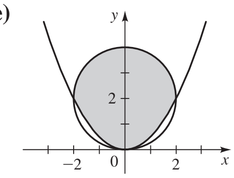
13. $\dfrac{(x-2)^2}{7} + \dfrac{y^2}{16} = 1$ **14.** $(x-2)^2 = 8(y-2)$ **15.** $\frac{3}{4}$ in.

FOCUS ON MODELING ■ PAGE 565

5. (c) $x^2 - mx + (ma - a^2) = 0$, discriminant $m^2 - 4ma + 4a^2 = (m - 2a)^2$, $m = 2a$

CUMULATIVE REVIEW TEST FOR CHAPTERS 5, 6, AND 7 ■ PAGE 567

1. (a) Nonlinear **(b)** $(0, 0)$, $(2, 2)$, $(-2, 2)$
(c) Circle, parabola **(d), (e)**



2. (a) $(3, 0, 1)$ **(b)** $x = t - 1$, $y = t + 2$, $z = t$
3. Xavier 4, Yolanda 10, Zachary 6
4. (a) $A + B$ impossible;

$$C - D = \begin{bmatrix} 0 & -4 & -2 \\ -1 & -4 & -4 \\ -1 & -1 & -1 \end{bmatrix}; \quad AB = \begin{bmatrix} -\frac{9}{2} & 1 & 5 \\ -4 & 2 & 0 \end{bmatrix};$$

CB impossible;

$$BD = \begin{bmatrix} -1 & -2 & -1 \\ -\frac{1}{2} & -1 & -\frac{1}{2} \end{bmatrix};$$

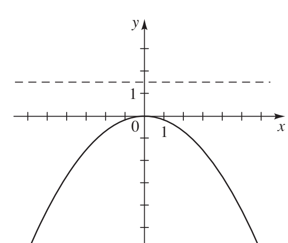
$\det(B)$ impossible; $\det(C) = 2$; $\det(D) = 0$

(b) $C^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}$ **5. (a)** $\begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

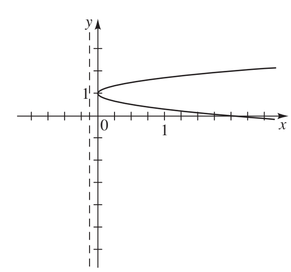
(b) $\begin{bmatrix} 2 & -\frac{3}{2} \\ 3 & -\frac{5}{2} \end{bmatrix}$ **(c)** $X = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$ **(d)** $x = 10$, $y = 15$

6. $\dfrac{1}{x} + \dfrac{2}{x^2} - \dfrac{x+2}{x^2+4}$ **7.** $x^2 = 12y$

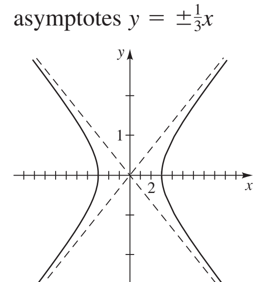
8. (a) $F\left(0, -\frac{3}{2}\right)$, directrix $y = \frac{3}{2}$



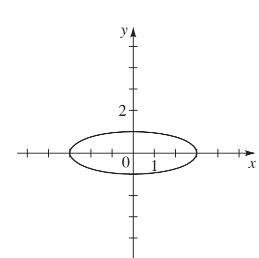
(b) $F\left(\frac{1}{8}, 1\right)$, directrix $x = -\frac{1}{8}$



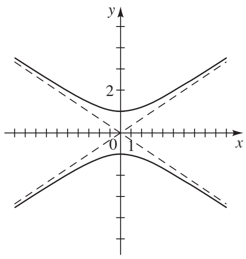
9. (a) Hyperbola; $V(\pm 3, 0)$, $F(\pm\sqrt{10}, 0)$; asymptotes $y = \pm\frac{1}{3}x$



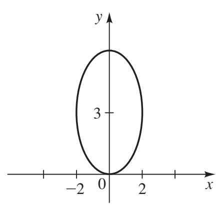
(b) Ellipse; $V(\pm 3, 0)$, $F(\pm 2\sqrt{2}, 0)$



(c) Hyperbola; $V(0, \pm 1)$, $F(0, \pm\sqrt{10})$; asymptotes $y = \pm\frac{1}{3}x$

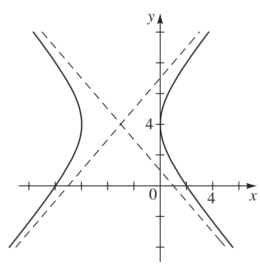


10. (a)



$F_1(0, 3 + \sqrt{5})$, $F_2(0, 3 - \sqrt{5})$; ellipse

(b)



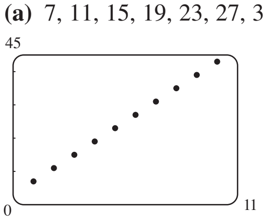
$F_1(-3 + 3\sqrt{2}, 4)$, $F_2(-3 - 3\sqrt{2}, 4)$; hyperbola

11. $\dfrac{(x-5)^2}{16} - \dfrac{y^2}{9} = 1$

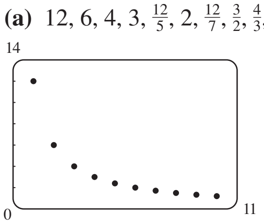
Chapter 8

SECTION 8.1 ■ PAGE 578

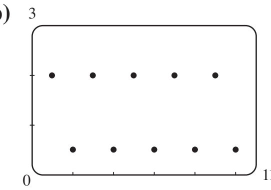
1. the natural numbers **2.** n; $1^2 + 2^2 + 3^2 + 4^2 = 30$
3. 2, 3, 4, 5; 101 **5.** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}; \frac{1}{101}$
7. 5, 25, 125, 625, 5^{100} **9.** $-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}; \frac{1}{10{,}000}$
11. 0, 2, 0, 2; 2 **13.** 1, 4, 27, 256; 100^{100}
15. 3, 2, 0, −4, −12 **17.** 1, 3, 7, 15, 31
19. 1, 2, 3, 5, 8
21. (a) 7, 11, 15, 19, 23, 27, 31, 35, 39, 43
(b)



23. (a) 12, 6, 4, 3, $\frac{12}{5}$, 2, $\frac{12}{7}$, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{6}{5}$
(b)



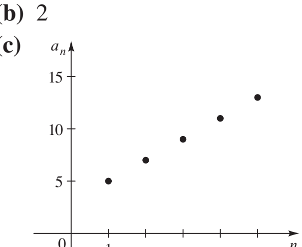
25. (a) 2, $\frac{1}{2}$, 2, $\frac{1}{2}$, 2, $\frac{1}{2}$, 2, $\frac{1}{2}$, 2, $\frac{1}{2}$
(b)



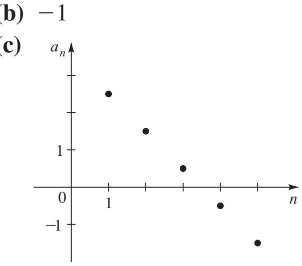
27. $2n$ **29.** 2^n **31.** $3n - 2$ **33.** $a_n = (-1)^{n+1}5^n$
35. $(2n-1)/n^2$ **37.** $1 + (-1)^n$ **39.** 1, 4, 9, 16, 25, 36
41. $\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}, \frac{121}{243}, \frac{364}{729}$
43. $\frac{2}{3}, \frac{8}{9}, \frac{26}{27}, \frac{80}{81}; S_n = 1 - \dfrac{1}{3^n}$
45. $1 - \sqrt{2}, 1 - \sqrt{3}, -1, 1 - \sqrt{5}; S_n = 1 - \sqrt{n+1}$
47. 10 **49.** $\frac{11}{6}$ **51.** 8 **53.** 31 **55.** 385 **57.** 46,438
59. 22 **61.** $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}$
63. $\sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{8} + \sqrt{9} + \sqrt{10}$
65. $x^3 + x^4 + \cdots + x^{100}$ **67.** $\sum_{k=1}^{100} k$ **69.** $\sum_{k=1}^{10} k^2$
71. $\sum_{k=1}^{999} \dfrac{1}{k(k+1)}$ **73.** $\sum_{k=0}^{100} x^k$ **75.** $2^{(2^n-1)/2^n}$
77. (a) 2004.00, 2008.01, 2012.02, 2016.05, 2020.08, 2024.12
(b) \$2149.16 **79. (a)** 35,700, 36,414, 37,142, 37,885, 38,643
(b) 42,665 **81. (b)** 6898 **83. (a)** $S_n = S_{n-1} + 2000$
(b) \$38,000

SECTION 8.2 ■ PAGE 584

1. difference **2.** common difference; 2, 5 **3.** True **4.** True
5. (a) 5, 7, 9, 11, 13
(b) 2
(c)



7. (a) $\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$
(b) −1
(c)



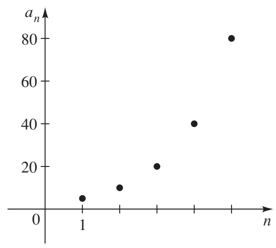
9. $a_n = 3 + 5(n-1)$, $a_{10} = 48$
11. $a_n = \frac{5}{2} - \frac{1}{2}(n-1)$, $a_{10} = -2$
13. Arithmetic, 3 **15.** Arithmetic, −25 **17.** Not arithmetic
19. Arithmetic, $-\frac{3}{2}$ **21.** Arithmetic, 1.7
23. 11, 18, 25, 32, 39; 7; $a_n = 11 + 7(n-1)$
25. $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}$; not arithmetic
27. −4, 2, 8, 14, 20; 6; $a_n = -4 + 6(n-1)$
29. 3, $a_5 = 14$, $a_n = 2 + 3(n-1)$, $a_{100} = 299$
31. −8, $a_5 = -11$, $a_n = 21 - 8(n-1)$, $a_{100} = -771$
33. 5, $a_5 = 24$, $a_n = 4 + 5(n-1)$, $a_{100} = 499$
35. 4, $a_5 = 4$, $a_n = -12 + 4(n-1)$, $a_{100} = 384$
37. 1.5, $a_5 = 31$, $a_n = 25 + 1.5(n-1)$, $a_{100} = 173.5$
39. s, $a_5 = 2 + 4s$, $a_n = 2 + (n-1)s$, $a_{100} = 2 + 99s$

41. $\frac{1}{2}$ **43.** $-100, -98, -96$ **45.** 30th **47.** 100 **49.** -660
51. 1090 **53.** 20,301 **55.** 1735 **57.** 832.3 **59.** 46.75
63. Yes **65.** 50 **67.** \$1250 **69.** \$403,500 **71.** 20 **73.** 78

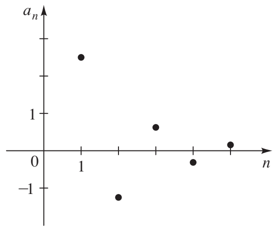
SECTION 8.3 ■ PAGE 591

1. ratio **2.** common ratio; 2, 5 **3.** True **4. (a)** $a\left(\dfrac{1-r^n}{1-r}\right)$
(b) geometric; converges, $a/(1-r)$; diverges

5. (a) 5, 10, 20, 40, 80
(b) 2
(c)



7. (a) $\frac{5}{2}, -\frac{5}{4}, \frac{5}{8}, -\frac{5}{16}, \frac{5}{32}$
(b) $-\frac{1}{2}$
(c)



9. $a_n = 3 \cdot 5^{n-1}$, $a_4 = 375$ **11.** $a_n = \frac{5}{2}\left(-\frac{1}{2}\right)^{n-1}$, $a_4 = -\frac{5}{16}$
13. Geometric, 2 **15.** Geometric, $\frac{2}{3}$ **17.** Geometric, $\frac{1}{2}$
19. Not geometric **21.** Geometric, 1.1
23. 6, 18, 54, 162, 486; geometric, common ratio 3; $a_n = 6 \cdot 3^{n-1}$
25. $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024}$; geometric, common ratio $\frac{1}{4}$; $a_n = \frac{1}{4}\left(\frac{1}{4}\right)^{n-1}$
27. 0, ln 5, 2 ln 5, 3 ln 5, 4 ln 5; not geometric
29. 3, $a_5 = 162$, $a_n = 2 \cdot 3^{n-1}$
31. -0.3, $a_5 = 0.00243$, $a_n = (0.3)(-0.3)^{n-1}$
33. $-\frac{1}{12}$, $a_5 = \frac{1}{144}$, $a_n = 144\left(-\frac{1}{12}\right)^{n-1}$
35. $3^{2/3}$, $a_5 = 3^{11/3}$, $a_n = 3^{(2n+1)/3}$ **37.** $s^{2/7}$, $a_5 = s^{8/7}$, $a_n = s^{2(n-1)/7}$
39. $\frac{1}{2}$ **41.** $a_1 = 25$, $a_2 = \frac{50}{3}$ **43.** $a_1 = 5$, $a_n = 5 \cdot 2^{n-1}$ **45.** $\frac{25}{4}$
47. 11th **49.** 315 **51.** 441 **53.** 3280 **55.** $\frac{6141}{1024}$ **57.** $\frac{3}{2}$
59. $\frac{3}{4}$ **61.** divergent **63.** 2 **65.** divergent **67.** $\sqrt{2} + 1$
69. $\frac{7}{9}$ **71.** $\frac{1}{33}$ **73.** $\frac{112}{999}$ **75.** 10, 20, 40
77. (a) $V_n = 160{,}000(0.80)^{n-1}$ **(b)** 4th year **79.** 19 ft, $80\left(\frac{3}{4}\right)^n$
81. $\frac{64}{25}, \frac{1024}{625}, 5\left(\frac{4}{5}\right)^n$ **83. (a)** $17\frac{8}{9}$ ft **(b)** $18 - \left(\frac{1}{3}\right)^{n-3}$
85. 2801 **87.** 3 m **89. (a)** 2 **(b)** $8 + 4\sqrt{2}$ **91.** 1

SECTION 8.4 ■ PAGE 598

1. amount **2.** present value **3.** \$13,180.79 **5.** \$360,262.21
7. \$5,591.79 **9.** \$572.34 **11.** \$13,007.94 **13.** \$2,601.59
15. \$307.24 **17.** \$733.76, \$264,153.60 **19.** \$583,770.65
21. \$9020.60 **23. (a)** \$859.15 **(b)** \$309,294.00
(c) \$1,841,519.29 **25.** 18.16% **27.** 11.68%

SECTION 8.5 ■ PAGE 605

1. natural; $P(1)$ **2.** (ii)
3. Let $P(n)$ denote the statement $2 + 4 + \cdots + 2n = n(n+1)$.

Step 1 $P(1)$ is true, since $2 = 1(1+1)$.
Step 2 Suppose $P(k)$ is true. Then

$$\begin{aligned} &2 + 4 + \cdots + 2k + 2(k+1) \\ &= k(k+1) + 2(k+1) \qquad \text{Induction hypothesis} \\ &= (k+1)(k+2) \end{aligned}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

5. Let $P(n)$ denote the statement

$$5 + 8 + \cdots + (3n+2) = \frac{n(3n+7)}{2}.$$

Step 1 $P(1)$ is true, since $5 = \dfrac{1(3 \cdot 1 + 7)}{2}$

Step 2 Suppose $P(k)$ is true. Then

$$\begin{aligned} &5 + 8 + \cdots + (3k+2) + [3(k+1)+2] \\ &= \frac{k(3k+7)}{2} + (3k+5) \qquad \text{Induction hypothesis} \\ &= \frac{3k^2 + 13k + 10}{2} \\ &= \frac{(k+1)[3(k+1)+7]}{2} \end{aligned}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

7. Let $P(n)$ denote the statement

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Step 1 $P(1)$ is true, since $1 \cdot 2 = \dfrac{1 \cdot (1+1) \cdot (1+2)}{3}$.

Step 2 Suppose $P(k)$ is true. Then

$$\begin{aligned} &1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \qquad \text{Induction hypothesis} \\ &= \frac{(k+1)(k+2)(k+3)}{3} \end{aligned}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

9. Let $P(n)$ denote the statement

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Step 1 $P(1)$ is true, since $1^3 = \dfrac{1^2 \cdot (1+1)^2}{4}$.

Step 2 Suppose $P(k)$ is true. Then

$$\begin{aligned} &1^3 + 2^3 + \cdots + k^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \qquad \text{Induction hypothesis} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \end{aligned}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

11. Let $P(n)$ denote the statement $2^3 + 4^3 + \cdots + (2n)^3 = 2n^2(n+1)^2$.

Step 1 $P(1)$ is true, since $2^3 = 2 \cdot 1^2(1+1)^2$.
Step 2 Suppose $P(k)$ is true. Then

$$\begin{aligned} &2^3 + 4^3 + \cdots + (2k)^3 + [2(k+1)]^3 \\ &= 2k^2(k+1)^2 + [2(k+1)]^3 && \text{Induction hypothesis} \\ &= (k+1)^2(2k^2 + 8k + 8) \\ &= 2(k+1)^2(k+2)^2 \end{aligned}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

13. Let $P(n)$ denote the statement $1 \cdot 2 + 2 \cdot 2^2 + \cdots + n \cdot 2^n = 2[1 + (n-1)2^n]$.

Step 1 $P(1)$ is true, since $1 \cdot 2 = 2[1 + 0]$.
Step 2 Suppose $P(k)$ is true. Then

$$\begin{aligned} &1 \cdot 2 + 2 \cdot 2^2 + \cdots + k \cdot 2^k + (k+1) \cdot 2^{k+1} \\ &= 2[1 + (k-1)2^k] + (k+1) \cdot 2^{k+1} && \text{Induction hypothesis} \\ &= 2 + (k-1)2^{k+1} + (k+1) \cdot 2^{k+1} \\ &= 2 + 2k2^{k+1} = 2(1 + k2^{k+1}) \end{aligned}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

15. Let $P(n)$ denote the statement $n^2 + n$ is divisible by 2.

Step 1 $P(1)$ is true, since $1^2 + 1$ is divisible by 2.
Step 2 Suppose $P(k)$ is true. Now

$$\begin{aligned} (k+1)^2 + (k+1) &= k^2 + 2k + 1 + k + 1 \\ &= (k^2 + k) + 2(k+1) \end{aligned}$$

But $k^2 + k$ is divisible by 2 (by the induction hypothesis), and $2(k+1)$ is clearly divisible by 2, so $(k+1)^2 + (k+1)$ is divisible by 2. So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

17. Let $P(n)$ denote the statement $n^2 - n + 41$ is odd.

Step 1 $P(1)$ is true, since $1^2 - 1 + 41$ is odd.
Step 2 Suppose $P(k)$ is true. Now

$$(k+1)^2 - (k+1) + 41 = (k^2 - k + 41) + 2k$$

But $k^2 - k + 41$ is odd (by the induction hypothesis), and $2k$ is clearly even, so their sum is odd. So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

19. Let $P(n)$ denote the statement $8^n - 3^n$ is divisible by 5.

Step 1 $P(1)$ is true, since $8^1 - 3^1$ is divisible by 5.
Step 2 Suppose $P(k)$ is true. Now

$$\begin{aligned} 8^{k+1} - 3^{k+1} &= 8 \cdot 8^k - 3 \cdot 3^k \\ &= 8 \cdot 8^k - (8-5) \cdot 3^k = 8 \cdot (8^k - 3^k) + 5 \cdot 3^k \end{aligned}$$

which is divisible by 5 because $8^k - 3^k$ is divisible by 5 (by the induction hypothesis) and $5 \cdot 3^k$ is clearly divisible by 5. So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

21. Let $P(n)$ denote the statement $n < 2^n$.

Step 1 $P(1)$ is true, since $1 < 2^1$.
Step 2 Suppose $P(k)$ is true. Then

$$\begin{aligned} k + 1 &< 2^k + 1 && \text{Induction hypothesis} \\ &< 2^k + 2^k && \text{Because } 1 < 2^k \\ &= 2 \cdot 2^k = 2^{k+1} \end{aligned}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

23. Let $P(n)$ denote the statement $(1+x)^n \geq 1 + nx$ for $x > -1$.

Step 1 $P(1)$ is true, since $(1+x)^1 \geq 1 + 1 \cdot x$.
Step 2 Suppose $P(k)$ is true. Then

$$\begin{aligned} (1+x)^{k+1} &= (1+x)(1+x)^k \\ &\geq (1+x)(1+kx) && \text{Induction hypothesis} \\ &= 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x \end{aligned}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

25. Let $P(n)$ denote the statement $a_n = 5 \cdot 3^{n-1}$.

Step 1 $P(1)$ is true, since $a_1 = 5 \cdot 3^0 = 5$.
Step 2 Suppose $P(k)$ is true. Then

$$\begin{aligned} a_{k+1} &= 3 \cdot a_k && \text{Definition of } a_{k+1} \\ &= 3 \cdot 5 \cdot 3^{k-1} && \text{Induction hypothesis} \\ &= 5 \cdot 3^k \end{aligned}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

27. Let $P(n)$ denote the statement $x - y$ is a factor of $x^n - y^n$.

Step 1 $P(1)$ is true, since $x - y$ is a factor of $x^1 - y^1$.
Step 2 Suppose $P(k)$ is true. Now

$$\begin{aligned} x^{k+1} - y^{k+1} &= x^{k+1} - x^k y + x^k y - y^{k+1} \\ &= x^k(x - y) + (x^k - y^k)y \end{aligned}$$

But $x^k(x-y)$ is clearly divisible by $x - y$, and $(x^k - y^k)y$ is divisible by $x - y$ (by the induction hypothesis), so their sum is divisible by $x - y$. So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

29. Let $P(n)$ denote the statement F_{3n} is even.

Step 1 $P(1)$ is true, since $F_{3 \cdot 1} = 2$, which is even.
Step 2 Suppose $P(k)$ is true. Now, by the definition of the Fibonacci sequence

$$\begin{aligned} F_{3(k+1)} &= F_{3k+3} = F_{3k+2} + F_{3k+1} \\ &= F_{3k+1} + F_{3k} + F_{3k+1} \\ &= F_{3k} + 2 \cdot F_{3k+1} \end{aligned}$$

But F_{3k} is even (by the induction hypothesis), and $2 \cdot F_{3k+1}$ is clearly even, so $F_{3(k+1)}$ is even. So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

31. Let $P(n)$ denote the statement $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$.

Step 1 $P(1)$ is true, since $F_1^2 = F_1 \cdot F_2$ (because $F_1 = F_2 = 1$).
Step 2 Suppose $P(k)$ is true. Then

$$\begin{aligned} F_1^2 + F_2^2 + \cdots + F_k^2 + F_{k+1}^2 &= F_k \cdot F_{k+1} + F_{k+1}^2 && \text{Induction hypothesis} \\ &= F_{k+1}(F_k + F_{k+1}) && \text{Definition of the Fibonacci sequence} \\ &= F_{k+1} \cdot F_{k+2} \end{aligned}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

33. Let $P(n)$ denote the statement $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$.

Step 1 $P(2)$ is true, since $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_3 & F_2 \\ F_2 & F_1 \end{bmatrix}$.
Step 2 Suppose $P(k)$ is true. Then

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k+1} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} && \text{Induction hypothesis} \\ &= \begin{bmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{bmatrix} \\ &= \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix} && \text{Definition of the Fibonacci sequence} \end{aligned}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all $n \ge 2$.

35. Let $P(n)$ denote the statement $F_n \ge n$.

Step 1 $P(5)$ is true, since $F_5 \ge 5$ (because $F_5 = 5$).
Step 2 Suppose $P(k)$ is true. Now

$$\begin{aligned} F_{k+1} &= F_k + F_{k-1} && \text{Definition of the Fibonacci sequence} \\ &\ge k + F_{k-1} && \text{Induction hypothesis} \\ &\ge k + 1 && \text{Because } F_{k-1} \ge 1 \end{aligned}$$

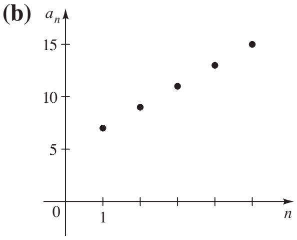
So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all $n \ge 5$.

SECTION 8.6 ■ PAGE 614

1. binomial **2.** Pascal's; 1, 4, 6, 4, 1
3. $\dfrac{n!}{k!(n-k)!}$; $\dfrac{4!}{3!(4-3)!} = 4$
4. Binomial; $\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4}$
5. $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$
7. $x^4 + 4x^2 + 6 + \dfrac{4}{x^2} + \dfrac{1}{x^4}$
9. $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$
11. $x^{10}y^5 - 5x^8y^4 + 10x^6y^3 - 10x^4y^2 + 5x^2y - 1$
13. $8x^3 - 36x^2y + 54xy^2 - 27y^3$
15. $\dfrac{1}{x^5} - \dfrac{5}{x^{7/2}} + \dfrac{10}{x^2} - \dfrac{10}{x^{1/2}} + 5x - x^{5/2}$
17. 15 **19.** 4950 **21.** 18 **23.** 32
25. $x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$
27. $1 + \dfrac{6}{x} + \dfrac{15}{x^2} + \dfrac{20}{x^3} + \dfrac{15}{x^4} + \dfrac{6}{x^5} + \dfrac{1}{x^6}$
29. $x^{20} + 40x^{19}y + 760x^{18}y^2$
31. $25a^{26/3} + a^{25/3}$ **33.** $48{,}620x^{18}$ **35.** $300a^2b^{23}$ **37.** $100y^{99}$
39. $13{,}440x^4y^6$ **41.** $495a^8b^8$ **43.** $(x+y)^4$ **45.** $(2a+b)^3$
47. $3x^2 + 3xh + h^2$

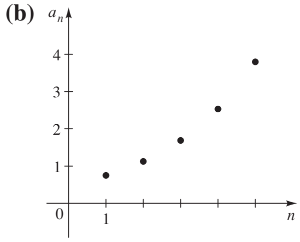
CHAPTER 8 REVIEW ■ PAGE 617

1. $\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}; \frac{100}{11}$ **3.** $0, \frac{1}{4}, 0, \frac{1}{32}; \frac{1}{500}$ **5.** 1, 3, 15, 105; 654,729,075
7. 1, 4, 9, 16, 25, 36, 49 **9.** 1, 3, 5, 11, 21, 43, 85
11. **(a)** 7, 9, 11, 13, 15
(b)



(c) 55
(d) Arithmetic, common difference 2

13. **(a)** $\frac{3}{4}, \frac{9}{8}, \frac{27}{16}, \frac{81}{32}, \frac{243}{64}$
(b)



(c) $\frac{633}{64}$
(d) Geometric, common ratio $\frac{3}{2}$

15. Arithmetic, 7 **17.** Arithmetic, $t + 1$ **19.** Geometric, $\dfrac{1}{t}$
21. Geometric, $\frac{4}{27}$ **23.** $2i$ **25.** 5 **27.** $\frac{81}{4}$
29. **(a)** $A_n = 32{,}000(1.05)^{n-1}$ **(b)** \$32,000, \$33,600, \$35,280, \$37,044, \$38,896.20, \$40,841.01, \$42,883.06, \$45,027.21
31. 12,288 **35.** **(a)** 9 **(b)** $\pm 6\sqrt{2}$ **37.** 126
39. 384 **41.** $0^2 + 1^2 + 2^2 + \cdots + 9^2$
43. $\dfrac{3}{2^2} + \dfrac{3^2}{2^3} + \dfrac{3^3}{2^4} + \cdots + \dfrac{3^{50}}{2^{51}}$
45. $\displaystyle\sum_{k=1}^{33} 3k$ **47.** $\displaystyle\sum_{k=1}^{100} k2^{k+2}$ **49.** Geometric; 4.68559
51. Arithmetic, $5050\sqrt{5}$ **53.** Geometric, 9831
55. $\frac{5}{7}$ **57.** Divergent **59.** Divergent **61.** 13 **63.** 65,534
65. \$2390.27
67. Let $P(n)$ denote the statement $1 + 4 + 7 + \cdots + (3n-2) = \dfrac{n(3n-1)}{2}$.

Step 1 $P(1)$ is true, since $1 = \dfrac{1(3 \cdot 1 - 1)}{2}$.
Step 2 Suppose $P(k)$ is true. Then

$$\begin{aligned} 1 + 4 + 7 + \cdots + (3k-2) + [3(k+1) - 2] &= \frac{k(3k-1)}{2} + [3k+1] && \text{Induction hypothesis} \\ &= \frac{3k^2 - k + 6k + 2}{2} \\ &= \frac{(k+1)(3k+2)}{2} \\ &= \frac{(k+1)[3(k+1) - 1]}{2} \end{aligned}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

69. Let $P(n)$ denote the statement $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)=n+1$.

Step 1 $P(1)$ is true, since $\left(1+\frac{1}{1}\right)=1+1$.
Step 2 Suppose $P(k)$ is true. Then

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)$$
$$=(k+1)\left(1+\frac{1}{k+1}\right) \quad \text{Induction hypothesis}$$
$$=(k+1)+1$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

71. Let $P(n)$ denote the statement $a_n = 2\cdot 3^n - 2$.

Step 1 $P(1)$ is true, since $a_1 = 2\cdot 3^1 - 2 = 4$.
Step 2 Suppose $P(k)$ is true. Then

$$a_{k+1} = 3a_k + 4$$
$$= 3(2\cdot 3^k - 2) + 4 \quad \text{Induction hypothesis}$$
$$= 2\cdot 3^{k+1} - 2$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

73. 100 **75.** 32 **77.** $A^3 - 3A^2B + 3AB^2 - B^3$
79. $1 - 6x^2 + 15x^4 - 20x^6 + 15x^8 - 6x^{10} + x^{12}$
81. $1540a^3b^{19}$ **83.** $17{,}010A^6B^4$

CHAPTER 8 TEST ■ PAGE 619

1. 1, 6, 15, 28, 45, 66; 161 **2.** 2, 5, 13, 36, 104, 307 **3. (a)** 3 **(b)** $a_n = 2 + (n-1)3$ **(c)** 104 **4. (a)** $\frac{1}{4}$ **(b)** $a_n = 12\left(\frac{1}{4}\right)^{n-1}$ **(c)** $3/4^8$ **5. (a)** $\frac{1}{5}, \frac{1}{25}$ **(b)** $\dfrac{5^8-1}{12{,}500}$ **6. (a)** $-\frac{8}{9}, -78$ **(b)** 60
8. (a) $(1-1^2)+(1-2^2)+(1-3^2)+(1-4^2)+(1-5^2)=-50$
(b) $(-1)^3 2^1 + (-1)^4 2^2 + (-1)^5 2^3 + (-1)^6 2^4 = 10$
9. (a) $\frac{58{,}025}{59{,}049}$ **(b)** $2+\sqrt{2}$
10. Let $P(n)$ denote the statement

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Step 1 $P(1)$ is true, since $1^2 = \dfrac{1(1+1)(2\cdot 1+1)}{6}$.

Step 2 Suppose $P(k)$ is true. Then

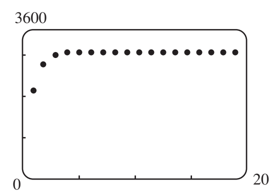
$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{Induction hypothesis}$$
$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$
$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$
$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$
$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

So $P(k+1)$ follows from $P(k)$. Thus by the Principle of Mathematical Induction $P(n)$ holds for all n.

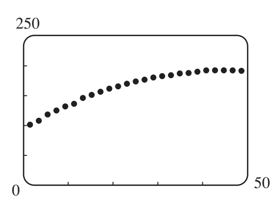
11. $32x^5 + 80x^4y^2 + 80x^3y^4 + 40x^2y^6 + 10xy^8 + y^{10}$
12. $\binom{10}{3}(3x)^3(-2)^7 = -414{,}720x^3$
13. (a) $a_n = (0.85)(1.24)^n$ **(b)** 3.09 lb **(c)** Geometric

FOCUS ON MODELING ■ PAGE 622

1. (a) $A_n = 1.0001A_{n-1}, A_0 = 275{,}000$ **(b)** $A_0 = 275{,}000$, $A_1 = 275{,}027.50$, $A_2 = 275{,}055.00$, $A_3 = 275{,}082.51$, $A_4 = 275{,}110.02$, $A_5 = 275{,}137.53$, $A_6 = 275{,}165.04$, $A_7 = 275{,}192.56$ **(c)** $A_n = 1.0001^n(275{,}000)$
3. (a) $A_n = 1.0025A_{n-1} + 100, A_0 = 100$ **(b)** $A_0 = 100$, $A_1 = 200.25$, $A_2 = 300.75$, $A_3 = 401.50$, $A_4 = 502.51$
(c) $A_n = 100[(1.0025^{n+1} - 1)/0.0025]$ **(d)** \$6580.83
5. (b) $A_0 = 2400, A_1 = 3120, A_2 = 3336, A_3 = 3400.8$, $A_4 = 3420.2$ **(c)** $A_n = 3428.6(1 - 0.3^{n+1})$
(d) 3427.8 tons, 3428.6 tons
(e)



7. (b) In the 35th year
9. (a) $R_1 = 104, R_2 = 108, R_3 = 112, R_4 = 116, R_5 = 120$, $R_6 = 124, R_7 = 127$ **(b)** It approaches 200.



CHAPTER 9

SECTION 9.1 ■ PAGE 633

1. $m \times n$; $2 \times 3 = 6$ **2.** permutations, $n!/(n-r)!$
3. combinations, $n!/[r!(n-r)!]$ **4. (a)** False **(b)** True **(c)** False **(d)** True **5.** 336 **7.** 7920 **9.** 100 **11.** 56
13. 330 **15.** 100 **17.** 12 **19. (a)** 40,320 **(b)** 336
21. 8,000,000 **23.** 60 **25.** 32 **27.** 216 **29.** 158,184,000
31. 208,860 **33.** 24,360 **35.** 700,000,000 **37. (a)** 56 **(b)** 256 **39.** 1024 **41. (a)** 3,628,800 **(b)** 151,200
43. 2730 **45.** 336 **47.** 362,880 **49.** 997,002,000
51. 24 **53.** 15 **55.** 277,200 **57.** 2,522,520 **59.** 168
61. 2300 **63.** 220 **65.** 2,598,960 **67.** 120 **69.** 495
71. 120 **73.** 13,983,816 **75. (a)** 15,504 **(b)** 792 **(c)** 6160
77. 1,162,800 **79.** 104,781,600 **81.** 6600 **83.** 182 **85.** 48
87. (a) 20,160 **(b)** 8640 **89.** 17,813,250

SECTION 9.2 ■ PAGE 646

1. sample space; event; $S = \{HH, HT, TH, TT\}$; $E = \{HH, HT, TH\}$; $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{3}{4}$ **2. (a)** $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
(b) mutually exclusive; mutually exclusive
(c) $P(E \cup F) = P(E) + P(F)$
3. $P(E \mid F) = \dfrac{n(E \cap F)}{n(F)}$; $P(E \mid F) = \dfrac{1}{3}$

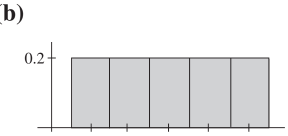
4. **(a)** $P(E \cap F) = P(E)P(F \mid E)$ **(b)** independent; independent **(c)** $P(E \cap F) = P(E) \cdot P(F)$
5. **(a)** $\{1, 2, 3, 4, 5, 6\}$ **(b)** $\{2, 4, 6\}$ **(c)** $\{5, 6\}$
7. **(a)** $S = \{HH, HT, TH, TT\}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{3}{4}$ **(d)** $\frac{1}{2}$
9. **(a)** $\frac{1}{6}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{6}$
11. **(a)** $\frac{1}{13}$ **(b)** $\frac{3}{13}$ **(c)** $\frac{10}{13}$
13. **(a)** $\frac{5}{8}$ **(b)** $\frac{7}{8}$ **(c)** 0
15. **(a)** $\dfrac{C(13, 5)}{C(52, 5)} \approx 0.000495$ **(b)** $\dfrac{4 \cdot C(13, 5)}{C(52, 5)} \approx 0.00198$
(c) $\dfrac{C(12, 5)}{C(52, 5)} \approx 0.000305$ **(d)** $\dfrac{4}{C(52, 5)} \approx 0.00000154$
17. **(a)** $\dfrac{C(3, 2)}{C(8, 2)} \approx 0.11$ **(b)** $\dfrac{C(5, 2)}{C(8, 2)} \approx 0.36$
19. **(a)** $1 - \dfrac{C(39, 5)}{C(52, 5)} \approx 0.778$ **(b)** $1 - \dfrac{C(40, 5)}{C(52, 5)} \approx 0.747$
21. **(a)** $\frac{3}{4}$ **(b)** $\frac{1}{2}$ **(c)** 1 **23.** **(a)** Mutually exclusive; 1 **(b)** Not mutually exclusive; $\frac{2}{3}$ **25.** **(a)** Not mutually exclusive; $\frac{11}{26}$ **(b)** Mutually exlusive; $\frac{1}{2}$ **27.** **(a)** $\frac{1}{3}$ **(b)** $\frac{1}{3}$ **29.** $\frac{1}{3}$ **31.** **(a)** $\frac{1}{2}$ **(b)** 1 **(c)** $\frac{2}{5}$ **(d)** $\frac{2}{5}$ **33.** **(a)** $\frac{7}{30}$ **(b)** $\frac{7}{15}$ **35.** **(a)** $\frac{4}{663}$ **(b)** $\frac{1}{221}$
37. $\frac{1}{12}$ **39.** **(a)** Yes **(b)** $\frac{1}{8}$
41. **(a)** $S = \{GGGG, GGGB, GGBG, GBGG, BGGG, GGBB, GBGB, BGGB, BGBG, BBGG, GBBG, GBBB, BGBB, BBGB, BBBG, BBBB\}$ **(b)** $\frac{1}{16}$ **(c)** $\frac{3}{8}$ **(d)** $\frac{1}{8}$ **(e)** $\frac{11}{16}$
43. $\frac{9}{19}$ **45.** $1/C(49, 6) \approx 7.15 \times 10^{-8}$
47. $\frac{1}{1024}$ **49.** **(a)** $1/48^6 \approx 8.18 \times 10^{-11}$ **(b)** $1/48^{18} \approx 5.47 \times 10^{-31}$ **51.** $\dfrac{1}{P(8, 8)} + \dfrac{1}{P(8, 8)} \approx 0.0000496$
53. **(a)** $\frac{3}{4}$ **(b)** $\frac{1}{4}$ **55.** $\frac{1}{1444}$ **57.** $1/36^3 \approx 2.14 \times 10^{-5}$
59. **(a)** $\frac{3}{8}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{11}{16}$ **(d)** $\frac{13}{16}$ **61.** (i)
63. $600/P(40, 3) = 5/494$ **65.** $\frac{1}{10}$ **67.** $\frac{1}{9{,}979{,}200}$

SECTION 9.3 ■ PAGE 653

1. two; success, failure **2.** $1 - p$; $C(n, r)p^r(1 - p)^{n-r}$
3. $C(5, 2)(0.7)^2(0.3)^3 = 0.1323$
5. $C(5, 0)(0.7)^0(0.3)^5 = 0.00243$
7. $C(5, 1)(0.7)^1(0.3)^4 = 0.02835$
9. $C(5, 4)(0.7)^4(0.3)^1 + C(5, 5)(0.7)^5(0.3)^0 = 0.52822$
11. $C(5, 5)(0.7)^5(0.3)^0 + C(5, 4)(0.7)^4(0.3)^1 = 0.52822$
13. $1 - C(5, 0)(0.7)^0(0.3)^5 - C(5, 1)(0.7)^1(0.3)^4 = 0.96922$
15. **(a)**

Outcome	Probability
1	0.2
2	0.2
3	0.2
4	0.2
5	0.2

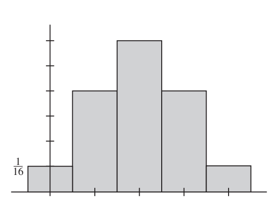
(b)



17. **(a)**

r	Probability
0	$\frac{1}{16}$
1	$\frac{4}{16}$
2	$\frac{6}{16}$
3	$\frac{4}{16}$
4	$\frac{1}{16}$

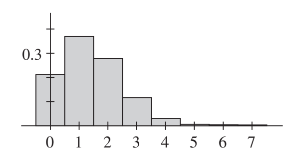
(b)



19. **(a)**

r	Probability
0	0.2097
1	0.3670
2	0.2753
3	0.1147
4	0.0287
5	0.0043
6	0.00036
7	0.000013

(b)



21. $C(6, 2)\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4 \approx 0.20094$
23. $C(10, 4)(0.4)^4(0.6)^6 \approx 0.25082$
25. **(a)** $C(10, 5)(0.45)^5(0.55)^5 \approx 0.23403$
(b) $1 - C(10, 0)(0.45)^0(0.55)^{10} - C(10, 1)(0.45)^1(0.55)^9 - C(10, 2)(0.45)^2(0.55)^8 \approx 0.90044$
27. **(a)** $1 - C(4, 0)(0.75)^0(0.25)^4 \approx 0.99609$
(b) $C(4, 2)(0.75)^2(0.25)^2 + C(4, 3)(0.75)^3(0.25)^1 + C(4, 4)(0.75)^4(0.25)^0 \approx 0.94922$
(c) $C(4, 4)(0.75)^4(0.25)^0 \approx 0.31641$
29. **(a)** $(0.52)^{10} \approx 1.4456 \times 10^{-3}$
(b) $(0.48)^{10} \approx 6.4925 \times 10^{-4}$
(c) $C(10, 5)(0.52)^5(0.48)^5 \approx 0.24413$
31. **(a)** $(0.005)^3 = 1.25 \times 10^{-7}$
(b) $1 - (0.995)^3 \approx 0.014925$
33. $1 - C(8, 0)(0.04)^0(0.96)^8 - C(8, 1)(0.04)^1(0.96)^7 \approx 0.038147$
35. **(a)** $(0.75)^6 \approx 0.17798$ **(b)** $(0.25)^6 \approx 2.4414 \times 10^{-4}$
(c) $C(6, 3)(0.75)^3(0.25)^3 \approx 0.13184$
(d) $1 - C(6, 6)(0.25)^0(0.75)^6 - C(6, 5)(0.25)^1(0.75)^5 \approx 0.46606$
37. **(a)** $1 - (0.75)^4 \approx 0.68359$
(b) $C(4, 3)(0.25)^3(0.75)^1 + C(4, 4)(0.25)^4(0.75)^0 \approx 0.05078$
39. **(a)** $C(4, 1)(0.3)^1(0.7)^3 = 0.4116$
(b) $1 - (0.7)^4 = 0.7599$
41. **(a)** $C(10, 8)(0.4)^8(0.6)^2 + C(10, 9)(0.4)^9(0.6)^1 + C(10, 10)(0.4)^{10}(0.6)^0 \approx 0.0123$ **(b)** Yes

SECTION 9.4 ■ PAGE 658

1. $E = \$10 \times 0.9 + \$100 \times 0.1 = \$19$ **2.** \$19 **3.** \$1.50
5. \$0.94 **7.** \$0.92 **9.** 0 **11.** −\$0.30 **13.** −\$0.0526
15. −\$0.50 **17.** No, she should expect to lose \$2.10 per stock.
19. −\$0.93 **21.** 3.35 h **23.** 1.95 **25.** **(a)** No **(b)** \$25.50
27. **(a)** No **(b)** \$70 **29.** **(a)** No **(b)** \$623

CHAPTER 9 REVIEW ■ PAGE 661

1. 624 **3.** **(a)** 10 **(b)** 20 **5.** 120 **7.** 45 **9.** 17,576
11. 120 **13.** 5 **15.** 14 **17.** **(a)** 240 **(b)** 3360 **(c)** 1680
19. 720 **21.** 120 **23.** **(a)** 31,824 **(b)** 11,760 **(c)** 19,448 **(d)** 2808 **(e)** 2808 **(f)** 6,683,040
25. **(a)** $\frac{2}{3}$ **(b)** $\frac{8}{15}$ **(c)** $\frac{2}{15}$ **(d)** $\frac{4}{5}$
27. **(a)** $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ **(b)** $\frac{1}{8}$ **(c)** $\frac{1}{2}$ **(d)** $\frac{1}{2}$ **29.** $\frac{1}{78}$ **31.** **(a)** $\frac{1}{624}$ **(b)** $\frac{1}{48}$ **(c)** $\frac{3}{52}$
33. **(a)** $\dfrac{C(4, 4)}{C(52, 4)} \approx 3.69 \times 10^{-6}$ **(b)** $\dfrac{C(13, 4)}{C(52, 4)} \approx 0.00264$
(c) $\dfrac{2 \cdot C(26, 4)}{C(52, 4)} \approx 0.11044$ **35.** $\frac{1}{24}$ **37.** **(a)** 3 **(b)** 0.51

39. (a) 10^9 **(b)** 10^5 **(c)** 10^{-4}
41. (a) $\frac{1}{13}$ **(b)** $\frac{2}{13}$ **(c)** $\frac{1}{3}$ **(d)** $\frac{1}{12}$
43. (a) $(0.3)^5 = 0.00243$ **(b)** $(0.7)^5 = 0.16807$
(c) $C(5, 2)(0.3)^2(0.7)^3 = 0.3087$
(d) $C(5, 3)(0.3)^2(0.7)^3 + C(5, 4)(0.3)^1(0.7)^4 + C(5, 5)(0.3)^0(0.7)^5 = 0.83692$

45.

Outcome (heads)	Probability
0	0.0081
1	0.0756
2	0.2646
3	0.4116
4	0.2401

47. \$0.00016

CHAPTER 9 TEST ▪ PAGE 664

1. 81 **2.** 72 **3. (a)** 456,976,000 **(b)** 258,336,000
4. (a) $P(30, 4) = 657{,}720$ **(b)** $C(30, 4) = 27{,}405$ **5.** 12
6. $4 \cdot 2^{14} = 65{,}536$ **7. (a)** $4! = 24$ **(b)** $6!/3! = 120$
8. $30 \cdot 29 \cdot 28 \cdot C(27, 5) = 1{,}966{,}582{,}800$ **9. (a)** $\frac{1}{2}$ **(b)** $\frac{1}{13}$
(c) $\frac{1}{26}$ **10. (a)** $\frac{5}{13}$ **(b)** $\frac{6}{13}$ **(c)** $\frac{9}{13}$
11. $C(5, 3)/C(15, 3) \approx 0.022$ **12.** $\frac{1}{6}$
13. $1 - 1 \cdot \frac{11}{12} \cdot \frac{10}{12} \cdot \frac{9}{12} \approx 0.427$
14. (a) $C(10, 6)(0.55)^6(0.45)^4 \approx 0.23837$
(b) $C(10, 0)(0.55)^0(0.45)^{10} + C(10, 1)(0.55)^1(0.45)^9 + C(10, 2)(0.55)^2(0.45)^8 \approx 0.02739$ **15.** \$0.65

FOCUS ON MODELING ▪ PAGE 666

1. (b) $\frac{9}{10}$ **3. (b)** $\frac{7}{8}$ **7. (b)** $\frac{1}{2}$

CUMULATIVE REVIEW TEST FOR CHAPTERS 8 AND 9 ▪ PAGE 669

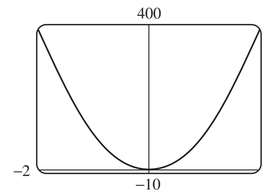
1. (a) $\frac{7}{15}, \frac{20}{41}$ **(b)** $\frac{99}{340}, \frac{801}{7984}$ **(c)** $\frac{37}{2}, \frac{115}{2}$ **(d)** $12\left(\frac{5}{6}\right)^6, 12\left(\frac{5}{6}\right)^{19}$
(e) $(-2)^6 \cdot 0.01 = 0.64, (-2)^{19} \cdot 0.01 = -5242.88$
2. (a) 41.4 **(b)** 88,572 **(c)** $\frac{5115}{512}$ **(d)** 9 **3.** \$2658.15
4. Hint: Induction step is
$a_{n+1} = a_n + 2(n + 1) - 1 = n^2 + 2n + 1 = (n + 1)^2$
5. (a) $32x^5 - 40x^4 + 20x^3 - 5x^2 + \frac{5}{8}x - \frac{1}{32}$ **(b)** $\frac{495}{16}x^4$
6. (a) $26^3 \cdot 10^4 = 175{,}760{,}000$
(b) $P(26, 3) \cdot P(10, 4) = 78{,}624{,}000$
(c) $C(26, 3) \cdot C(10, 4) = 546{,}000$ **7. (a)** $\frac{1}{36}$ **(b)** $\frac{1}{8}$ **(c)** $\frac{5}{108}$
8. -0.56 dollar **9. (a)** Getting 3 heads and 2 tails
(b) $10\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3 \approx 0.16$ **10. (a)** The event that a randomly selected insect has at least one spot; $\frac{1023}{3072}$ **(b)** $\frac{2049}{3072}$

APPENDIX A ▪ PAGE 672

1. 3.09 **2.** 129.4 **3.** 14,220 **4.** 38.41 **5.** 2.52 **6.** 20.67
7. 2300 **8.** -75.9 **9.** 3.80 **10.** 506.6 **11.** 33.1 ft, 87.3 ft^2
12. 997 cm^3 **13.** 2.66×10^{-12} N **14. (a)** 3.52×10^{22} N
(b) 7.93×10^{21} lb

APPENDIX B ▪ PAGE 677

1. (c) **2.** (c) **3.** (c) **4.** (d) **5.** (c) **6.** (d)

7.


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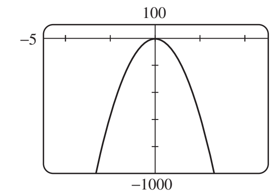

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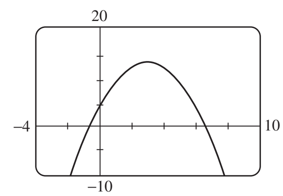

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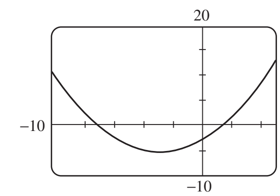

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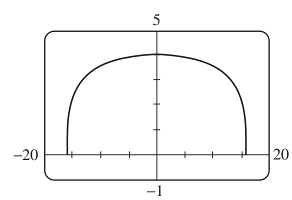

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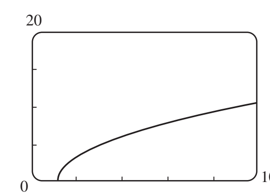

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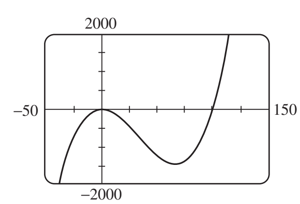

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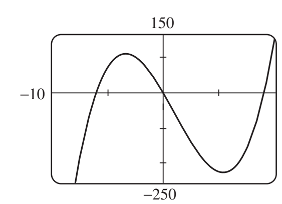

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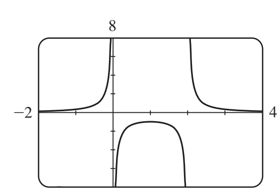

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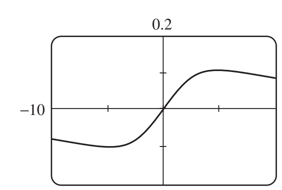

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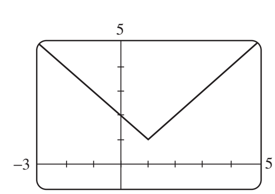

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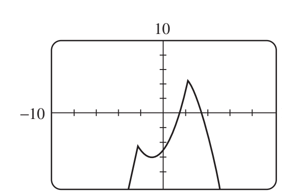
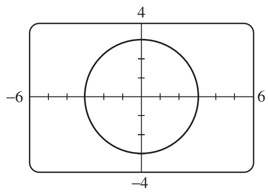


19. No **20.** No **21.** Yes, 2 **22.** Yes, 1

23.



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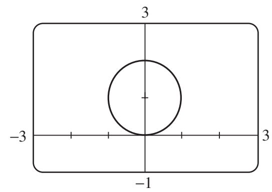

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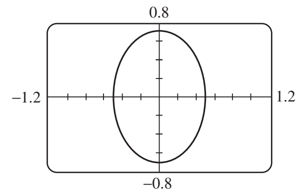
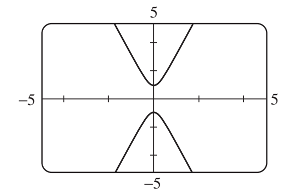


26.



INDEX

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SEQUENCES AND SERIES

Arithmetic

$a, a + d, a + 2d, a + 3d, a + 4d, \ldots$

$a_n = a + (n - 1)d$

$S_n = \sum_{k=1}^{n} a_k = \frac{n}{2}[2a + (n - 1)d] = n\left(\frac{a + a_n}{2}\right)$

Geometric

$a, ar, ar^2, ar^3, ar^4, \ldots$

$a_n = ar^{n-1}$

$S_n = \sum_{k=1}^{n} a_k = a\frac{1 - r^n}{1 - r}$

If $|r| < 1$, then the sum of an infinite geometric series is

$$S = \frac{a}{1 - r}$$

THE BINOMIAL THEOREM

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

FINANCE

Compound interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where A is the amount after t years, P is the principal, r is the interest rate, and the interest is compounded n times per year.

Amount of an annuity

$$A_f = R\frac{(1 + i)^n - 1}{i}$$

where A_f is the final amount, i is the interest rate per time period, and there are n payments of size R.

Present value of an annuity

$$A_p = R\frac{1 - (1 + i)^{-n}}{i}$$

where A_p is the present value, i is the interest rate per time period, and there are n payments of size R.

Installment buying

$$R = \frac{iA_p}{1 - (1 + i)^{-n}}$$

where R is the size of each payment, i is the interest rate per time period, A_p is the amount of the loan, and n is the number of payments.

COUNTING

Fundamental counting principle

Suppose that two events occur in order. If the first can occur in m ways and the second can occur in n ways (after the first has occurred), then the two events can occur in order in $m \times n$ ways.

Permutations and combinations

The number of **permutations** of n objects taken r at a time is

$$P(n, r) = \frac{n!}{(n - r)!}$$

The number of **combinations** of n objects taken r at a time is

$$C(n, r) = \frac{n!}{r!(n - r)!}$$

The number of **subsets** of a set with n elements is 2^n.

The number of **distinguishable permutations** of n elements, with n_i elements of the ith kind (where $n_1 + n_2 + \cdots + n_k = n$), is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

PROBABILITY

Probability of an event:

If S is a sample space consisting of equally likely outcomes, and E is an event in S, then the probability of E is

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of elements in } E}{\text{number of elements in } S}$$

Complement of an event:

$$P(E') = 1 - P(E)$$

Union of two events:

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Conditional probability of *E* given *F*:

$$P(E|F) = \frac{n(E \cap F)}{n(F)}$$

Intersection of two events:

$$P(E \cap F) = P(E)P(F|E)$$

Intersection of two independent events:

$$P(E \cap F) = P(E)P(F)$$

Binomial Probability: If an experiment has the outcomes "success" and "failure" with probabilities p and $q = 1 - p$ respectively, then

$$P(r \text{ successes in } n \text{ trials}) = C(n, r)p^r q^{n-r}$$

If a game gives payoffs of $a_1, a_2, \ldots, a_n$ with probabilities $p_1, p_2, \ldots, p_n$, respectively, then the **expected value** is

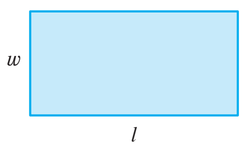
$$E = a_1 p_1 + a_2 p_2 + \cdots + a_n p_n$$

GEOMETRIC FORMULAS

Formulas for area A, perimeter P, circumference C, volume V:

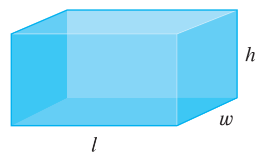
Rectangle

$A = lw \qquad P = 2l + 2w$



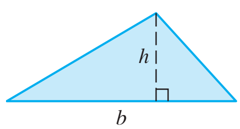
Box

$V = lwh$



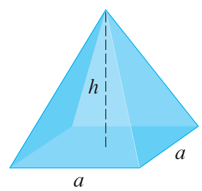
Triangle

$A = \frac{1}{2}bh$



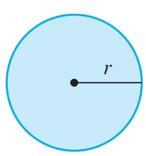
Pyramid

$V = \frac{1}{3}ha^2$



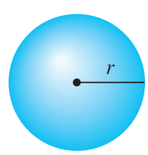
Circle

$A = \pi r^2 \qquad C = 2\pi r$



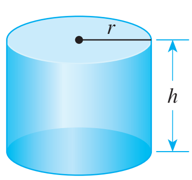
Sphere

$V = \frac{4}{3}\pi r^3 \qquad A = 4\pi r^2$



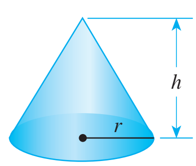
Cylinder

$V = \pi r^2 h$



Cone

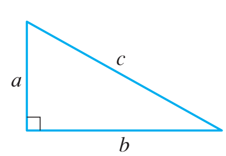
$V = \frac{1}{3}\pi r^2 h$



PYTHAGOREAN THEOREM

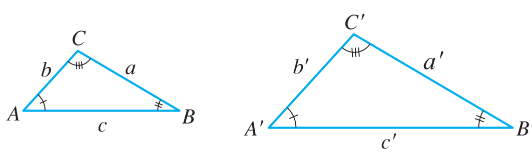
In a right triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

$$a^2 + b^2 = c^2$$



SIMILAR TRIANGLES

Two triangles are similar if corresponding angles are equal.



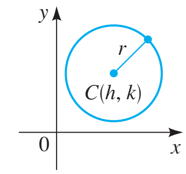
If $\triangle ABC$ is similar to $\triangle A'B'C'$, then ratios of corresponding sides are equal:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

CONIC SECTIONS

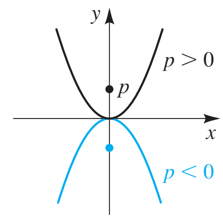
Circles

$(x - h)^2 + (y - k)^2 = r^2$



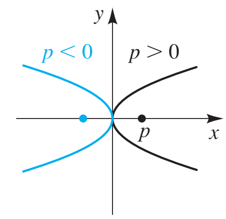
Parabolas

$x^2 = 4py$

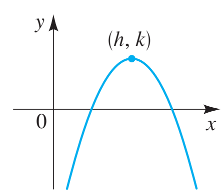


Focus $(0, p)$, directrix $y = -p$

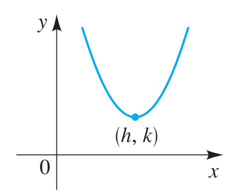
$y^2 = 4px$



Focus $(p, 0)$, directrix $x = -p$



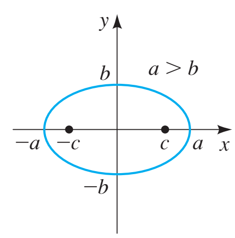
$y = a(x - h)^2 + k,$
$a < 0, \quad h > 0, \quad k > 0$



$y = a(x - h)^2 + k,$
$a > 0, \quad h > 0, \quad k > 0$

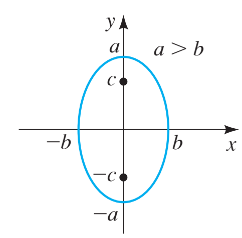
Ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Foci $(\pm c, 0)$, $c^2 = a^2 - b^2$

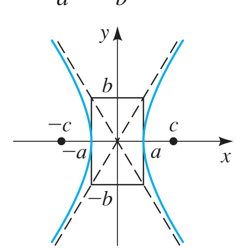
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$



Foci $(0, \pm c)$, $c^2 = a^2 - b^2$

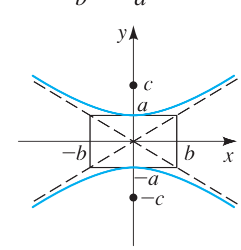
Hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Foci $(\pm c, 0)$, $c^2 = a^2 + b^2$

$$-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$



Foci $(0, \pm c)$, $c^2 = a^2 + b^2$